Wavelet Bases in Few-Body Scattering Calculations

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Outline

• Background

• Scaling functions

• Multiscale analysis and wavelets

• Wavelet numerical analysis

• Applications
What are wavelets?

They are orthonormal basis functions that are used in data compression algorithms.

JPEG digital images are tables of expansion coefficients in a wavelet basis.

FBI fingerprint files are stored as expansion coefficients in a wavelet basis.
Our interest in wavelets?

Scattering problems in Poincaré invariant quantum mechanics can be formulated as integral equations with compact kernels.

Compact operators can be uniformly approximated by finite-dimensional matrices.

Large matrices can be uniformly approximated by sparse matrices in a wavelet basis.

Wavelet bases can be used to improve the efficiency of momentum-space scattering calculations.
Interesting considerations

Wavelet bases are natural for problems (quantum field theory, phase transitions) where many scales are strongly coupled.

Basis functions are related to fixed points of a linear renormalization group equation.

Basis functions have a fractal structure, and are not amenable to standard numerical methods.
Elements of wavelet numerical analysis

Unitary operators, $D$ and $T$ on $L^2(\mathbb{R})$:

$$D(f)(x) := \sqrt{\frac{1}{2}} f\left(\frac{x}{2}\right) \quad T(f)(x) := f(x - 1)$$

Scaling equation:

$$D(\phi)(x) = \sum_{l=0}^{2k-1} h_l T^l(\phi)(x). \quad (1)$$

The scaling function, $\phi(x)$, is a fixed point of (1) with normalization

$$\int \phi(x) dx = 1.$$
$h_l$ are constant coefficients that determine the type of wavelet (we use Daubechies’ wavelets).

$k$ is a finite integer.

A necessary condition for the existence of a fixed point of the scaling equation is:

$$
2k - 1 \sum_{l=1}^{2k-1} h_l = \sqrt{2}.
$$
Daubechies’ scaling coefficients, $k = 1, 2, 3$

<table>
<thead>
<tr>
<th>$h_l$</th>
<th>$k=1$</th>
<th>$k=2$</th>
<th>$k=3$</th>
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<td>$h_0$</td>
<td>$1/\sqrt{2}$</td>
<td>$(1 + \sqrt{3})/4\sqrt{2}$</td>
<td>$(1 + \sqrt{10 + \sqrt{5 + 2\sqrt{10}}})/16\sqrt{2}$</td>
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<td>$h_1$</td>
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<td>$(3 + \sqrt{3})/4\sqrt{2}$</td>
<td>$(5 + \sqrt{10 + 3\sqrt{5 + 2\sqrt{10}}})/16\sqrt{2}$</td>
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<td>$(3 - \sqrt{3})/4\sqrt{2}$</td>
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<td>$h_3$</td>
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</table>
Computing $\phi(x)$

**Theorem:** support $[\phi(x)] \in [0, 2k - 1]$

**Theorem:** $k > 1 \implies \phi(x)$ continuous.

$$\phi(n) = \sqrt{2} \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(2n - l).$$

$$\sum_n \phi(n) = 1.$$  

Solving above equations gives $\phi(n)$, $n \in [0, 2k - 1]$.

$$\phi\left(\frac{n}{2}\right) = \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(n - l).$$

Induction gives exact values at all dyadic rationals.
Daubechies’ K=3 scaling function
Approximation spaces

- \( \phi_{mn}(x) := D^m T^n \phi(x) \).

- \((\phi_{mn}, \phi_{mk}) = \delta_{nk}\) (fixed \(m\)).

- \(\{\phi_{mn}\}_n\) can locally pointwise represent polynomials of degree \(< k\) for each \(m\).

- \(\mathcal{V}_m := \text{span}\{\phi_{mn}\}_n \cap L^2(\mathbb{R}) = \text{“scale } m\text{ subspace” of } L^2(\mathbb{R}).\)
Daubechies-2 Scaling Functions Translated
Summing Daubechies−2 Wavelets to Represent a Constant Function
Summing Daubechies−2 Wavelets to Represent a Linear Function
Multiscale analysis

\[ m > n \quad \Rightarrow \quad V_n \supset V_m \]

\[ L^2(\mathbb{R}) \supset \cdots \supset V_{n-1} \supset V_n \supset V_{n+1} \supset \cdots \supset \emptyset \]

