The light-front vacuum

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Why is the light-front vacuum different than the canonical vacuum if both describe the same theory?

Outline

- Triviality of vacuum
- Annihilation operators / role of algebras
- Light-front Fock algebra
- Extension of algebras
- Interacting fields
- Zero modes

Notation

Light front: $x^+ = x^0 + \hat{\mathbf{n}} \cdot \mathbf{x} = 0$

Translation generators

$$P^\pm := P^0 \pm \hat{\mathbf{n}} \cdot \mathbf{P} \geq 0 \qquad \mathbf{P}_\perp := \mathbf{P} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{P})$$

$$P^- = P_0^- + V \qquad P^+ = P_0^+ = \sum_i p_i^+$$

$$p_i^+ = \omega_i(\mathbf{p}) + \hat{\mathbf{n}} \cdot \mathbf{p}_{m_i} \geq 0 \qquad \omega_{m_i}(\mathbf{p}) = \sqrt{\mathbf{p}_i^2 + m_i^2}$$

Light-front 3 vectors

$$\tilde{\mathbf{p}} = (p^+, \mathbf{p}_\perp) \qquad \tilde{\mathbf{x}} = (x^-, \mathbf{x}_\perp)$$

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{x}} = -\frac{1}{2} p^+ x^- + \mathbf{p}_\perp \cdot \mathbf{x}_\perp$$

Triviality of the light-front vacuum (spectral condition $p_i^+ \geq 0$)

$$[P^+, P^-] = [P^+, P_0^-] = 0 \qquad \Rightarrow \qquad [P^+, V] = 0$$

$$P^+ V|0\rangle = VP^+|0\rangle = 0|0\rangle$$

$$: V := \int \mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) a^\dagger(\tilde{\mathbf{p}}_1) \cdots a^\dagger(\tilde{\mathbf{p}}_n) d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n + \cdots$$

$[P^+, a^\dagger(\tilde{\mathbf{p}})] = p^+ a^\dagger(\tilde{\mathbf{p}})$ $a^\dagger(\tilde{\mathbf{p}})$ **increases** P^+ **for** $p^+ > 0$

$$\sum_i p_i^+ = 0 \qquad p_i^+ \geq 0 \qquad \Rightarrow \qquad \mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) = 0$$

unless $p_1^+ = p_2^+ \cdots p_n^+ = 0$**.**

$\therefore$ **if** $\mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n)$ **is continuous and normal ordered then** $\mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) = 0$ **and the vacuum is unchanged.**

Single oscillator

"Vacuum" determined by annihilation operator

$$\langle x|a|0\rangle = 0 \qquad \Rightarrow \qquad \langle x|0\rangle$$

Linear canonical transformation changes vacuum

$$a' = \cosh(\eta)a + \sinh(\eta)a^\dagger$$

$$\langle x|a'|0'\rangle = 0 \qquad \Rightarrow \qquad \langle x|0'\rangle$$

Implemented by unitary transformation

$$a' = e^{iG} a e^{-iG} \qquad |0'\rangle = e^{iG}|0\rangle$$

$$e^{iG} \qquad \textbf{where} \qquad G = G^\dagger = -i\frac{\eta}{2}(aa - a^\dagger a^\dagger)$$

Free fields of different mass (simplest example)

Interaction = mass difference:

$$V = \frac{1}{2}(m_1^2 - m_2^2)\int : \phi(\mathbf{x})^2 : d\mathbf{x}$$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\mathbf{p}}{\sqrt{2\omega_m(\mathbf{p})}}(e^{ip\cdot x}a(\mathbf{p}) + e^{-ip\cdot x}a(\mathbf{p}))$$

$$= \frac{1}{(2\pi)^{3/2}}\int \frac{d\tilde{\mathbf{p}}}{\sqrt{2p^+}}(e^{ip\cdot x}a(\tilde{\mathbf{p}}) + e^{-ip\cdot x}a(\tilde{\mathbf{p}}))$$

Vacuum

$a|0\rangle = 0 \quad \rightarrow \quad a(\mathbf{p})|0\rangle = 0$ **or** $a(\tilde{\mathbf{p}})|0\rangle = 0$

