Scattering in a Euclidean formulation of relativistic quantum mechanics

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Useful discussions with colleagues at Iowa

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Motivation and Observations

• Constructing relativistic quantum mechanical models satisfying cluster properties is complicated.

• Locality is logically independent of the rest of the axioms of Euclidean field theory → Euclidean formulation of relativistic quantum theory satisfying cluster properties.

• Reconstruction theorem: The physical Hilbert space and a unitary representation of the Poincaré group can be directly formulated in the Euclidean representation. Analytic continuation is not necessary.

• Given these elements it should be possible to formulate a relativistic treatment of scattering in a Euclidean representation using standard quantum mechanical methods.
Elements of relativistic quantum mechanics

\[ \langle \psi | \phi \rangle \text{ Hilbert space} \]

\[ U(\Lambda, a) \leftrightarrow \{ P^\mu, J^{\mu\nu} \} \quad \text{Relativity} \]

\[ P^0 = H \quad \text{Dynamics} \]

\[ P^0 = H \geq 0 \quad \text{Spectral condition} \rightarrow \text{stability} \]

\[ [U(\Lambda, a) - \bigotimes U_i(\Lambda, a)]|\psi\rangle \rightarrow 0 \quad (x_i - x_j)^2 \rightarrow \infty \]

Cluster properties: scattering asymptotic conditions
Osterwalder-Schrader (Euclidean) reconstruction

Input: \[ \{ G_{E_n}(x_1, \cdots, x_n) \} \]

Relevant properties

- Euclidean covariant (invariant)
- Cluster property
- Reflection positivity
Construction of the physical Hilbert space: $\mathcal{H}_M$

**Vectors (dense set)**

$$\psi(x) := (\psi_1(x_{11}), \psi_2(x_{21}, x_{22}), \cdots)$$

$$\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = 0 \quad \text{unless} \quad 0 < x_{n1}^0 < x_{n2}^0 < \cdots < x_{nn}^0.$$  

$$\theta x := \theta(\tau, x) = (-\tau, x) \quad \text{Euclidean time reflection}$$

**Physical Hilbert space inner product**

$$\langle \psi | \phi \rangle_M = (\theta \psi, G_E \phi)_E = \sum_{kn} \int d^4k x d^4n y \psi^*_n(\theta x_{n1}, \theta x_{n2}, \cdots, \theta x_{nn}) \times$$

$$G_{E,n+k}(x_{nn}, \cdots, x_{1n}; y_{1k}, \cdots, y_{kk}) \phi_k(y_{k1}, y_{k2}, \cdots, y_{kk})$$

All variables are Euclidean - no analytic continuation.
Reflection positivity - property of \( \{ G_{En} \} \)

\[ \langle \psi | \psi \rangle_M = (\psi, \Pi_+ \Theta G_E \Pi_+ \psi)_E \geq 0 \]

\[ \Downarrow \]

Gives the physical Hilbert space and spectral condition.
Illustration

Two-point Green function: Euclidean $\rightarrow$ Minkowski

$$\langle \phi | \psi \rangle_M = \int \phi^*(\tau_x, x) \frac{d^4 p \rho(m) dm}{(2\pi)^4} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2} \psi(\tau_y, y) d^4 x d^4 y$$

$$= \int \xi^*_m(p) \frac{d^4 p \rho(m) dm}{2e_m(p)} \chi_m(p)$$

Euclidean wave function $\rightarrow$ Minkowski wave function

$$\chi_m(p) := \int \frac{d^4 y}{(2\pi)^{3/2}} e^{-e_m(p)\tau_y - ip \cdot y} \psi(\tau_y, y)$$

$$\xi_m(p) := \int \frac{d^4 x}{(2\pi)^{3/2}} e^{-e_m(p)\tau_x - ip \cdot x} \phi(\tau_x, x)$$

$$m^2 \psi(\tau_x, x) = \nabla^2_4 \psi(\tau_x, x)$$
Euclidean invariance $\rightarrow$ Poincaré invariance

