Cluster Properties and Relativistic Quantum Mechanics

Wayne Polyzou
polyzou@uiowa.edu

The University of Iowa
Why is quantum field theory difficult?

- $\infty$ number of degrees of freedom.
- No rigorous ab-initio error bounds.
- Fock-space generators not generally densely defined

\[ H = \int d^3 x T^{00}(x) \quad \langle \psi | H^2 | \psi \rangle = \infty \]

regularization destroys commutation relations.

- No few-body problem.
Why is non-relativistic few-body quantum mechanics successful?

- Meaningful few-body problem directly related to experiment.
- Solutions with ab-initio error bounds possible.
- Cluster properties relate the few- and many-body problem.
Relativistic few-body quantum mechanics?

- Determine degrees of freedom by experiment.
- Construct the most general class quantum models with desired degrees of freedom and exact Poincaré symmetry.
- Constrain few-body dynamics by experiment.
- Use cluster properties to relate few and many-body problem.
It is possible?

- (Weinberg) “There have been many attempts to formulate a relativistically invariant theory that would not be a local field theory, and it is indeed possible to construct theories that are not field theories and yet yield a Lorentz invariant S-matrix for two-particle scattering, but such efforts have always run into trouble in sectors with more than two particles; either the three particle S-matrix is not Lorentz invariant, or else it violates the cluster decomposition principle.”

- (Schroer) An anthology of non-local QFT and QFT on noncommutative spacetime, Bert Schroer, hep-th/0405105 “This raised the question whether there exist consistent relativistic unitary and macro-causal particle theories are all. · · · In particular these theories fulfill the very non-trivial cluster separability properties of the associated Poincaré invariant unitary S-matrix; as a result their existence contradicts a dictum (which is ascribed to S. Weinberg) saying that a Poincaré invariant unitary S-matrix with these properties is characteristic for a (local) QFT.”
Relativity and Quantum Theory

- Inertial coordinate systems are coordinate systems where free particles move with constant velocity.
- Experiments on isolated systems cannot make an absolute determination of inertial coordinate system.
Relativity

- Because solutions of the Schrödinger equation are not observable in quantum theory, relativity does not require that solutions of the Schrödinger equation transform covariantly.

- This is an important difference between the classical and quantum formulation of relativistic invariance.
Relativistic Invariance

- **Galilean Relativity:** \( X \) and \( X' \) inertial \( \Rightarrow \)

\[
|\Delta \vec{x}_{ij}| = |\Delta \vec{x}_{ij}'|
\]

\[
\Delta t_{ij} = \Delta t'_{ij}
\]

- **Special Relativity:** \( X \) and \( X' \) inertial \( \Rightarrow \)

\[
|\Delta \vec{x}_{ij}|^2 - c^2 |\Delta t_{ij}|^2 = |\Delta \vec{x}_{ij}'|^2 - c^2 |\Delta t'_{ij}|^2
\]
Relativity

- The Michelson-Morley experiment verified that special relativity gives the observed relation between inertial coordinate systems.
- The most general transformation relating two inertial coordinate systems is a Poincaré transformation:

\[
x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\nu} x^\nu + a^\mu
\]

\[
g^{\mu\nu} = \Lambda^\mu_{\alpha} \Lambda^\nu_{\beta} g^{\alpha\beta}
\]
Quantum Theory

- States are represented by unit vectors (rays), $|\psi\rangle$, in a complex vector space.

\[ \langle \phi | \psi \rangle = \langle \psi | \phi \rangle^* \]

- The predictions of a quantum theory are the probabilities

\[ P_{\phi \psi} = |\langle \phi | \psi \rangle|^2 \]
Relativity in Quantum Theory

\[ X \rightarrow X' \]

\[ |\phi\rangle, |\psi\rangle \rightarrow |\phi'\rangle, |\psi'\rangle \]

\[ \Downarrow \]

\[ \left| \langle \phi | \psi \rangle \right|^2 = \left| \langle \phi' | \psi' \rangle \right|^2 \]