Observations

- $m_1 \neq m_2$ **fields restricted to** $t = 0$ **related by linear canonical transformation:**

$$a_2(\mathbf{p}) = \underbrace{\frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}}{\omega_{m_1}}} + \sqrt{\frac{\omega_{m_1}}{\omega_{m_2}}}\right)}_{\cosh(\eta(\mathbf{p}))} a_1(\mathbf{p}) + \underbrace{\frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}}{\omega_{m_1}}} - \sqrt{\frac{\omega_{m_1}}{\omega_{m_2}}}\right)}_{\sinh(\eta(\mathbf{p}))} a_1^\dagger(\mathbf{p})$$

- $m_1 \neq m_2$ **fields restricted to** $x^+ = 0$ **identical:**

$$a_2(\tilde{\mathbf{p}}) = a_1(\tilde{\mathbf{p}})$$

Relations: $t = 0$ **vs** $x^+ = 0$

$$a_i(\mathbf{p}) = a_i(\tilde{\mathbf{p}})\sqrt{\frac{p^+}{\omega_{m_i}(\mathbf{p})}}$$

Observations

Relations: $x^+ = 0 \;\; m_1 \neq m_2$

Free fields of different mass restricted to light front are unitarily equivalent

$$ {}_1\langle 0|\phi_1(\tilde{\mathbf{p}}_1)\cdots\phi_1(\tilde{\mathbf{p}}_n)|0\rangle_1 = {}_2\langle 0|\phi_2(\tilde{\mathbf{p}}_1)\cdots\phi_2(\tilde{\mathbf{p}}_n)|0\rangle_2 $$

$$\Downarrow$$

- **The correspondence** $U|0\rangle_1 = |0\rangle_2$ **and** $U\phi_1(\tilde{\mathbf{x}})U^\dagger = \phi_2(\tilde{\mathbf{x}})$ **preserves all Hilbert space inner products.**

Observations

Relations: $t = 0 \;\; m_1 \neq m_2$

Comparison with the single oscillator suggests

$$|0_2\rangle = U|0_1\rangle \qquad a_2(\mathbf{p})|0_2\rangle = 0 \qquad a_1(\mathbf{p})|0_1\rangle = 0$$

$$a_2(\mathbf{p}) = U a_1(\mathbf{p}) U^\dagger \qquad U = e^{iG}$$

$$G = -i \frac{\int d\mathbf{p}\eta(\mathbf{p})}{2}(a_1(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}))$$

but

$$\|G|0\rangle\|^2 = \frac{1}{4}\int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty$$

Domain of G is empty!

- $|0_1\rangle$ **and** $|0_2\rangle$ **not in the same Hilbert space in this representation (Haag 1955).**

Characterization of vacuum by an annihilation operator?

$$0 = a_1(\mathbf{p})|0\rangle_1 = \sqrt{\frac{p^+}{\omega_{m_1}(\mathbf{p})}}a_1(\tilde{\mathbf{p}})|0\rangle_1$$

$$= \sqrt{\frac{p^+}{\omega_{m_1}(\mathbf{p})}}a_2(\tilde{\mathbf{p}})|0\rangle_1 = \sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}}a_2(\mathbf{p})|0\rangle_1$$

contradicts

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p}))a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p}))a_1^\dagger(\mathbf{p})$$

$$\Downarrow$$

$$a_2(\mathbf{p})|0\rangle_1 \neq 0$$

???

How do we reconcile this apparent contradiction?

- **The annihilation operators do not define the vacuum!**
- **The vacuum is an invariant linear functional on an algebra of operators.**
- **The algebra matters!**
  - $a(\mathbf{p})$: **algebra = field and time derivatives restricted to** $t = 0$ **(canonical algebra).**
  - $a(\tilde{\mathbf{p}})$: **algebra = fields restricted to** $x^+ = 0$ **(light-front algebra).**
  - $\phi(x)$: **algebra = fields smeared against functions of four space-time variables (local algebra).**

- **The local algebra is invariant under Poincaré transformations and includes all local observables.**

- **Schlieder and Seiler give an example of sub-algebras of the local algebras of different mass local field algebras that are (1) irreducible and (2) unitarily equivalent. This illustrates the essential role of the algebra in defining the vacuum**

