Few Body Currents Generated by Cluster Separability Constraints

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April 13, 2008
Outline

1. Poincaré Invariant Quantum Mechanics
   - Dynamics
   - Constraints

2. Clustering
   - Definition
   - Clustering Problems, Solutions

3. Testing the Effects
   - A Simple Model
   - Some Results

4. Conclusion
Poincaré invariant quantum theory:
- Unitary representation, $U(\Lambda, a)$, of the Poincaré group.
- Dynamics given by $U(\Lambda, a)$.
- Infinitesimal generators $H$, $P$, $J$, and $K$. Specifically,

**Definition**

$$H := i \left( \frac{\partial}{\partial a} U(I, a) \right) U^\dagger(I, a)$$

**Definition**

$$P := -i \left( \frac{\partial}{\partial a} U(I, a) \right) U^\dagger(I, a)$$
Dynamical constraints on current operators:

- Depend on representation of the Poincaré group, $U(\Lambda, a)$:

**Definition**

**Current conservation:** $[H, J^0(0)] - \sum_i [P^i, J^i(0)] = 0$

**Definition**

**Current covariance:** $U(\Lambda, a) J^\mu(x) U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu J^\nu(\Lambda x + a)$
U(Λ, a) is cluster separable if,

\[
\lim_{(a_i - a_j)^2 \to +\infty} \left\| [U(\Lambda, a) - \bigotimes U_i(\Lambda, a)] \prod U_i(I, a_i) |\psi\rangle \right\| = 0
\]

This ensures that the Poincaré invariance of the system also holds for isolated subsystems (large space-like separation).
In the clustering limit, the current should break up into a sum:

\[
\left[H, J^0(0)\right] - \sum_i \left[P^i, J^i(0)\right] = 0
\]

\[
\sum_a \left(\left[H_a, J_a^0(0)\right] - \sum_i \left[P^i_a, J^i_a(0)\right]\right) = 0
\]

\[
\left[H_a, J_a^0(0)\right] - \sum_i \left[P^i_a, J^i_a(0)\right] = 0 \quad \text{for each } a.
\]

Poincaré invariance, current covariance, and current conservation should all hold for isolated subsystems.
Construction of BT, TP Representations

Bakamjian-Thomas Construction

Bakamjian-Thomas, Coester, Sokolov:

- Add interactions to mass operator to construct $U(\Lambda, a)$.
- Kinematic spin; Does not satisfy cluster separability.

Tensor Product Construction

Derivative of BT constructed to satisfy cluster separability.

- Interaction-dependent spin.
- Difficult to add more than one two body interaction.
In the Bakamjian-Thomas construction,

- Systems with $N > 2$ fail to satisfy cluster properties.
- For $N = 3$, scattering equivalent to one that clusters.
  
  $\implies$ there is a unitary operator $A$ such that
  
  $$AU(\Lambda, a)A^\dagger = U'(\Lambda, a)$$
  
  where $U'(\Lambda, a)$ satisfies cluster properties.

- The operator, $A$, is called a packing operator.

- Essentially, $A$ restores cluster properties for $N = 3$. 
Electromagnetic observables can be calculated from current matrix elements ($|\psi_i\rangle$ and $|\psi_f\rangle$ are eigenstates of $H$):

$$\langle \psi_f | J^{\mu}(0) | \psi_i \rangle$$

Using $A$,

- $H \rightarrow H' = AH\mathcal{A}^\dagger$
- $|\psi_i\rangle \rightarrow |\psi'_i\rangle = \mathcal{A}|\psi_i\rangle$
- $|\psi_f\rangle \rightarrow |\psi'_f\rangle = \mathcal{A}|\psi_f\rangle$
- $\langle \psi_f | J^{\mu}(0) | \psi_i \rangle \rightarrow \langle \psi'_f | J^{\mu}(0) | \psi'_i \rangle = \langle \psi_f | \mathcal{A}^\dagger J^{\mu}(0) \mathcal{A} | \psi_i \rangle$
How can we Test the Effects of $A$?

