Exchange currents in null-plane Poincaré invariant quantum mechanics

Y. Huang and W. N. Polyzou
F. Coester

The University of Iowa

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Elastic electron-deuteron scattering observables

\[
\frac{d\sigma}{d\Omega}(Q^2, \theta) = \frac{\alpha^2 \cos^2(\theta/2)}{4E_i^2 \sin^4(\theta/2)} \frac{E_f}{E_i} [A(Q^2) + B(Q^2) \tan^2(\theta/2)]
\]

\[
T_{20}(Q^2, \theta) = \sqrt{2} \frac{\frac{d\sigma}{d\Omega}}{\frac{d\sigma}{d\Omega}}(Q^2, \theta) - \frac{d\sigma}{d\Omega}(Q^2, \theta)
\]
Electron-deuteron scattering observables 
\(A(Q^2), B(Q^2), T_{20}(Q^2)\) are functions of the deuteron current 
matrix elements.

\[ \langle P', \mu', d | I^{\nu}(0) | P, \mu, d \rangle \]

Theoretical problem: compute current matrix elements.
Features - null plane kinematic subgroup

- All current matrix element are linear functions of matrix elements of $l^+(0) \ (Q^+ = 0)$.

- All current matrix element of $l^+(0)$ are invariant with respect to kinematic Lorentz transformations.

- Matrix elements of the one-body part of $l^+(0)$ factor out of current matrix elements with physical momentum transfer.

- Boosts are kinematic and form a subgroup.
Observables and current matrix elements

\[ A(Q^2) = G_0^2(Q^2) + \frac{2}{3} \eta G_1^2(Q^2) + G_2^2(Q^2) \]

\[ B(Q^2) = \frac{4}{3} \eta (1 + \eta) G_1^2(Q^2) \]

\[ T_{20}(Q^2, \theta) = \]

\[ \frac{G_2^2(Q^2) + \sqrt{8} G_0(Q^2) G_2(Q^2) + \frac{1}{3} \eta G_1^2(Q^2)[1 + 2(1 + \eta) \tan^2(\theta/2)]}{\sqrt{2}[A(Q^2) + B(Q^2) \tan^2(\theta/2)]} \]
\[ l^\mu_{\nu\nu'} := c \left\langle \frac{Q}{2}, \nu | l^\mu(0) \right| - \frac{Q}{2}, \nu', d \right\rangle_c \]

\[ G_0(0) = \frac{1}{3} (2l^0_{00} + l^0_{11}) \]

\[ G_1(Q^2) = \sqrt{\frac{2}{\eta}} l^1_{-10} \]

\[ G_2(Q^2) = \frac{\sqrt{2}}{3} (l^0_{00} - l^0_{11}) \]
Symmetry considerations

**State covariance**

\[ U(\Lambda, a)|P, \mu, d\rangle = |\Lambda P, \mu', d\rangle J(\Lambda, P)e^{i\Lambda P \cdot a}D_{\mu' \mu}^{1}[R_w(\Lambda, P)] \]

**Current covariance**

\[ U(\Lambda, a)I^\mu(x)U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu I^\nu(\Lambda x + a) \]

**Current conservation**

\[ g_{\mu\nu}[P^\mu, I^\nu(0)] = 0 \]
State covariance, current covariance, current conservation

All current matrix elements linear functions of 3 independent matrix elements

Null plane kinematics, $Q^+ = 0 \Rightarrow$ independent matrix elements can be chosen as matrix elements of $l^+(0)$ with light-front spin
Model construction

