Osterwalder-Schrader Stability

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Quasi-Schwinger Functions

\[ S_n(\bar{x}_1, x_1^0, \cdots, \bar{x}_n, x_n^0) = \]

\[ = \lim_{\phi \to \frac{\pi}{2}} G_n(\bar{x}_1, e^{i\phi}x_1^0, \cdots, \bar{x}_n, e^{i\phi}x_n^0) \]

\[ G_n \Rightarrow S_n \]
Quasi-Schwinger Functions

- Analyticity of $G_n$ in times follows from spectral properties:

\[
\hat{f}(t) = \int_0^\infty e^{iEt} f(E) dE
\]

\[
\text{supp}(f) \in [0, \infty)
\]

\[
\hat{f}(t) \rightarrow \hat{f}(t + i\tau) \quad \tau > 0
\]

\[
\hat{f}(t) \rightarrow \hat{f}(e^{i\phi} t) \quad t > 0, \quad 0 \leq \phi \leq \frac{\pi}{2}
\]
Quasi-Wightman Functions

- quasi-Wightman functions can be recovered from quasi-Schwinger functions as boundary values of the analytic functions

\[ W_n(\bar{x}_1, x_1^0, \cdots, \bar{x}_N, x_N^0) = \lim_{x_1^0 > x_2^0 > \cdots > x_N^0 \to 0} S_n(\bar{x}_1, x_1^0 + ix_1^0, \cdots, \bar{x}_N, x_N^0 + ix_N^0) \]

- Order of Euclidean times = order of fields in \( W \)

\[ S_n \Rightarrow W_n \]
Physical Hilbert Space

\[ \langle f | g \rangle = (\Theta f, Sg) = (f, \Theta Sg) = \]

\[ \sum_{m,n} \int d^4x_1 \cdots d^4x_{m+n} f_n^*(\theta x_n, \cdots, \theta x_1) \times \]

\[ S_{m+n}(x_1, \cdots, x_{m+n}) g_m(x_{n+1}, \cdots, x_{n+m}). \]
Reflection Positivity

\[
\text{supp}(x|f) : t_{k1} > t_{k2} \cdots > t_{kk} > 0 \quad (\langle x|f \rangle \in \mathcal{S}_>)
\]

\[
(\Theta f, Sg) \geq 0
\]
Poincaré Group

\[
\langle x|\vec{P}|f\rangle := \{0, -i \frac{\partial}{\partial \vec{x}_{11}} f_1(x_{11}), (-i \frac{\partial}{\partial \vec{x}_{21}} - i \frac{\partial}{\partial \vec{x}_{22}}) f_2(x_{21}, x_{22}), \cdots \}\]

\[
\langle x|\vec{J}|f\rangle := \{0, -i \vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} f_1(x_{11}), \]

\[
(-i \vec{x}_{21} \times \frac{\partial}{\partial \vec{x}_{21}} - i \vec{x}_{22} \times \frac{\partial}{\partial \vec{x}_{22}}) f_2(x_{21}, x_{22}), \cdots \}\]
Poincaré Group

$$\langle x | H | f \rangle := \{ 0, \frac{\partial}{\partial x_{11}^0} f_1(x_{11}), \frac{\partial}{\partial x_{21}^0} + \frac{\partial}{\partial x_{22}^0} f_2(x_{21}, x_{22}), \cdots \}$$

$$\langle x | \vec{B} | f \rangle := \{ 0, \vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} f_1(x_{11}),$$

$$(\vec{x}_{21} \frac{\partial}{\partial x_{21}^0} - x_{21}^0 \frac{\partial}{\partial \vec{x}_{21}} + \vec{x}_{22} \frac{\partial}{\partial x_{22}^0} - x_{22}^0 \frac{\partial}{\partial \vec{x}_{22}}) f_2(x_{21}, x_{22}), \cdots \}.$$
Dynamics - Euclidean RQM