Schlieder-Seiler example

Linear subspace ($\mathcal{L}$) of test functions satisfying

$$\frac{f(\sqrt{m_1^2+\mathbf{p}^2},\mathbf{p})}{(m_1^2+\mathbf{p}^2)^{1/4}} = \frac{f(\sqrt{m_2^2+\mathbf{p}^2},\mathbf{p})}{(m_2^2+\mathbf{p}^2)^{1/4}}.$$

$$\langle 0_1|\phi_1(f_1)\cdots\phi_1(f_n)|0_1\rangle = \langle 0_2|\phi_2(f_1)\cdots\phi_2(f_n)|0_2\rangle$$

$$U|0_1\rangle := |0_2\rangle \qquad U\phi_1(f)U^\dagger := \phi_2(f)$$

U **preserves all scalar products** $\rightarrow$ U **is unitary**

Irreducibility: For any $g\,\exists\, f \in \mathcal{L}$ **satisfying** $f = g$ **on mass shell.**

$$\|(\phi(f)-\phi(g))|A\rangle\| = 0.$$

The light-front Fock algebra

Generated by

$$e^{i\phi(\tilde{f})} = e^{i\int d\tilde{\mathbf{x}}\phi(\tilde{\mathbf{x}},x^+=0)f(\tilde{\mathbf{x}})}$$

Operator products

$$e^{i\phi(\tilde{f})}e^{i\phi(\tilde{g})} = e^{i\phi(\tilde{f}+\tilde{g})}e^{-\frac{1}{2}((\tilde{f},\tilde{g})-(\tilde{g},\tilde{f}))}$$

Light-front inner product

$$(\tilde{f},\tilde{g}) = \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+}f(-\tilde{\mathbf{p}})g(\tilde{\mathbf{p}})$$

- $(\tilde{f},\tilde{g})$ **is log divergent if** $\tilde{f}(\tilde{\mathbf{p}}) \neq 0$ **for** $p^+ = 0$**; however** $((\tilde{f},\tilde{g}) - (\tilde{g},\tilde{f}))$ **is defined**

Irreducibility

$$\tilde{f}(\tilde{\mathbf{p}}) = \tilde{f}_r(\tilde{\mathbf{p}}) + \tilde{f}_i(\tilde{\mathbf{p}})$$

$$U(\tilde{f}_r, \tilde{f}_i) := e^{i\phi(\tilde{f})}$$

$$U(\tilde{f}_r, \tilde{f}_i)U(\tilde{g}_r, \tilde{g}_i) = U(\tilde{f}_r + \tilde{g}_r, \tilde{f}_i + \tilde{g}_i)e^{-\frac{1}{2}((\tilde{f}_r,\tilde{g}_i)-(\tilde{f}_i,\tilde{g}_r))}$$

Same algebraic structure (Weyl algebra) as canonical equal time algebra.

$$U(f, g) = e^{i\phi(f,t=0)+i\pi(f,t=0)}$$

- **Light-front Fock algebra is irreducible**

- **Light-front Fock algebra is kinematically invariant but not Poincaré invariant**

- **Light-front Fock algebra does not contain all local observables.**

- **The light-front Fock algebra and light-front vacuum do not determine the dynamics (mass).**

Algebraic normal ordering

$$\phi_+(\tilde{\mathbf{p}}) := \theta(p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(p^+)}{\sqrt{p^+}} a(\tilde{\mathbf{p}})$$

$$\phi_-(\tilde{\mathbf{p}}) := \theta(-p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(-p^+)}{\sqrt{-p^+}} a^\dagger(-\tilde{\mathbf{p}})$$

$$\phi(\tilde{\mathbf{p}}) = \phi_+(\tilde{\mathbf{p}}) + \phi_-(\tilde{\mathbf{p}})$$

$$: e^{i\phi(\tilde{f})} := e^{i\phi_-(\tilde{f})} e^{i\phi_+(\tilde{f})}$$

$$\boxed{e^{i\phi(\tilde{f})} =: e^{i\phi(\tilde{f})} : e^{\frac{1}{2}(\tilde{f},\tilde{f})}}$$