Single nucleon Hilbert space - null plane basis

\[ \psi(\tilde{p}, \mu) = \langle \tilde{p}, \mu | \psi \rangle \quad \tilde{p} := (p^1, p^2, p^+ = p^0 + p^3) \]

\[ \int_0^\infty dp^+ \int_{\mathbb{R}^2} dp_\perp \sum_{\mu = -j}^{j} |\psi(\tilde{p}, \mu)|^2 < \infty. \]
Covariance of nucleon states - null plane basis

\[ \langle \tilde{p}, \mu | U_1(\Lambda, a) | \psi \rangle = \]

\[ \int_0^\infty dp^+ \int_{\mathbb{R}^2} dp'_\perp \sum_{\mu' = -j}^j D^{m,j}_{\tilde{p}, \mu; \tilde{p}', \mu'}[\Lambda, a] \langle \tilde{p}', \mu' | \psi \rangle, \]

\[ D^{m,j}_{\tilde{p}, \mu; \tilde{p}', \mu'}[\Lambda, a] := \langle \tilde{p}, \mu | U(\Lambda, a) | \tilde{p}', \mu' \rangle = \]

\[ \delta(\tilde{p} - \tilde{\Lambda}(p')) \sqrt{\frac{p^+}{p^+_f}} D^j_{\mu \mu'} [\Lambda^{-1}_f(\tilde{p}/m) \Lambda_f(\tilde{p}'/m)] e^{ip \cdot a} \]
Two-nucleon Hilbert space

\[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1 \]

Kinematic representation of Poincaré group

\[ U_0(\Lambda, A) := U_1(\Lambda, A) \otimes U_1(\Lambda, A). \]
Poincaré Clebsch-Gordan coefficients - null plane basis

\[ \langle m_1, j_1, \tilde{p}_1, \mu_1; m_2, j_2, \tilde{p}_2, \mu_2 | k, j(l, s) \tilde{P}, \mu \rangle = \]

\[ \delta(\tilde{P} - \tilde{p}_1 - \tilde{p}_2) \frac{\delta(k - k(\tilde{p}_1, \tilde{p}_2))}{k^2} \sqrt{\frac{P^+ \omega_1(k) \omega_2(k)}{(\omega_1(k) + \omega_2(k)) p_1^+ p_2^+}} \times \]

\[ \sum D_{\mu_1, \mu_1'}^{1/2} [\Lambda_f^{-1}(k/m_1) \Lambda_c(k/m_1)] D_{\mu_2, \mu_2'}^{1/2} [\Lambda_f^{-1}(-k/m_2) \Lambda_c(-k/m_2)] \times \]

\[ Y_{lm}(\hat{k}(\tilde{p}_1, \tilde{p}_2)) \langle \frac{1}{2}, \mu_1', \frac{1}{2}, \mu_2' | s, \mu_s \rangle \langle l, m, s, \mu_s | j, \mu \rangle \]
Irreducible representations - null plane basis

\[
\langle k, j(l, s) \tilde{P}, \mu | U_0(\Lambda, a) | \psi \rangle = \\
\int_0^\infty dP^+ \int_{\mathbb{R}^2} dP'_{\perp} \sum_{\mu' = -j}^{j} D_{\tilde{P}, \mu, \tilde{P}', \mu'}^{M_0(k), j} [\Lambda, a] \langle k, j(l, s) \tilde{P}', \mu' | \psi \rangle
\]
Model input
Dynamical 2 nucleon model

\[ M^2 = M_0^2 + 4m\nu_{nn} \]

\[ \langle k', j', (l', s')|\tilde{P}', \mu'|\nu_{nn}|k, j, (l, s)|\tilde{P}, \mu \rangle \]

\[ \delta (\tilde{P}' - \tilde{P}) \delta_{j' j} \delta_{\mu' \mu} \langle k', l', s'|v^j||k, l, s \rangle \]
Mass eigenvalue problem

\[(4k^2 + 4m^2 - \lambda^2)\phi_{\lambda,j}(k, l, s) = \]

\[-\sum_{s=0}^{1} \sum_{l=|j-s|}^{|j+s|} \int_{0}^{\infty} 4m \langle k, l, s \| v^{i}_{nn} \| k', l', s' \rangle k'^2 dk' \phi_{\lambda,j}(k', l', s') \]
Dynamical representation of Poincaré group

\[\langle k', j', (l', s') \tilde{P}', \mu' | U(\Lambda, a) | \lambda, j, \tilde{P}, \mu \rangle = \]

\[\int \sum_{\mu'' = -j}^{j} \langle k', j', (l', s') \tilde{P}', \mu' | \lambda, j, \tilde{P}'', \mu'' \rangle \times \]