Model:
- First assume 3 particles interacting in Bakamjian-Thomas representation.
- Turn off 13- and 23- pair interactions.
- $A$ connects resulting and TP representations.
- Scatter an electron off of particle 3.
- Can construct both BT and TP exactly.
Illustrate the Two Calculations: Details

\( A \) changes a delta function in \( p_3 \) to a delta function in relative momentum, \( q_3 \), obtained by boosting \( p_3 \) to rest frame of non-interacting 2+1 body system, using

\[
q_3 = B^{-1}(P/M_0)p_3.
\]

Realistic three body calculations are now being performed using this formalism and the issues being discussed are very relevant if these solutions are used to calculate electromagnetic observables!

\[
\langle \psi_f | A^\dagger J^\mu (0) A | \psi_i \rangle \quad \text{– good}
\]
\[
\langle \psi_f | J^\mu (0) | \psi_i \rangle \quad \text{– bad}
\]
Recap:

- Will calculate $\langle \psi_f | J^0(0) | \psi_i \rangle$ and $\langle \psi_f | A^\dagger J^0(0) A | \psi_i \rangle$.
- Here there is a difference since $N = 4$.

Recall

Current matrix elements...

TP: $\langle \psi_f | A^\dagger J^\mu(0) A | \psi_i \rangle$

BT: $\langle \psi_f | J^\mu(0) | \psi_i \rangle$
TP: $Q$ and $p_{12}$ vs $\langle \psi_f | A^\dagger J^0(0) A | \psi_i \rangle$ (good)

\[ \int \langle Q/2 \ p_{12} | J^0 | - Q/2 \ p'_{12} \rangle dp'_{12} \]
BT: $Q$ and $p_{12}$ vs $\langle \psi_f | J^0(0) | \psi_i \rangle$ (bad)

$$\int \langle p_{12} \ P \ D | J^0 | p'_{12} \ P' \rangle dp'_{12}$$

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Difference: BT - TP

Perpendicular BT - TP

Parallel BT - TP

Up to 6% difference.
Conclusion

- Two constructions of $U(\Lambda, a) - \text{BT, TP}$.
  - BT does not cluster for $N > 2$.
  - $N = 3$: scattering equivalent (special case).
  - $N > 3$: $\exists$ measurable differences.

- EM observables from current matrix elements.

- Clustering should hold for Poincaré invariance, current conservation, and current covariance.

- In BT representation, clustering should be restored using $A$.

- Electromagnetic calculations being done without clustering.

- Calculations with and without $A$ differ by up to 6%.
Goals

- Repeat with Light Front and Point forms.
- Use more complicated models.
Here the Minkowski metric is taken to have signature \((-,+,+,+):\)

\[
\eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

so that \(\lim (a_i - a_j)^2 \to +\infty\) means space-like separation.
Current Covariance

\[ U(\Lambda, a) J^\mu(x) U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu J^\nu(\Lambda x + a) \]

\[ \downarrow \]

\[ \sum_a U_a(\Lambda, a) J^\mu_a(x) U_a^\dagger(\Lambda, a) = \sum_a (\Lambda^{-1})^\mu_\nu J^\nu_a(\Lambda x + a) \]

\[ \downarrow \]

\[ U_a(\Lambda, a) J^\mu_a(x) U_a^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu J^\nu_a(\Lambda x + a) \quad \text{for each } a. \]
Illustration of BT and TP Constructions

\[
\begin{align*}
\left| (12) \otimes (3) \right\rangle & \xrightarrow{\langle AB|C \rangle_0} \left| ((12)(3)) \right\rangle \\
V_{(12)(3)} \downarrow & \\
\left| (12)_I \otimes (3) \right\rangle & \xrightarrow{\langle AB|C \rangle_I} \left| ((12)_I(3)) \right\rangle \sim \left| ((12)(3))_I \right\rangle \\
A_{(12)(3)} & \\
\end{align*}
\]
**Methods**

**Wave Functions**

Gaussian

\[ |\phi(k^2)|^2 = \left| \exp \left( -\frac{k^2}{k_0^2} \right) \right|^2 \]

Malfliet-Tjon

\[
\begin{align*}
a &= 1438.4812 \\
b &= -626.893 \\
c_1 &= 3.11 \\
c_2 &= 1.55
\end{align*}
\]

**Form Factor**

Monopole:

\[
F(q^2) = \left( \frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2
\]
Difference: BT - TP

Perpendicular BT - TP

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