**Theorem:** \( \lim_{n \to -\infty} V_n = L^2(\mathbb{R}) \)

\[ V_n = V_{n+1} \oplus W_{n+1} \]

\[ \downarrow \]

\[ V_n = W_{n+1} \oplus W_{n+2} \oplus \cdots \oplus W_{n+m} \oplus V_{n+m} \]

\[ L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} W_n = V_m \oplus \left( \bigoplus_{n=-\infty}^{m} W_n \right) \]
Wavelets

$\mathcal{W}_n$ are wavelet spaces

$$D\psi(x) = \sum_{l=0}^{2k-1} (-1)^l h_{2k-l-1} T^l \phi(x)$$

$$\psi_{ml}(x) = D^m T^l \psi(x)$$

$\{\psi_{ml}\}_l$ orthonormal basis for $\mathcal{W}_m$

$\psi(x)$ is called the “Mother” wavelet
Daubechies’ $k = 3$ mother wavelet
support \([\psi(x)] = \text{support} \[\phi(x)]\)

\(h_l\) are determined up to space reflection by the requirements

\[(\psi, x^n) = 0, \quad n = 0, \ldots, k - 1 \quad (\phi, T^m\phi) = \delta_{m0}\]

\[
\downarrow
\]

\[
\sum_m m^n (-)^m h_{l-m} = 0 \quad \sum_{l=0}^{2k-1} h_{l-2m} h_l = \delta_{m0}
\]

\[
\sum_{l=1}^{2k-1} h_l = \sqrt{2}
\]
Properties

\( \phi_m(x) \) can locally pointwise represent polynomials of degree \( < k \).

\( \psi_{mn}(x) \) are orthogonal to polynomials of degree \( < k \).

\[
\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+l} \oplus \mathcal{V}_{n+l}
\]
Approximations

Fix scale $n$ of approximate solution

$$v_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus v_{n+m}$$

$$\mathcal{W} : v_n \rightarrow \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus v_{n+m}$$

Wavelet transform $\mathcal{W}$ ($O(N)$ orthogonal transformation)

$$f(x) = \sum_m a_m \phi_{nm}(x) = \sum_m a'_m \phi_{n+1,m}(x) + \sum_m \sum_{k=1}^I b_{n+k,m} \psi_{n+k,m}(x)$$
Sparse matrices

\[ a_n \leftrightarrow a'_n, b_{nm} \]

\[ W M W^T = M_1 + M_2 \quad \| M_2 \| < \epsilon, \ M_1 \text{ sparse} \]

\( b_{nm} \text{ small if } f(x) \text{ can be well approximated by degree } k - 1 \)
\( \text{polynomial on the support of } \psi_{nm}(x). \)

\[ \supp(\psi_{nm}(x)) \subseteq [2^n m, 2^n (m + 2k - 1)] \]
Wavelet numerical analysis

Moments

\[ \langle x^0 \rangle_\phi := (x^0, \phi) = 1 \]

\[ \langle x^m \rangle_\phi := (x^m, \phi) = (Dx^m, D\phi) = 2^{-m-1/2} \sum_l h_l (x^m, T^l \phi) \Rightarrow \]

\[ \langle x^m \rangle_\phi = \frac{1}{\sqrt{2}} \frac{1}{2^m - 1} \sum_{l=0}^{2k-1} \sum_{m=0}^{m-1} h_l l^{m-n} \frac{m!}{n!(m-n)!} \langle x^n \rangle_\phi \]

recursion

\[ \downarrow \]

\[ \langle x^m \rangle_\phi, \quad \langle x^n \rangle_{\phi lm}, \quad \langle x^n \rangle_{\psi lm} \quad \text{exact} \]
Low-degree polynomials

\[ x^m = \sum_n c_n^m \phi_n(x) \]

**exact for** \( m < k \)

\[ c_n^m = (T^n \phi, x^m) = \sum_{l=0}^{m} \frac{m! n^{m-l}}{l! (m-l)!} \langle x^l \rangle_\phi \]

\[ c_n^0 = 1; \quad c_n^1 = n + \langle x^1 \rangle_\phi, \quad \ldots \]
Smoothness

\[ \phi(x) = \sqrt{2} \sum_{l=0}^{2k-1} h_l \phi(2x - l) \]

\[ \downarrow \]

\[ \phi(n) = \sqrt{2} \sum_m H_{nm} \phi(m) \quad H_{mn} := h_{2n-m} \]

\[ \frac{d^l \phi}{dx^l}(n) = 2^l \sqrt{2} \sum_m H_{nm} \frac{d^l \phi}{dx^l}(m) \]