\[d\tilde{P}'' \langle \lambda, j, \tilde{P}'', \mu'' | U(\Lambda, A) | \lambda, j, \tilde{P}, \mu \rangle = \]

\[\phi_{\lambda, j}(k', l', s') D_{\tilde{p}', \mu'; p, \mu}^{\lambda, j'} [\Lambda, a] \]
Realistic two-body model

\[ S(H_{nr}, H_{0nr}) = \Omega^\dagger_+ (H_{nr}, H_{0nr}) \Omega_- (H_{nr}, H_{0nr}) = \]

\[ \Omega^\dagger_+ (H_r, H_{0r}) \Omega_- (H_r, H_{0r}) = S(H_r, H_{0r}). \]
Nucleon currents

\[
\langle \tilde{p}', \nu' | l_1^\mu(0) | \tilde{p}, \nu \rangle = \sqrt{m_p + \bar{u}(p')\Gamma^\mu u(p)} \sqrt{m_p}
\]

where

\[
\Gamma^\mu = \gamma^\mu F_1(Q^2) + \frac{1}{2} [\gamma^\mu, \frac{1}{2m} \gamma \cdot Q] F_2(Q^2)
\]
Exchange current

\[ \langle \tilde{P}', \nu', d| l_{\text{ex}}^+ (0)| \tilde{P}, \nu, d \rangle := \]

\[ \int \langle \tilde{P}', \nu', d| \tilde{p}_1', \nu_1', \tilde{p}_2', \nu_2' \rangle \times \]

\[ (-\frac{1}{2m}) \sqrt{\frac{m}{p_1^+}} \bar{u}_f (p_1') \Gamma^+ \gamma_5 \left( \frac{P \cdot \gamma}{M_d} \right) u_f (p_1'') \sqrt{\frac{m}{p_1''}} d\tilde{p}_1' d\tilde{p}_1'' d\tilde{p}_2' \]

\[ \langle \tilde{p}_1'', \nu_1'', \tilde{p}_2', \nu_2' | U(A_f (P/M_d)) \tilde{\nu}_{\text{ope}} | \tilde{P}_0, \nu, d \rangle + (1 \leftrightarrow 2) + hc \]
Independent matrix element of $I^+(0)$

9 matrix elements - 3 independent matrix elements

$I_{ex}^{+}(0)$ constructed to be kinematically covariant

kinematic symmetries eliminate 5 matrix elements

$I_{11}^{+}(0), I_{10}^{+}(0), I_{1-1}^{+}(0), I_{00}^{+}(0)$ kinematically independent

(dynamical) rotational symmetry eliminates 1 matrix element.
Choosing independent matrix elements

I. Count spin flips.

II. (FFS) Use linear combinations of canonical spin matrix elements that are largest in the infinite momentum frame.

III. (FC) Use linear combinations null plane matrix elements constructed to minimize dependence of matrix elements on mass.
Rotational covariance - Poincaré Wigner Eckart theorem

Independent matrix element - choice I:

\[(1 + \eta) G_0(Q^2) = \left(\frac{1}{2} - \frac{\eta}{3}\right)(l_{11} + l_{00}) + \frac{5\sqrt{2}\eta}{3} l_{10} + \left(\frac{2\eta}{3} - \frac{1}{6}\right) l_{1-1}\]

\[(1 + \eta) G_1(Q^2) = l_{11} + l_{00} - l_{1-1} - (1 - \eta) \sqrt{\frac{2}{\eta}} l_{10}\]

\[(1 + \eta) G_2(Q^2) = -\frac{\sqrt{2}\eta}{3}(l_{11} + l_{00}) + \frac{4\sqrt{\eta}}{3} l_{10} - \frac{\sqrt{2}}{3}(2 + \eta) l_{1-1}\]
Rotational covariance - Poincaré Wigner Eckart theorem

Independent matrix element - choice II \((G_0(Q^2)\) of choice I replaced by:

\[
(1 + \eta) G_0(Q^2) = \left(\frac{2\eta}{3} + 1\right) l_{1,1}^+ - \frac{\eta}{3} I_{00}^+ + \frac{2\sqrt{2}\eta}{3} I_{1,0}^+ + \left(\frac{2\eta + 1}{3}\right) I_{1,-1}^+
\]