- Must hold for all \( |\psi\rangle, |\phi\rangle \) and all inertial coordinate systems \( X \) and \( X' \)
Wigner’s Theorem

\[ |\langle \phi | \psi \rangle|^2 = |\langle \phi' | \psi' \rangle|^2 \]

\[ \Downarrow \]

\[ |\phi'\rangle = U|\phi\rangle \quad |\psi'\rangle = U|\psi\rangle \quad \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle \]

or

\[ |\phi'\rangle = A|\phi\rangle \quad |\psi'\rangle = A|\psi\rangle \quad \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle^* \]
Wigner’s Theorem

- For rotations, translations, and rotationless Lorentz transformations:

\[ U = U^{1/2}U^{1/2} \quad A = A^{1/2}A^{1/2} = U \]

- The correspondence between states must be unitary!

\[ |\psi'\rangle = U|\psi\rangle \]
Wigner’s Theorem

\[ X \rightarrow X' \rightarrow X'' \]

\( (\Lambda, a) \) \( (\Lambda', a') \) \( (\Lambda'', a'') \)

\[ \downarrow \]

\[ U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda'a + a') \]

- \( U(\Lambda, a) \) is a unitary representation of the Poincaré group
Elements of RQM

- Model Hilbert space: $\mathcal{H}$
- Unitary representation of Poincaré group: $U(\Lambda, a) : \mathcal{H} \rightarrow \mathcal{H}$
- Spectral condition: $H \geq 0 \quad U(I, t) = e^{-iHt}$
- Cluster properties:

$$\lim_{|\vec{d}_1 - \vec{d}_2| \rightarrow \infty} \|[U(\Lambda, a) - U_1(\Lambda, a) \otimes U_2(\Lambda, a)]U_1(I, \vec{d}_1) \otimes U_2(I, \vec{d}_2) |\psi\rangle\| = 0$$
Possible approaches

- Covariant wave functions $\rightarrow$ quasi-Wightman functions.
- Covariant constraint dynamics $\rightarrow$ quasi-Wightman functions based on first class constraints.
- Euclidean relativistic quantum theory - based on reflection positive quasi-Schwinger functions.
- Directly interacting particles (this talk).
Difficulties

Space Time Diagram
Difficulties

\[ \Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta & 0 & 0 & \gamma \end{pmatrix} \]

\[ \vec{a} = (0, 0, 0, z) \quad \vec{b} = (0, 0, 0, -\gamma z) \quad t := (\gamma \beta z, \vec{0}) \]

\[ (\Lambda^{-1}, \vec{b})(\Lambda, \vec{a}) = (I, (t, \vec{0})) \]
Difficulties

\[ U(\Lambda, a) = e^{-i \sum \lambda_i G_i} \]

\[ G_i = G_i^\dagger \quad G_i \in \{ H, \vec{P}, \vec{J}, \vec{K} \} \]

\[ [G_i, G_j] = i g_{ijk} G_k \]

\[ \downarrow \]

\[ [P_i, K_j] = i \delta_{ij} H \]
Dynamics

- Galilean Invariant Dynamics \( \{H, \vec{P}, \vec{J}, \vec{K}, M\} \)

\[
[\vec{P}_i, \vec{K}_j] = i\delta_{ij}M
\]

- Poincaré Invariant Dynamics \( \{H, \vec{P}, \vec{J}, \vec{K}\} \)

\[
[\vec{P}_i, \vec{K}_j] = i\delta_{ij}(H_0 + V)
\]

- Puts non-linear constraints on \( \{H, \vec{P}, \vec{J}, \vec{K}\} \) for consistent initial value problem!
The Currie-Jordan-Sudarshan Theorem

\[ \{X_i, P_j\} = \delta_{ij} \]

\[ \{X_i, J_j\} = \epsilon_{ijk} X_k \]

\[ \{X_i, K_j\} = X_j \{X_i, H\} \quad \text{world-line condition} \]

