Euclidean relativistic quantum mechanics

W. Polyzou
Midwest Theory Get Together

8 October 2005

Goals

E1. **Model Hilbert space (N particles)**
E2. **Unitary representation of the Poincaré group**
E3. **Spacelike cluster properties**
E4. **Spectral condition**

P1. **Connection to formal Lagrangian field theories**
P2. **Ability to perform computations**

Why a Euclidean approach?

Lagrangian

$$\Downarrow$$

Green functions

$$\Downarrow$$

Quantum theory (OS Axioms)

Input: Model 2N-point Euclidean covariant Green functions

$$G_{2N}(\mathrm{X} : \mathrm{Y})$$

$$\mathrm{X} = (\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_N)$$

$$\mathrm{x}_i = (\mathrm{x}_i^0, \vec{\mathrm{x}}_i, s_i)$$

Model Hilbert space

Vectors

$$F(\mathrm{X})$$

Smooth
Permutation symmetric
Support:

$$F(\mathrm{X}) = 0 \qquad \mathrm{x}_{\mathrm{i}}^0 \leq 0$$

Physical inner product $\langle\cdot|\cdot\rangle$

$$\langle F|F'\rangle := (\Theta\, F, G_{2N}\, F')_E$$

$$\Theta \mathrm{X} := (\theta \mathrm{x}_1, \cdots, \theta \mathrm{x}_N)$$

$$\theta \mathrm{x}_i = (-\mathrm{x}_i^0, \vec{\mathrm{x}}_i, \beta s_i)$$

Poincaré symmetry:

$$H := -\sum_{i=1}^{N} \frac{\partial}{\partial \mathrm{x}_i^0} \qquad \vec{P} := -i \sum_{i=1}^{N} \frac{\partial}{\partial \vec{\mathrm{x}}_i}$$

$$\vec{J} := -i \sum_{i=1}^{N} \left(\vec{\mathrm{x}}_i \times \frac{\partial}{\partial \vec{\mathrm{x}}_i} + \vec{\Sigma}_i \right)$$

$$\vec{K} := \sum_{i=1}^{N} \left(\mathrm{x}_i^0 \frac{\partial}{\partial \vec{\mathrm{x}}_i} - \vec{\mathrm{x}}_i \frac{\partial}{\partial \mathrm{x}_i^0} + \vec{\mathrm{B}}_i \right)$$

$$H = H^\dagger,\ \vec{P} = \vec{P}^\dagger,\ \vec{J} = \vec{J}^\dagger,\ \vec{K} = \vec{K}^\dagger$$

$$\{H, \vec{P}, \vec{J}, \vec{K}\}$$

Satisfy Poincaré CR

Reflection Positivity

$$\langle F|F\rangle \geq 0$$

$$\Downarrow$$

$$H \geq 0$$

Holds for free Green functions

Holds for lattice QCD (F Gauge invariant)

Cluster Properties

Positive time constraints symmetric

$$+$$

$$G_0 = G(1) \otimes G(2) \qquad G_0^{-1} = G(1)^{-1} \otimes G(2)^{-1}$$

$$G(12) = G_0 + G_0 K(12) G(12)$$

$$G(12)^{-1} = G_0^{-1} - K(12)$$

$$G(123)^{-1} = G(12)^{-1} G(3)^{-1} + G(23)^{-1} G(1)^{-1} +$$

$$G(31)^{-1} G(2)^{-1} - 2 G(1)^{-1} G(2)^{-1} G(3)^{-1} - K(123)$$

$$G(N)^{-1} = \sum_a (-)^{n_a} (n_a - 1)! \prod_{i=1}^{n_a} G(a_i)^{-1} - K(N)$$

$$\Downarrow$$

$$U(\Lambda, a) \to U_A(\Lambda, a) \otimes U_B(\Lambda, a) \otimes \cdots$$

Conservation of trouble

$$\langle F|F\rangle \geq 0 \qquad ?$$

$$G = G_0 + G_0 K G \qquad G_0 = G_1 \otimes G_2$$

$$(F, \Theta G_0 F) \geq 0 \qquad K \quad \text{“small”}$$

$$?$$

$$(F, \Theta G F) \geq 0$$

$$\Pi^+ \Theta G \Pi^+ = \Pi^+ G_0 \Theta \Pi^+ +$$

$$\Pi^+ G_0 K \Pi^+ (I - \Pi^+ G_0 K \Pi^+)^{-1} \Pi^+ G_0 \Theta \Pi^+ +$$

$$(I - \Pi^+ G_0 K \Pi^+)^{-1} \Pi^+ G_0 K \Pi^- (I - \Pi^- G_0 K \Pi^-)^{-1} \Pi^- G_0 \Theta \Pi^+ +$$

$$(I - \Pi^+ G_0 K \Pi^+)^{-1} \Pi^+ G_0 K \Pi^- (I - \Pi^- G_0 K \Pi^-)^{-1} \Pi^- G_0 K (\Pi^+ \Theta G \Pi^+)$$

$$(\Pi^- := I - \Pi^+)$$

Computations

Quantities that can be calculated easily

$$\{F_n\} \qquad \langle F_m | F_n \rangle = \delta_{mn}$$

$$\langle F_m | e^{-\beta H} | F_n \rangle = \delta_{mn}$$

$$\Downarrow$$

$$P_N e^{-\beta H} P_N$$

Scattering

$$S = \Omega_+^\dagger(H, H_{\mathcal{A}})\Omega_-(H, H_{\mathcal{A}})$$

$$\Omega_\pm(H, H_{\mathcal{A}}) := s - \lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_{\mathcal{A}}t}$$

Theorem

$$\Omega_\pm(H, H_{\mathcal{A}}) := \Omega_\mp(e^{-\beta H}, e^{-\beta H_{\mathcal{A}}})$$

Theorem

$$s - \lim_{n\to\infty} P_n = I$$

$$\downarrow$$

$$s - \lim_{n\to\infty} P_n e^{-\beta H} P_n \to e^{-\beta H}$$

Theorem

$$s - \lim_{n\to\infty} P_n e^{-\beta H} P_n \to e^{-\beta H}$$

$$\Downarrow$$

$$s - \lim_{n\to\infty} e^{iP_n e^{-\beta H} P_n t} \to e^{ie^{-\beta H} t}$$

$$\Downarrow$$

$$s - \lim_{n\to\infty} e^{iP_n e^{-\beta H} P_n t} \Phi e^{-ie^{-\beta H_{\mathcal{A}}} t} = e^{ie^{-\beta H} t} \Phi e^{-ie^{-\beta H_{\mathcal{A}}} t}$$

time-dependent calculations based on related theorems have been used in scattering computations

Conclusion - Summary - Outlook

1. No analytic continuation needed.
2. Given reflection positive Euclidean Green functions, Poincaré invariance, spectral properties, and cluster properties are easy to formulate.
3. Relation to Lagrangian-based “field theories” is apparent.
4. The theoretical challenge is to find a large class of Euclidean Bethe-Salpeter kernels that preserve reflection positivity.
5. The formalism easily produces matrix elements of $e^{-\beta H}$ in a basis of normalizable states.