Independent matrix element - choice III \((G_0(Q^2)\) and \(G_2(Q^2)\) of choice I replaced by

\[
G_0(Q^2) = (1 + \frac{2\eta}{3}) l_{11}^+ + \frac{1}{3} I_{1,-1}^+ - \frac{2\eta}{3} G_1(Q^2)
\]

\[
G_2(Q^2) = \frac{2\sqrt{2}}{3} (\eta l_{1,1}^+ - l_{1,-1}^+ - \eta G_1(Q^2))
\]
The model input includes the realistic nucleon-nucleon interaction, nucleon form factors, the choice of independent matrix elements, and the model exchange current.

Without the model exchange current the results are not very sensitive to the choice of independent current matrix elements.

The model exchange current increases the sensitivity to the choice of independent current matrix elements.

Summary of results
Neutron Electric Form Factors

\[ G_{E_{n}} / G_{d} \]

\[ Q^{2}[(GeV)^{2}] \]
Isoscalar Nucleon Form Factor: $F_{1N}$
Isoscalar Nucleon Form Factor: $F_{2N}$
One-Pion Exchange Potential

\[ v(k) = \frac{f^2}{m^2(m^2 + k^2)} \]

**V18**

*One-Pion Exchange Potential*
Deuteron Form Factor: $G_0(Q^2)$
Deuteron Form Factor: $G_1(Q^2)$

![Graph showing the deuteron form factor $G_1(Q^2)$ with various models and their legends.](image-url)

- IM BBBA
- IM BI
- IM+Exchange BBBA
- IM+Exchange BI

The graph plots $G_1(Q^2)$ against $Q^2$ (in GeV$^2$) with different models indicated by different lines and colors.
Deuteron Form Factor: $G_2(Q^2)$
A(Q^2): Impulse Approximation
$B(Q^2)$: Impulse Approximation

![Graph showing $B(Q^2)$ as a function of $Q^2$](image)

- SLAC NPSA NE4
- Martin
- Bonn
- Saclay ALS
- Mainz
- Stanford Mark III
- BBA
- BBBA
- BI
- Lomon
- Kelly

$Q^2 [\text{GeV}^2]$ vs. $B$
Deuteron Structure Function $T_{20}$

![Graph of Deuteron Structure Function $T_{20}$](image)
$A(Q^2)$: Impulse + Exchange Current

![Graph showing the relationship between $Q^2$ and $A$ for various experiments and institutions.](image-url)
B(Q^2) : Impulse + Exchange Current
$T_{20}(Q^2, 70^\circ)$: Impulse + Exchange Current
\[ I_{++}^{+}(Q^2) - I_{00}^{+}(Q^2) \]
A($Q^2$): Impulse + Exchange; II

![Graph showing the relationship between $Q^2$ and $A$ with various data points from different institutions.](image-url)
$T_{20}(Q^2,70^\circ)$: Impulse + Exchange; II
A(Q^2): Impulse + Exchange; III
$T_{20}(Q^2,70^\circ)$: Impulse + Exchange; III
$A(Q^2)$ : CD Bonn interaction
$T_{20}(Q^2,70^\circ)$: CD Bonn Interaction
$A(Q^2)$: BBBA; II, III

![Graph showing the relationship between $A(Q^2)$ and $Q^2$]
B(Q^2): BBBA; II, III
$T_{20}(Q^2, 70^\circ)$: BBBA; II, III

![Graph showing $T_{20}(Q^2, 70^\circ)$: BBBA; II, III with data points and curves for different experiments such as Novosibirsk-85, Novosibirsk-90, Bates-84, Bates-91, JLab Hall C, IM II, IM III, IM+Exchange II, and IM+Exchange III.](image-url)
Conclusion

I Model exchange current explains difference between measurement and impulse approximation.

II Sensitivity to implementation of the Wigner-Eckart theorem

III Model can be generalized to treat more complex systems