- Poisson brackets of \( \{H, \vec{P}, \vec{J}, \vec{K}\} \) satisfy Lie algebra of Poincaré group.
The Currie-Jordan-Sudarshan Theorem

- Can only be satisfied for free particles!
- The CJS theorem suggests that it might be difficult to formulate relativistic models for systems of interacting particles.
- The out in relativistic quantum mechanics is that there is no position operator!
Position

- Let $|\vec{0}, 0\rangle$ be a vector corresponding to a particle at $\vec{x} = \vec{0}$ at time $t = 0$

\[
\langle \vec{p} | \vec{0}; 0 \rangle = \langle \vec{p} | U^\dagger(\Lambda(\vec{p}), 0) | \vec{0}; 0 \rangle
\]

\[
= \frac{\sqrt{m}}{(\vec{p}^2 + m^2)^{1/4}} \langle \vec{p} = 0 | \vec{0}; 0 \rangle
\]

\[
= \frac{N}{(\vec{p}^2 + m^2)^{1/4}}
\]

\[
\langle \vec{p} | \vec{x}; t = 0 \rangle = \langle \vec{p} | U(I, \vec{x}) | \vec{0}; 0 \rangle
\]
Position

\[ \langle \vec{p} | \vec{x} \rangle = N \frac{e^{-i\vec{p} \cdot \vec{x}}}{(\vec{p}^2 + m^2)^{1/4}} \]

\[ \langle 0 | \vec{x} \rangle = N^2 \int \frac{d^3p}{(\vec{p}^2 + m^2)^{1/2}} e^{-i\vec{p} \cdot \vec{x}} = \]

const \( \times \) \( D_+(0, |\vec{x}|) \sim \left( \frac{mc|\vec{x}|}{\hbar} \right) e^{-\frac{mc|\vec{x}|}{\hbar}} \neq 0 \)
Cluster Properties

\[ \tilde{K} = \tilde{K}_{(12)(3)} + \tilde{K}_{(23)(1)} + \tilde{K}_{(31)(2)} - 2\tilde{K}_0 \]

\[ [K^1, K^2] = -iJ^3 = -iJ^3_0 \]

\[ [K^1, K^2] = \cdots [K^1_{(12)(3)}, K^2_{(23)(1)}] + \cdots \]

- Commutator of interacting generators create many-body interactions that need to be canceled.
- Result is that either some cluster limits do not exist or quantities that should survive cluster limits actually vanish.
Cluster Properties

\[
\lim_{\lambda \to \infty} e^{i\vec{p}_3 \cdot \lambda \vec{a}} V_{12} e^{-i\vec{p}_3 \cdot \lambda \vec{a}} = \frac{1}{?}
\]

\[
\vec{p}_3 = \vec{q}_3 + \vec{P} F(m_{012}, \vec{P}, \vec{q}) = 
\]

\[
\lim_{\lambda \to \infty} e^{\vec{P} \cdot \lambda \vec{a}} F(m_{012}, \vec{P}, \vec{q}) V_{12} e^{-i\vec{P} \cdot \lambda \vec{a}} F(m_{012}, \vec{P}, \vec{q}_3) = 
\]

\[
[V_{12}, \vec{q}_3] = [V_{12}, \vec{P}] = 0; \quad [V_{12}, m_{120}] \neq 0
\]

\[
\downarrow
\]

\[
\lim_{\lambda \to \infty} e^{i\vec{p}_3 \cdot \lambda \vec{a}} V_{12} e^{-i\vec{p}_3 \cdot \lambda \vec{a}} = 0
\]
Model Hilbert space

- Determine a complete measurement
- Defines a complete set of commuting observables
- The model Hilbert space is the space of square summable functions of the eigenvalues of the complete set of commuting observables.