Relativity and $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$

$$X_m := \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \quad X_e := \begin{pmatrix} i\tau + z & x - iy \\ x + iy & i\tau - z \end{pmatrix}$$

$$\det(X_M) = t^2 - x^2 \quad \det(X_E) = -(\tau^2 + x^2)$$

$$X \rightarrow X' = AXB^t \quad \det(A) = \det(B) = 1$$

Preserves both $t^2 - x^2$ and $\tau^2 + x^2$

Complex Lorentz group = complex orthogonal group

Real orthogonal group = subgroup of complex Lorentz group

Real Lorentz $(A, B) = (A, A^*)$, $A \in SL(2, \mathbb{C})$;
Real orthogonal $(A, B) \in SU(2) \times SU(2)$
Relation between Euclidean and Poincaré generators

- **Euclidean time translations** → contractive Hermitian semigroup on $\mathcal{H}_M$:
  $$H_E = P^0_E = -iH_M = -iP^0_M$$

- **Euclidean space-time rotations** → local symmetric semigroup on $\mathcal{H}_M$:
  $$J^0_{Eij} = iJ^0_{Mij} = iK^j$$

- **Euclidean space rotations** → unitary one parameter groups on $\mathcal{H}_M$:
  $$J^ij_E = J^ij_M$$

- **Euclidean space translations** → unitary one parameter groups on $\mathcal{H}_M$:
  $$P^i_E = P^i_M$$

$\{P^\mu_M, J^\mu\nu_M\} = 10$ self-adjoint generators satisfying the Poincaré commutation relations on $\mathcal{H}_M$
Spinless case

\[ H\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = \sum_{k=1}^{n} \frac{\partial}{\partial x_{nk}^0} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \]

\[ P\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = -i \sum_{k=1}^{n} \frac{\partial}{\partial x_{nk}} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \]

\[ J\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = -i \sum_{k=1}^{n} x_{nk} \times \frac{\partial}{\partial x_{nk}} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \]

\[ K\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = \sum_{k=1}^{n} (x_{nk} \frac{\partial}{\partial x_{nk}^0} - x_{nk}^0 \frac{\partial}{\partial x_{nk}}) \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}). \]

All integration variables are Euclidean; Minkowski time is a parameter.
Cluster properties

\[ G_{E,n+m} \rightarrow G_{E,n}G_{E,m} \]

\[ \Downarrow \]

Generators become additive in asymptotically separated subsystems

Used to formulate scattering asymptotic conditions.
Multichannel scattering theory

Scattering probability \( = |S_{fi}|^2 = |\langle \psi_+ | \psi_- \rangle|^2 \)

\( |\psi_\pm \rangle = \Omega_\pm |\psi_{0\pm} \rangle \)

\( \Omega_\pm |\psi_{0\pm} \rangle = \lim_{t \to \pm \infty} \sum e^{iHt} \prod \langle \phi_n, p_n, \mu_n \rangle e^{-ie_n t} f_n(p_n, \mu_n) d p_n \)

\( e^{-iH_0 t} |\psi_{0\pm} \rangle \)

\( = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_0 t} |\psi_{0\pm} \rangle \)

Elements: Cluster properties, subsystem bound states: \( |\phi_n \rangle \),
wave packets: \( f_n \), dynamics: \( H \), strong limits.
Field theoretic implementation: Haag-Ruelle scattering

(Minkowski case)

\[ \Phi(x) = \text{interpolating field} \]

\[ \tilde{\Phi}(p) = \frac{1}{(2\pi)^2} \int e^{-ip \cdot x} \Phi(x) d^4 x \]

\[ \tilde{\Phi}_m(p) = h(p^2) \tilde{\Phi}(p), \ h(-m^2) = 1, \ h(p^2) = 0, \ -p^2 \notin (m^2 - \epsilon, m^2 + \epsilon) \]

\[ \Phi_m(x) = \frac{1}{(2\pi)^2} \int e^{ip \cdot x} \tilde{\Phi}_m(p) d^4 p \]

\[ f_m(x) = \frac{i}{(2\pi)^{3/2}} \int e^{-i\sqrt{p^2 + m^2 t + p \cdot x}} \tilde{f}(p) dp \]

\[ a^\dagger_m(f_m, t) = -i \int d\mathbf{x} \left( \frac{\partial \Phi_m(t, \mathbf{x})}{\partial t} f_m(t, \mathbf{x}) - \Phi_m(t, \mathbf{x}) \frac{\partial f_m(t, \mathbf{x})}{\partial t} \right) \]

\[ \Omega_\pm |\psi_{0\pm}\rangle = s - \lim_{t \to \pm \infty} \prod_i a^\dagger_{m_i}(f_{m_i}, t) |0\rangle \]
Euclidean formulation of HR scattering - technical issues

