Poincaré invariant quantum mechanics: an alternative to integrating relativity with quantum mechanics

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11/28/06
Introduction to Quantum Theory

- Complex vector space
- Scalar product

\[ (\bar{a})^* \cdot \vec{b} := \langle a | b \rangle \]
Example: electron at rest

- **Basis vectors**

  \[ |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

- **Scalar product**

  \[ \langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 1 \quad \langle \downarrow | \uparrow \rangle = \langle \uparrow | \downarrow \rangle = 0 \]
Measurements

The state of an electron is represented by a unit vector

\[ |e\rangle = \cos(\theta) |\uparrow\rangle + \sin(\theta) |\downarrow\rangle \]

\[ P_{\uparrow} := |\langle e | \uparrow \rangle|^2 = |\cos(\theta)|^2 = \cos^2(\theta) \]

\[ P_{\downarrow} := |\langle e | \downarrow \rangle|^2 = |\sin(\theta)|^2 = \sin^2(\theta) \]

\[ P_{\downarrow} + P_{\uparrow} = \sin^2(\theta) + \cos^2(\theta) = 1 \]
$P_{a,b} = |\langle a | b \rangle|^2$
Key Observations

- The Schrödinger equation is not needed to discuss quantum measurements.

- The result of any measurement is a probability.
A vector correspondence $|a\rangle \rightarrow |a'\rangle$ is a symmetry of a quantum theory if

$$P_{ab} = |\langle a|b\rangle|^2 = |\langle a'|b'\rangle|^2 = P_{a'b'}$$
If $|a\rangle \rightarrow |a'\rangle$ is a Symmetry

Physics in the unprimed world is indistinguishable from physics in the primed world.
Relativity = existence of inertial reference frames

Physics in different inertial reference frames is indistinguishable

\[ |a\rangle \rightarrow |a'\rangle \]
\[ \Downarrow \]
\[ P_{ab} = P_{a'b'} \]

Differs from the historical development
How are inertial reference frames related?

Use classical physics

- By coordinate transforms that preserve the form of Newton’s second law? (Galilean relativity)

- By coordinate transforms that preserve the form of Maxwell’s equations of electricity and magnetism? (Special relativity)
Galilean Relativity preserves

\[
\frac{d^2 \vec{x}}{dt'^2} = 0 \quad \Leftrightarrow \quad \frac{d^2 \vec{x}'}{dt^2} = 0
\]

\[
\vec{x}' = R\vec{x} + \vec{v}t + \vec{a} \quad t' = t + c
\]

Special Relativity preserves

\[
|\vec{x} - \vec{y}|^2 - c^2 |t_x - t_y|^2 = |\vec{x}' - \vec{y}'|^2 - c^2 |t_x' - t_y'|^2
\]

\[
c = \text{speed of light}
\]
Which is the correct symmetry of nature?

Michelson-Morley Experiment

↓

Special relativity
Poincaré Transformations

\[ |\vec{x} - \vec{y}|^2 - c^2|t_x - t_y|^2 = |\vec{x}' - \vec{y}'|^2 - c^2|t_x' - t_y'|^2 \]

\[ x^\mu := (ct, x^1, x^2, x^3) \]

Differentiate above \( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu} \)

\[ \Lambda_{\mu \nu} := \frac{\partial x'{}^\mu}{\partial x^\nu}(0) \quad a^\mu = x'{}^\mu(0) \]

\[ \downarrow \]

\[ x'{}^\mu = \sum_{\nu=0}^{3} \Lambda_{\mu \nu} x^\nu + a^\mu \quad \Lambda_\mu{}^\alpha \eta_{\mu \nu} \Lambda_\nu{}^\beta = \eta_{\alpha \beta} \]

\[ \eta^{\mu \nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
Note the similarity to

\[ \ddot{x}' = O\ddot{x} + \bar{a} \quad Q^t I O = I \]

which are the corresponding equations for rigid-body motion
Basic Poincaré Transformations

Rotations (3):

\[ \vec{x}' = O\vec{x} \quad O^t O = I \]

Translations (4):

\[ \vec{x}' = \vec{x} + \vec{a}, \quad ct' = ct + a^0 \]

Rotationless Lorentz transformations (3):

\[ x' = \gamma(x + vt) \quad t' = \gamma(t + vx/c^2) \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \]

All Poincaré transformations can be generated from these ten elementary transformations
All 10 elementary Poincaré transformations have inverses

\[ O \rightarrow O^t \]

\[ \vec{a} \rightarrow -\vec{a}; \quad a^0 \rightarrow -a^0 \]

\[ \nu \rightarrow -\nu \]
\[ x' = \Lambda_1 x + a_1 \]
\[ x'' = \Lambda_2 x' + a_2 \]
\[ \Downarrow \]
\[ x'' = (\Lambda_2 \Lambda_1) x + (\Lambda_2 a_1 + a_2) \]
\[ \Lambda_3 := \Lambda_2 \Lambda_1 \] \[ a_3 = \Lambda_2 a_1 + a_2 \]
\[ \Downarrow \]
\[ x'' = \Lambda_3 x + a_3 \]
• \( \Lambda^\mu_\nu \) and \( a^\mu \) are labels for distinct inertial reference frames.