- Adequate to describe results of any experiment
One-Particle Relativistic Quantum Mechanics

\[ U_1(\Lambda, a) \Rightarrow \{ H, \vec{P}, \vec{J}, \vec{K} \} \]

\[ \Rightarrow \{ M, j, \eta_1, \cdots, \eta_4, \Delta \eta_1, \cdots, \Delta \eta_4 \} \]

\[ \Rightarrow \]

\[ \mathcal{H}_1 : \langle \psi | \phi \rangle = \sum \int \psi^* (\eta) \phi (\eta) d\eta \]

\[ \Rightarrow \]

\[ U_1(\Lambda, a) |(m, j)\eta\rangle = \sum \int |(m, j)\eta'\rangle D_{\eta'\eta}^{m, j} (\Lambda, a) \]
Two-particle Hilbert space

\[ \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_1 \quad U_0(\Lambda, a) := U_1(\Lambda, a) \otimes U_1(\Lambda, a) \]

\[ \downarrow \]

\[ G^i_0 = G^i_1 \otimes I_2 + I_1 \otimes G^i_2 \]

\[ \downarrow \]

\[ \{ H_0, \vec{P}_0, \vec{J}_0, \vec{K}_0 \} \]

\[ \downarrow \]
Two-particle Hilbert space

\[ \{ M, j, \eta_1, \cdots, \eta_4, \Delta \eta_1, \cdots, \Delta \eta_4 \} \]

\[
U_1(\Lambda, a) \otimes U_1(\Lambda, a) = \int \bigoplus U_{0,mj}(\Lambda, a)
\]

\[
\langle (m_1, j_1)\eta_1; (m_2, j_2)\eta_2 | (m, j)\eta, \{d\} \rangle
\]

\[
U_0(\Lambda, a) | (m, j)\eta, \{d\} \rangle = \sum \int | (m, j)\eta', \{d\} \rangle D_{\eta'\eta}^{m,j}(\Lambda, a)
\]
Two particle Hilbert space

- The form of the Clebsch-Gordan coefficients depend on choice of vector labels \( \eta \). There are an infinite number of choices.
- The two-body \( m \) has continuous spectrum; it is often replaced by

\[
k \quad \Leftrightarrow \quad m = \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2}
\]

- Poincaré Clebsch-Gordan coefficients have multiplicity quantum numbers \( \{d\} \).
Two-Particle Dynamics

\[ \{ M_0, j_0, \eta_{01}, \cdots, \eta_{04}, \Delta \eta_{01}, \cdots, \Delta \eta_{04} \} \]

\[ \downarrow \]

\[ M = M_0 + V; \quad 0 = [V, j_0] = [V, \eta_{0i}] = [V, \Delta \eta_{0i}] \]

- Find simultaneous eigenstates of \( \{ M, j_0, \eta_{0i} \} \)
- Can be solved in basis of eigenstates of \( \{ M_0, j_0, \eta_{0i} \} \)
Two-Particle Dynamics

\[ \langle (m_0, j)\eta, \{d\} | (m, j')\eta' \rangle = \delta(\eta-\eta')\delta_{jj'}\phi_{m,j}(m_0, \{d\}) \]

\[ \Downarrow \]

\[ U(\Lambda, a) | (m, j)\eta \rangle = \sum \int | (m, j)\eta' \rangle D_{\eta'\eta}^{m,j}(\Lambda, a) \]
2+1-Particle Dynamics

\[ U_{12}(\Lambda, a) \otimes U_3(\Lambda, a) \]

\[ U_0(\Lambda, a) := U_{012}(\Lambda, a) \otimes U_3(\Lambda, a) \]

\[ \downarrow \]

\[ \left| (m_{0123}, j_{0123}) \eta, \{k_{12}, \{d_{12}\} \cdots \} \right| \]

\[ \bar{V} : [\bar{V}, k_{12}] \neq 0, \quad [\bar{V}, \{d_{12}\}] \neq 0, \quad [\bar{V}, \cdots] = 0 \]

\[ \downarrow \]
2+1-Particle Dynamics

\[ \tilde{m}_{12} = m_{012} + \tilde{V} \]

\[ \tilde{m}_{(12)(3)} = \sqrt{k_0^{2(12)(3)} + \tilde{m}_{12}^2} + \sqrt{k_0^{2(12)(3)} + m_3^2} \]

\[ \downarrow \]

\[ U_{12}(\Lambda, a) \otimes U_3(\Lambda, a) \quad \tilde{U}_{(12)(3)}(\Lambda, a) \]
2+1-Particle Dynamics