Implications

- **if** $f(\tilde{\mathbf{p}}) = 0$ **for** $p^+ = 0$

$$\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle = 1 \quad \langle 0| e^{i\phi(\tilde{f})} |0\rangle = e^{\frac{1}{2}(\tilde{f},\tilde{f})}$$

  - **Light-front vacuum fixed!**

- **if** $f(\tilde{\mathbf{p}}) \neq 0$ **for** $p^+ = 0$ **then**

  - $(\tilde{f}, \tilde{f})$ **is divergent and** $\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle \neq 1$

  - **All contributions to** $\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle$ **have** $p^+ = 0$ **(zero modes).**

Zero modes

$$\langle 0| : e^{i\phi(\tilde{f})} : |0\rangle = \sum \frac{i^n}{n!} \int z_n(\mathbf{p}_{1\perp}, \cdots, \mathbf{p}_{n\perp}) \times$$

$$\prod_{i=1}^{n} \delta(p_i^+) \tilde{f}(\tilde{\mathbf{p}}_1) \cdots \tilde{f}(\tilde{\mathbf{p}}_n) d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n$$

- **Regulation of inner product breaks kinematic scale invariance; zero modes are needed to restore the full kinematic symmetry, positivity, $\cdots$.**

Extension to local algebra

$$\phi(x) = \frac{1}{(2\pi)^2} \int \frac{d\tilde{\mathbf{p}} d\tilde{\mathbf{y}}}{2} e^{-i\frac{p_\perp^2+m^2}{p^+}x^+ + i\tilde{\mathbf{p}}\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})} \phi(\tilde{\mathbf{y}})$$

structure of mapping

$$\phi(x) = \int F_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}}$$

- **Extension still utilizes light-front creation operators and light-front Fock vacuum.**
- **Mass dependence (dynamics) is in the extension. Inner products with different mass extensions are not preserved in the extension to the local algebra.**

Mapping local test functions to light-front test functions

$$f(x) = f_+(x^+)\tilde{f}(\tilde{\mathbf{x}})$$

$$\phi(f) = \int f(x) F_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}} d^4x =$$

$$\int \tilde{f}_+\left(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}\right)\tilde{f}(\tilde{\mathbf{p}})\phi(\tilde{\mathbf{p}})d\tilde{\mathbf{p}} = \int \tilde{g}(\tilde{\mathbf{p}})\phi(\tilde{\mathbf{p}})d\tilde{\mathbf{p}}$$

- $\tilde{f}_+(\frac{\mathbf{p}_\perp^2 + m^2}{p^+})$ **vanishes faster than any power of** p^+ **near** $p^+ = 0$ **for** $f_+(x^+)$ **a Schwartz function.**

- $F_m(\cdot)$ **maps local algebra to a sub algebra of the light-front algebra.**

- **Vacuum is the light-front Fock vacuum.**

Interacting fields

Irreducibility of the asymptotic fields

(Haag expansion)

$$\Downarrow$$

$$\phi(x) = \sum \int L(x; x_1, \cdots, x_n) : \phi_{in}(x_1) \cdots \phi_{in}(x_n) : d^4 x_1 \cdots d^4 x^n$$

$$L(f, x_1 \cdots x_n) = \int f(x) L(x; x_1, \cdots, x_n) d^4 x$$

$$f(x) \in \mathcal{S}(\mathbb{R}^4) \to L(f, x_1, \cdots, x_n) \in \mathcal{S}(\mathbb{R}^{4n})$$

$$\phi_{in}(x) = \int zF_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})d\tilde{\mathbf{y}}$$

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) : \phi_0(\tilde{\mathbf{y}}_1) \cdots \phi_0(\tilde{\mathbf{y}}_n) : d\tilde{\mathbf{y}}_1 \cdots d\tilde{\mathbf{y}}_n$$

$$\mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) = \int L(x; x_1, \cdots, x_n) \prod z_i F_m(x_i, \tilde{\mathbf{y}}) d^4x_1 \cdots d^4x_n$$

expect

$$\int f(x)\tilde{\mathcal{L}}(x; \tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n)d^4x \to 0 \quad \text{any} \quad p_i^+ \to 0$$

