Cluster Properties and Particle Production in Relativistic Quantum Mechanics

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Relevance

- The cluster property is the justification for few-body physics

Few-body measurements/calculations

\[ \Downarrow \]

Few-body operators

\[ \Downarrow \]

Many-body Hamiltonian
Cluster properties and production

- Physics at \textit{sub-nucleon} scales $\Rightarrow$ Poincaré invariant models
- Cluster properties are \textit{difficult to satisfy} in Poincaré invariant models
- In models with particle production there is \textit{no few-body problem}
Cluster Properties (fixed $N$)

$$\lim_{|r_i - r_j| \to \infty} \left[ U(\Lambda, y) - U_a(\Lambda, y) \right] T_a(r_1, \cdots, r_n) \to 0$$

$$U_a(\Lambda, y) := \bigotimes_i U_{a_i}(\Lambda, y)$$

$$T_a(r_1, \cdots, r_n) := \bigotimes_i U_{a_i}(I, r_i)$$
Goal

- Construct Poincaré invariant quantum models of reactions which change particle number
- Construct few degree-of-freedom models which are (i) directly comparable to experiment (ii) decoupled from the many-body problem (iii) related to the many-body problem by cluster properties
Strategy - start with simple models

- Separate the difficulties due to infinite number of degrees of freedom from the difficulties due to particle production
- Limit the number of degrees of freedom using conservation laws which restrict the production channels

Models in this class include generalizations of the relativistic Lee model and relativistic isobar models
Conservation laws

• Introduce conserved positive “charges”

\[(n_1, n_2); \quad N = (1, 0), \quad \pi = (0, 1), \quad \Delta = (1, 1), \quad \rho = (0, 2), \quad \cdots\]

• Introduce an ordering on charge number

\[(n_1, n_2) \leq (n'_1, n'_2) \iff n_1 \leq n'_1, n_2 \leq n'_2\]

• Use induction on charge number
Factorization into Tensor Products

\[ \mathcal{H}_{(1,2)} = \mathcal{H}_{N\pi\pi'} \oplus \mathcal{H}_{\Delta\pi'} \oplus \mathcal{H}_{\Delta'\pi} \oplus \mathcal{H}_{N\rho} \]

\[ \mathcal{H}_{(1,2)} = \left[ (\mathcal{H}_{N\pi} \oplus \mathcal{H}_{\Delta}) \otimes \mathcal{H}_{\pi'} \right] \oplus \left[ \mathcal{H}_{\Delta'\pi} \oplus \mathcal{H}_{N\rho} \right] \]

\[ \mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}^a = \left( \otimes_i \mathcal{H}_{a_i} \right) \oplus \mathcal{H}^a \]
\[ H = \mathcal{H}_a \oplus \mathcal{H}^a \]

- Cluster property for separation \( a \) only makes sense on the subspace \( \mathcal{H}_a \)
- The residual spaces \( \mathcal{H}^a \) cause technical difficulties
- General construction is treated in detail in
  1. WP, nucl-th/0201013; JMP 43,6024(2002)
  2. WP, nucl-th/0302023; to appear PRC
Cluster Condition on $\mathcal{H}_a \oplus \mathcal{H}^a$

$$\lim_{|r_i-r_j| \to \infty} [U(\Lambda, y) - U_a(\Lambda, y)]T_a(r_1, \cdots, r_n)\Pi_a = 0$$

"$N$-charge" interaction $V_N \Rightarrow \forall a$

$$\lim_{|r_i-r_j| \to \infty} V_N T_a(r_1, \cdots, r_n)\Pi_a = 0$$

- Cluster condition fixes $U(\Lambda, y)$, and generators, up to "$N$-charge" operators
Construction Summary

1. Use Wigner’s form of dynamics for Poincaré invariant addition of interactions
2. Use Poincaré Clebsch-Gordan coefficients and Wigner’s form of dynamics in different orders to generate scattering equivalences
3. Use algebraic properties of scattering equivalences to restore cluster properties
Wigner’s form of dynamics