- \((M^2 = \nabla^2)\) One-body solutions must satisfy the time support condition:

\[
\text{support}(h(\nabla^2)\langle x|\psi\rangle) = \text{support}(\langle x|\psi\rangle)
\]

- Products of one-body solutions must satisfy the relative time support condition \((n = 2, \text{ no spin})\).

\[
J : \langle x_1|\phi_1, p_1\rangle\langle x_2|\phi_2, p_2\rangle = h_1(\nabla_1^2)\delta(x_1^0 - \tau_1) h_2(\nabla_2^2)\delta(x_2^0 - \tau_2) \frac{1}{(2\pi)^3} e^{ip_1 \cdot x_1 + ip_2 \cdot x_2}
\]

\[\tau_2 > \tau_1\]
• Delta functions in Euclidean time $\times f(x)$ are square integrable in $\mathcal{H}_M$!

• A sufficient condition for $h_i(\nabla^2)$ to preserve the support condition is for polynomials in $\nabla^2$ to be complete with respect to the inner product on $\mathcal{H}_M$

$$h_i(\nabla^2) \approx P(\nabla^2)$$

• The $J$ defined on the previous slide can be used to satisfy the time-support conditions.
Completeness of $P_n(\nabla^2)$ sufficient to construct $h(m^2)$ without violating positive Euclidean time-support condition.

Proving completeness - Stieltjes moment problem

$G_{E2}$ moments

$$
\gamma_n := \int_0^\infty \frac{e^{-(\sqrt{m^2+p^2}\tau)}}{2\sqrt{m^2+p^2}} \rho(m)m^{2n} dm
$$

where $\tau = \tau_1 + \tau_2 > 0$.

Carleman’s condition

$$
\sum_{n=0}^{\infty} |\gamma_n|^{-\frac{1}{2n}} > \infty
$$

Satisfied for $\rho(m^2)$ a tempered distribution $\Rightarrow P(\nabla^2)$ complete.

$$
|\gamma_n|^{-\frac{1}{2n}} \sim \frac{1}{n + c}
$$
Existence - sufficient condition (Cook)

\[ \int_a^\infty \| (HJ - JH_0) U_0(\pm t) |\psi_0\rangle \|_M dt < \infty \]

\[ \| (HJ - JH_0) \Phi U_0(\pm t) |\psi_0\rangle \|^2_M = \]

\[ (\psi_0 U_0(\mp t) (J^\dagger H - H_0 J^\dagger) \theta G_E (HJ - JH_0) U_0(\pm t) |\psi_0\rangle_E \]

The effect of using one-body solutions for 2-2 scattering is that the contribution from the disconnected part of \( G_E \) to the above is zero. This fails for LSZ scattering.

The connected part is expected to behave like \( ct^{-3} \) for large \( t \), satisfying the Cook condition.
Computational tricks for scattering

Invariance principle:

\[
\lim_{t \to \pm \infty} e^{iHt} Je^{-iH_0t} |\psi\rangle = \lim_{t \to \pm \infty} e^{if(H)t} Je^{-i(H_0)t} |\psi\rangle
\]

\[f(x) = -e^{-\beta x}\]

\[
\lim_{t \to \pm \infty} e^{iHt} Je^{-iH_0t} |\psi\rangle = \lim_{n \to \infty} e^{\pm ine^{-\beta H}} Je^{i\pm ne^{-\beta H_0}} |\psi\rangle
\]

\[\sigma(e^{-\beta H}) \in [0, 1] \to\]

\[|e^{inx} - P(x)| < \epsilon \quad x \in [0, 1]\]

\[\|e^{ine^{-\beta H}} - P(e^{-\beta H})\| < \epsilon\]