Light-front Haag expansion

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{x}}_1, \cdots, \tilde{\mathbf{x}}_n) : \phi_0(\tilde{\mathbf{x}}_1) \cdots \phi_0(\tilde{\mathbf{x}}_n) : d\tilde{\mathbf{x}}_1 \cdots d\tilde{\mathbf{x}}_n$$

- **Vacuum is light-front Fock vacuum**

- **Test functions on the local algebra are mapped into functions on the light front with Fourier transforms that vanish at $p^+ = 0$.**

- **$\mathcal{L}$ maps the local algebra on to a sub-algebra of light-front algebra with no zero-mode contributions.**

- **Locality and Poincaré symmetry recovered by extension.**

Origin of zero modes (local operator products)

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$\int F_m(x^+;\tilde{\mathbf{x}}-\tilde{\mathbf{z}})F_m(y^+;\tilde{\mathbf{y}}-\tilde{\mathbf{w}})\langle 0|\phi(0,\tilde{\mathbf{z}})\phi(0,\tilde{\mathbf{w}})|0\rangle =$$

$$\frac{1}{(2\pi)^3}\int \frac{d\tilde{\mathbf{p}}}{p^+} e^{-i\frac{\mathbf{p}_\perp^2+m^2}{p^+}(x^+-y^+)+i\tilde{\mathbf{p}}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})} =$$

$$\int \theta(p^+)\delta(p^2+m^2)e^{ip\cdot(x-y)}d^4p$$

Gives the two-point function on the local algebra in terms of the two-point function on light-front algebra. $(x^+ - y^+) \neq 0$ **regularizes two-point function**

$$\int_0^a \frac{e^{ic/p^+}}{p^+}dp^+ = \int_{1/a}^\infty \frac{e^{iu}}{u}du = \frac{\pi}{2} - (Ci(1/a) + iSi(1/a))$$

- **Local operator products have $x^+ - y^+ = 0$.**

$$\Downarrow$$

- **Turns off the term the regulates the $p^+ = 0$ singularity.**

$$\Downarrow$$

- **Light front scalar product becomes divergent.**

$$\Downarrow$$

- **Regularization at $p^+ = 0$ needed; breaks kinematic symmetry.**

$$\Downarrow$$

Regularization

$$(f,g) \to \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+}(\tilde{f}(-\tilde{\mathbf{p}})\tilde{g}(\tilde{\mathbf{p}}) - \tilde{f}(0,-\mathbf{p}_\perp)\tilde{g}(0,\mathbf{p}_\perp)e^{-p^+/b})$$

Removes log divergence.

Breaks longitudinal boost invariance.

Positivity of scalar product?

Additional terms $\sim \delta^{(n)}(p^+)$ (zero modes) allowed.

- **Zero modes needed to recover broken kinematic symmetry, full rotational covariance, and positivity of the Hilbert space norm of the extension.**

- **Zero modes play a role in the proper definition of local operator products, but they do not play a role in the Haag expansion of the Heisenberg field.**

- **Generators: Algebraic normal ordering replaced by 0-mode normal ordering.**

- **They will appear in the local products of Heisenberg fields since the leading term in the Haag expansion is**

$$\phi(x) \sim Z \int F_m(x, \tilde{\mathbf{y}}) \phi_0(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}} + \cdots$$

Conclusions

- **A vacuum is an invariant linear functional on an algebra; the definition of the vacuum depends on the algebra.**
- **The relevant algebra is the algebra of local observables.**
- **For both free and interacting fields there are dynamical maps from the local algebra to a sub algebra of the light-front Fock algebra.**
- **This leads to a formulation of full Poincaré invariance and locality on a sub algebra of the light front Fock algebra.**

- **Models with different maps lead to inequivalent representations of the local algebra.**

- **Zero modes play no role in these mappings.**

- **The mappings carry the dynamics. Zero modes can play a role in the explicit construction of these mappings.**

- **There is a unitary map, $U_0(R)U(R)^\dagger$, that relates theories with different light fronts - $p^+ = 0$ singularities $\leftrightarrow$ ultraviolet singularities. $\therefore$ $3+1$ is more complicated that $1+1$.**

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