- Construct irreps of the Poincaré group using $j, M$, four commuting functions of generators $h_i$, four complementary observables $\Delta h_i$

\[ |(j, m), h \rangle \]

\[
U(\Lambda, y) |(j, m)h, d\rangle = \sum_{h'} |(j, m)h', d\rangle D_{h'h}(\Lambda, y)
\]
Wigner’s form of dynamics

- Use \((j, m)h\) Clebsch-Gordan coefficients

\[ \langle (j, m)h, d | (j_1, m_1)h_1 (j_2, m_2)h_2 \rangle \]

to decompose \(\mathcal{H}_1 \otimes \mathcal{H}_2\) into direct integral of free-particle irreducible representations

\[ U_0(\Lambda, y) |(j, m)h, d\rangle = \sum_{h'} |(j, m)h', d\rangle D_{h'h}^{jm}(\Lambda, y) \]
Dynamical perturbations of irreps

\[ M = M_0 + V \]

\[ [V, h_i] = [V, j^2] = [V, \Delta h_i] = 0 \]

\[ M|(j, m_n)h\rangle = m_n|(j, m_n)h\rangle \]

\[ U(\Lambda, y)|(j, m_n)h\rangle = \sum_{h'}|(j, m_n)h'\rangle D_{h'h}^{jm_n}(\Lambda, y) \]
Cluster properties \( \Leftrightarrow \) irreps

- Representations with \( j = j_0 \) do not cluster for more than two charges.

\[
\begin{align*}
\langle 12 \otimes 3 \rangle & \xrightarrow{V_{(12)(3)}} \langle 12 \rangle_I \otimes (3) \\
\langle AB|C\rangle_0 & \quad \langle(12)(3)\rangle \\
\langle(12)I(3)\rangle & \xrightarrow{V_{((12)(3))}} \langle(12)(3)I\rangle \\
\langle(12)(3)\rangle & \sim A_{(12)(3)}
\end{align*}
\]

- \( A_a \) scattering equivalence
Scattering Equivalences

- Scattering equivalences $A_a$ are unitary elements of a $C^*$ algebra of asymptotic constants.
- The $C^*$ algebra provides a functional calculus to construct functions of non-commuting scattering equivalences.
- The operators in this $C^*$ algebra relate Wigner’s forms of dynamics to representations that satisfy the cluster condition.
Example - three-charge problem

\[ H = H_{(1,2)} = H_{N\pi\pi'} \oplus H_{\Delta \pi'} \oplus H_{\Delta' \pi} \oplus H_{N\rho} \]

\[ H_{(N\pi)(\pi')} = \begin{pmatrix}
    K_{N\pi\pi'} + V_{N\pi} & V_{N\pi;\Delta} & 0 & 0 \\
    V_{\Delta;N\pi} & K_{\Delta\pi'} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix} \]
Example

\[ H = A^\dagger \left( \sum_a C_a A_a \left( \sum_i H_{a_i} \right) A_a^\dagger \right) A \]

\[ A = A(A_{a_1}, \ldots, A_{a_n}) \rightarrow A_a \]

\[ H \rightarrow H_a = \sum_i H_{a_i} \]
Observations

\[ H = H_{(N\pi)(\pi')} + H_{(N\pi')(\pi)} + H_{(\pi\pi')(N)} - 2H_{(N)(\pi)(\pi')} + V_3 \]

- The operators \( A \) generate many-body interactions and dynamical \( j^2, h_i, \Delta h_i \)
- The few-charge dynamics determines the many-charge dynamics up to “\( N \)-charge” operators
Summary

- Models have few-charge problems directly tied to experiment
- Models are Poincaré invariant
- Models satisfy cluster properties
- Many-body interactions and dynamical spin are unavoidable consequences of this construction
- Few-charge operators determine many-charge operators up to $N$-charge interactions
Outlook - Realistic Models

- Replace “conserved charges” by # of physical particles?
- Construction requires threshold-by-threshold block diagonalization of $U(\Lambda, a)$
- Simplest production interaction is a short-range $2 \rightarrow 3$ interaction
- Allows clustering and a “few-body” problem directly tied to experiment