Matrix elements of \(e^{-n\beta H}\) are easy to calculate:

\[\langle \tau, x | e^{-n\beta H} | \psi \rangle = \langle \tau - n\beta, x | \psi \rangle\]
Model tests (of computational methods)

\[ H = \frac{k^2}{m} - \lambda |g\rangle \langle g| \]

\[ (M^2 = 4k^2 + 4m^2 - 4m\lambda |g\rangle \langle g|) \]

\[ \langle k|g \rangle = \frac{1}{k^2 + m^2} \]

Attractive - one pion exchange range, bound state with deuteron mass.

\[ e^{-2ine^{-\beta H}} \approx P(e^{-\beta H}) \]

\[ \langle k_f| T(E + i0)|k_i \rangle \approx \]

\[ \langle \psi_f|(1 - e^{-ine^{-\beta M_0}} P(e^{-\beta H}e^{-ine^{-\beta H_0}})|\psi_i \rangle \]

\[ 2\pi i \langle \psi_f| \delta(E - H_0)|\psi_i \rangle \]
• Choose sufficiently narrow initial and final wave packets.

• Choose sufficiently large $n$.

• Replace $e^{2i\alpha - \beta H}$ by a polynomial approximation.

• Calculations formally independent of $\beta$, adjust $\beta$ for faster convergence.

• Model allows independent tests of each approximation.

• Approximations must be done in the proper order.
Results

- Converges to exact sharp momentum transition matrix elements.
- Tests converge for $0.050 - 2 \text{ GeV}$.  
- Biggest source of error is the wave packet width.
Convergence with respect to wave packet width

Table 1

<table>
<thead>
<tr>
<th>$k_0$ [GeV]</th>
<th>$k_w$ [GeV]</th>
<th>% error</th>
<th>$k_w/k_0$</th>
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<td>0.1</td>
<td>0.00308607</td>
<td>0.1</td>
<td>0.030</td>
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<td>0.3</td>
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<td>1.9</td>
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<td>0.1</td>
<td>0.042</td>
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</table>
Convergence with respect time “$n$”

Table 2: $k_0 = 2.0\text{[GeV]}$, $k_w = .09\text{[GeV]}$

| $n$ | $\text{Re} \langle \phi | (S_n - I) | \phi \rangle$ | $\text{Im} \langle \phi | (S_n - I) | \phi \rangle$ |
|-----|--------------------------------------|--------------------------------------|
| 50  | -2.60094316473225e-6                  | 1.94120750171791e-3                  |
| 100 | -2.82916859895010e-6                  | 2.35553585404449e-3                  |
| 150 | -2.83171624670953e-6                  | 2.37471383801820e-3                  |
| 200 | -2.83165946257657e-6                  | 2.37492460997990e-3                  |
| 250 | -2.83165905312632e-6                  | 2.37492527186858e-3                  |
| 300 | -2.83165905257121e-6                  | 2.37492527262432e-3                  |
| 350 | -2.83165905190508e-6                  | 2.37492527262493e-3                  |
| 400 | -2.83165905234917e-6                  | 2.37492527262540e-3                  |
| ex  | -2.83165905227843e-6                  | 2.37492527259701e-3                  |
Table 3: Parameter choices

<table>
<thead>
<tr>
<th>$k_0$ [GeV]</th>
<th>$\beta$ [GeV$^{-1}$]</th>
<th>$k_0 \times \beta$</th>
<th>$n$</th>
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<td>3.0</td>
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<td>0.7</td>
<td>1.6</td>
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<tr>
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<td>.945</td>
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Table 4: Convergence with respect to Polynomial degree $e^{inx}$

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<th>$n$</th>
<th>deg</th>
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<td>630</td>
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Table 5: Final calculation

<table>
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<th>$k_0$</th>
<th>Real T</th>
<th>Im T</th>
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Unfinished business/things to consider

Structure theorem for reflection-positive Euclidean-invariant n>2 point functions.

General form of Bethe-Salpeter kernels that lead to reflection positive four-point functions?

Formulate N-body scattering based on two-body Bethe-Salpeter kernels?

Numerical test of the polynomial approximation to the Haag-Ruelle function \( h(m^2) \) for a two-point function with a non-trivial Lehmann weight.

Scattering calculation based on a realistic four-point function.