• All Poincaré transformations can be generated from 10 elementary transformations.

• Compositions of Poincaré transformations are Poincaré transforms.

• Poincaré transformations are elements of a group (closed under composition, inverse, identity, associative).
Special relativity and quantum mechanics

The group of Poincaré transformations is a symmetry of quantum mechanics

\[ |v\rangle \rightarrow |v_{\Lambda, a}\rangle \]

\[ \downarrow \]

\[ P_{uv} = P_{u_{\Lambda, a}, v_{\Lambda, a}} \]
Eugene P. Wigner

\[ P_{uv} = P_{u\Lambda, a \nu \Lambda, a} \]

\[ \Downarrow \]

\[ \langle u\Lambda, a | v\Lambda, a \rangle = \langle u | v \rangle \]

\[ U(\Lambda, a)|v\rangle = |v\Lambda, a\rangle \]

\[ U := U(\Lambda, a) \quad U^\dagger U = (U^t)^* U = I \]

\[ U_3|v\rangle = U_2|v_1\rangle = U_2 U_1|v\rangle \]

\[ \Downarrow \]

\[ U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \]
Note that $U(\Lambda, a)$ has the multiplication properties of the Poincaré group but it acts on a vector space with different dimension than four.

$U(\Lambda, a)$ is a unitary representation of the Poincaré group.

How do we construct $U(\Lambda, a)$?
Rotating an electron at rest

\[ U(R)|\uparrow\rangle = |\uparrow\rangle u_{\uparrow\uparrow} + |\downarrow\rangle u_{\uparrow\downarrow} \]
\[ U(R)|\downarrow\rangle = |\uparrow\rangle u_{\uparrow\downarrow} + |\downarrow\rangle u_{\downarrow\downarrow} \]
\[ U(R)|\mu\rangle = \sum_{\nu=\uparrow,\downarrow} |\nu\rangle D_{\nu\mu}^2(R) \]

\( D_{\nu\mu}^2(R) \) is a two dimensional representation of the rotation group.
Translating and electron at rest

\[ U(a^0, \bar{a}) |\mu\rangle = |\mu\rangle e^{-imca^0} \]
Electron moving with velocity $\vec{v}$, \hspace{1cm} (\vec{p} = m_e \gamma \vec{v})$

\[ |\vec{p}, \mu\rangle := U(\vec{v})|\mu\rangle \]
Every Poincaré transformation can be expressed as

- Rotationless Lorentz transform to rest frame
- Rotation
- Translation of a rest state
- Rotationless Lorentz transform to final frame
\[ U(\Lambda, a)|\vec{p}, \mu\rangle = \sum_{\mu'} |\vec{p}', \mu'\rangle D^{2}_{\mu', \mu}(R(\Lambda, p))e^{-ip' \cdot a} \]
This transformation law is a mass $m_e$ spin $1/2$ irreducible representation for the Poincaré group.

This can be done for particles with any mass and spin.
Two free electrons in zero total momentum frame

\[ |\vec{k}, \mu_1; -\vec{k}, \mu_2 \rangle := |\vec{0}, \vec{k}, \mu_1, \mu_2 \rangle = \]

- Decompose into linear combinations that have definite angular momenta, \( J \)
- Resulting states look just like free particle states with mass and spin

\[ M = 2\sqrt{k^2 + m_e^2} \quad J \]

- Repeat construction to make irreducible two particle states

\[ |\vec{p}, \mu(M(k), J, d)\rangle \]

\[ U(\Lambda, a)|\vec{p}, \mu(M(k), J, d)\rangle = \sum_{\mu'} |\vec{p}', \mu'(M(k), J, d)\rangle D_{\mu' \mu}^{2J+1}(R(\Lambda, p))e^{-ip' \cdot a} \]
Two Interacting Electrons

Consistent initial value problem?
Solving the non-linear problem?

- Add interactions to mass
  \[ M = 2 \sqrt{k^2 + m_1^2} + V \]

- Make sure that \( V \) it does not mix different angular momenta

- Make sure that \( V \) is independent of the quantities that change under change of inertial coordinate system.

- Diagonalize the matrix \( M \) in the free two electron states.
\[ (2\sqrt{k^2 + m_e^2 + V})|\vec{p}, \mu, (M', J)\rangle = M'|\vec{p}, \mu, (M', J)\rangle \]

\[ U_I(\Lambda, a)|\vec{p}, \mu(M', J)\rangle = \]

\[ \sum_{\mu'} |\vec{p}', \mu'(M', J)\rangle D^{2J+1}_{\mu'\mu}(R(\Lambda, p))e^{-ip' \cdot a} \]

This defines a relativistic interacting two-electron quantum theory
Conclusion

• The realization of special relativity in quantum mechanics is very different than it is in classical physics.

• The Poincaré group provides alternate paths to the future.

• The Poincaré symmetry can be preserved with interactions if they are added to the two-body invariant mass.

• Framework is often used in nuclear physics.