- $H, \vec{P}, \vec{J}, \vec{B}$ well defined and self-adjoint on the Physical Hilbert space
- $H, \vec{P}, \vec{J}, \vec{B}$ satisfy Poincaré commutation relations
- $\langle \cdot | \cdot \rangle, H, \vec{P}, \vec{J}, \vec{B}$ defines a relativistic quantum theory.
Euclidean BS

\[ G_4 = G_0 + G_0 K G_4 \]
\[ \Downarrow \]
\[ S_4 = S_0 + S_0 K_e S_4 \]

\[ G_4 \iff S_4 \iff W_4 \]
OS Stability

\[ S_4 = S_0 + S_0 K_e S_4 \]

\[ \Pi_\succ : S \rightarrow S_\succ \]

\[ (f, \Pi_\succ \Theta S_0 \Pi_\succ f) \geq 0 \]

\( K_e \) small, Euclidean covariant

? 

\[ (f, \Pi_\succ \Theta S_4 \Pi_\succ f) \geq 0 \]
Theorems

Theorem 1: \((f, \Pi_\Theta S_0 \Pi_\Theta f) = 0\) for some \(f \in S_\theta\) implies the BS equation is OS unstable in a neighborhood of \(K = 0\).
Theorems

Theorem 2: If $S_0 = S_{10} S_{20}$ with $S_{i0}$ a free particle Schwinger functions, then there are non-zero $f \in S_>$ such that 
$(f, \Pi_> \Theta S_0 \Pi_> f) = 0.$
Theorems

Theorem 3: If $S_0 = S_{10}S_{20}$ and $S_{i0}$ has a Lehmann with non-empty continuous spectrum then there are no non-zero $f \in S_>$ such that $(f, \Pi_> \Theta S_0 \Pi_> f) = 0$. 
Null Spaces

\[(f, \Theta S_0 f) = \int \int d\tau \tilde{f}(p, t) \left( \frac{2\pi e^{-\omega_m(p)\tau}}{\sqrt{\omega_m(p)}} \right)^2 d^3 p d\rho m(m) \]

\[\tilde{f}(p, t) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{p} \cdot \vec{x}} f(\vec{x}, t) d^3 x \]
Null Spaces

\[(f, \Theta S_0 f) = 0\]

\[\downarrow\]

\[\int dt \tilde{f}(p, t) e^{-\omega_m(p)t} = 0\]

\[\downarrow\]

\[\forall p \quad \text{and all} \quad m \in \text{supp}(\rho).\]
Null Spaces

- $S_0$ free $\Rightarrow m$ is a fixed number

$$\tilde{f}(\vec{p}, t) = \tilde{g}(\vec{p})\xi(t) \quad \int_a^b \xi(t)dt = 1 \quad \text{supp}(\xi) \in [a, b]$$

$$\hat{\xi}(\vec{p}) := \int_a^b e^{-\omega_m(p)t} \xi(t)dt$$

$$\tilde{f}(\vec{p}, t) = \tilde{g}(\vec{p})\xi(t)[1 - e^{\omega_m(p)t} \hat{\xi}(\vec{p})]$$

$$\downarrow$$

$$(f, \Theta S_0 f) = 0$$
Null Spaces

- If $\rho(m)$ has absolutely continuous spectrum then the instability proof breaks down:

$$\int \tilde{f}(\vec{p}, t) e^{-\omega_m(\vec{p}) t} = 0.$$  

$$\Downarrow$$

$$\tilde{f}(\vec{p}, t) = 0$$
Conclusion

- Euclidean BS with free $S_0$ is OS unstable at $K = 0$.
- The BS equation with a realistic $S_0$ may be OS stable for a suitably restricted class of Kernels.
- OS positivity is essential for a quantum mechanical interpretation.
Future

- Models with $S_0$ having absolutely continuous Lehmann spectra?
- Case of Fermions (protons, neutrons, quarks)?