- $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ and $\bar{U}_{(12)(3)}(\Lambda, a)$ give identical scattering matrix elements.
- $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ clusters; combining different interactions destroys the Poincaré commutation relations.
- $\bar{U}_{(12)(3)}(\Lambda, a)$ violates cluster properties; Poincaré invariant addition of interactions for different interacting pairs possible in this representation.
Scattering Equivalences

\[
A^\dagger = A^{-1} \quad \lim_{t \to \pm \infty} \| (I - A) U_0(I, t) | \psi \rangle \| = 0
\]

\[
S(H, H_0) = \Omega_+^\dagger (H, H_0) \Omega_-(H, H_0)
\]

\[
\Downarrow
\]

\[
S(H, H_0) = S(H', H_0) \iff H' = AH A^\dagger
\]
Scattering Equivalences

- Scattering equivalences $A$ are unitary elements of a $*$ algebra of asymptotic constants.
- The $*$ algebra provides a functional calculus to construct functions of non-commuting scattering equivalences.
- Operators in this $*$ algebra relate $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ and $\bar{U}_{(12)(3)}(\Lambda, a)$
Cluster properties $\Leftrightarrow$ irreps

\[
\begin{align*}
\left| (12) \otimes (3) \right> & \xrightarrow{\langle AB|C \rangle_0} \left| ((12)(3)) \right> \\
V_{(12)(3)} \downarrow & \\
\left| (12)_I \otimes (3) \right> & \xrightarrow{\langle AB|C \rangle_I} \left| ((12)_I(3)) \right> \\
& \overset{\sim}{\underset{A_{(12)(3)}}{\xrightarrow{\bar{V}_{(12)(3)}}}} \left| ((12)(3))_I \right>
\end{align*}
\]

- $A_{(12)(3)}$ scattering equivalence
Three Particles

\[ M = M_{(12)(3)} + M_{(23)(1)} + M_{(31)(2)} - 2M_0 + V_{(123)} = \]

\[
A^\dagger \left[ \sum_{(ij)(k)} A^{(1j)(k)} M_{(ij)(k)} A^\dagger_{(ij)(k)} - 2M_0 + \bar{V}_{(123)} \right] A
\]

\[
\vdots
\]

\[
H_{(ij)(k)} := H_{ij} \otimes I_k + I_{ij} \otimes H_k
\]

\[ A \rightarrow A_{(ij)(k)} \quad A = \exp \left( \sum \ln( A_{(ij)(k)} ) \right) \]
Three Particles

- Result satisfies cluster properties and Poincaré Lie algebra.
- $A$ is also a scattering equivalence.
- $A$ and $A_{(ij)(k)}$ generate the many-body interactions needed to preserve commutation relations and cluster properties.
- Simplest non-trivial problem is electron scattering off of three-nucleon system.
Beyond-three

- Construction on previous slide is the first step in an inductive construction for $N$ particles.
- The input is Poincaré Clebsch-Gordan coefficients, irreducible representations of Poincaré group, Poincaré Racah coefficients ($N \geq 3$), Poincaré Wigner-Eckart theorem (tensor and spinor operators), and Scattering equivalences generated by adding interacting and reducing representations in different orders.
Beyond-three

- The construction has been generalized to treat N-particle systems and systems with limited particle production.
- The requirement of cluster properties adds new dynamical non-linear constraints for more than two particles. These require many body interactions.
Beyond-three

- $U(\Lambda, a)$ implies that spinor, tensor, and four vector operators are interaction dependent. These can be constructed using the Poincaré Wigner-Eckart theorem
- Applications to few-body systems exist.
Outlook

- True production is an open problem.
- Need few degree of freedom problem directly related to both experiment and many-body problem.
- Use physical particles as degrees of freedom?
- Use center of momentum energy to control number of degrees of freedom?