_\phi \text{ has } l \text{ derivatives if } H_{mn} \text{ has an eigenvalue } \lambda = 2^l \sqrt{2} \_
Derivatives

\[ x = \sum_n c_n^1 \phi_n(x) \Rightarrow 1 = \sum_n c_n^1 \frac{d\phi_n}{dx}(x) \quad \Rightarrow c_n^1 = n + \langle x^1 \rangle_\phi \]

\[ D(\frac{d\phi}{dx})(x) = 2 \sum_{l=0}^{2k-1} h_l T^l(\frac{d\phi}{dx})(x). \]

\[ \Downarrow \]

\[ \frac{d\phi}{dx}(x) \approx \sum_n \left( \frac{d\phi}{dx}, \phi_n \right) \phi_n(x) \]
Derivatives

\[
\frac{d\phi_{mn}}{dx}(x) \quad \frac{d\psi_{mn}}{dx}(x)
\]

can be calculated exactly at dyadic rationals.

\[
(\phi_{mn}, \frac{d\phi_{mk}}{dx}) \quad (\psi_{nl}, \frac{d\phi_{mk}}{dx}) \quad (\phi_{mn}, \frac{d\psi_{mk}}{dx}) \quad (\psi_{nl}, \frac{d\psi_{mk}}{dx})
\]

can all be computed exactly.
Nonlinearities

\[ \phi_{n_2}(x)\phi_{n_2}(x) \approx \sum_{n_3} \Gamma_{n_1, n_2}^{n_3} \phi_{n_3}(x) \]

\[ \Gamma_{n_1, n_2}^{n_3} = \int \phi_{n_1}(x)\phi_{n_2}(x)\phi_{n_3}(x)dx = \sum_{l_1, l_2, l_3} \sqrt{2} h_{l_1} h_{l_2} h_{l_3} \Gamma_{2n_3+l_3}^{2n_1+l_1, 2n_2+l_2} \]

\[ \sum_{n_3} \Gamma_{n_1, n_2}^{n_3} = \delta_{n_1, n_2} \quad \Gamma_{n_1, n_2}^{n_3} = \Gamma_{n_1-n_3, n_2-n_3}^0 \]

\[ \Downarrow \]

\[ \Gamma_{n_1, n_2}^{n_3}, \ldots \]
Boundary integrals

\[ B_m = \int_m^\infty \phi(x) \, dx = \frac{1}{2} \sum_{l=1}^{2k-1} h_l B_{2m-l} \]

\[ B_m = 0 \quad m \geq 2k - 1 \quad B_m = 1 \quad m \leq 0 \]

\[ B^m = \int_{-\infty}^m \phi(x) \, dx = 1 - B_m \]
Poles

\[ J_n := (\phi_n, \frac{1}{x - i0^+}) = (D\phi_n(x), D\frac{1}{x - i0^+}) = \]

\[ \sqrt{2} \sum_{l=0}^{2k-1} h_l(\phi_{2n+l}, \frac{1}{x}) = \sqrt{2} \sum_{l=0}^{2k-1} h_l J_{2n+l} \]

large \( n \)

\[ J_n = \frac{1}{n} \sum_{m=0}^{\infty} \frac{\langle x^m \rangle_{\phi}}{n^m} \]

\[ i\pi = \int_{-m}^{m} \frac{dx}{x - i0^+} = \sum_n J_n + \text{boundary terms} \]
Logarithmic singularities

\[
L_n := (\phi_n, \ln) = (D\phi_n, D\ln) = \\
\frac{1}{\sqrt{2}} \sum_{l=0}^{2k-1} h_l L_{2n-l} - \ln(2)
\]

large \(n\)

\[
L_n = \ln(n) - \sum_{m=1}^{\infty} (-1)^m \frac{\langle x^m \rangle_\phi}{mn^m}
\]
**One-point quadrature**

\[ k > 1 \Rightarrow \langle x^n \rangle_\phi = \langle x \rangle^n_\phi, \quad n = 0, 1, 2 \]

\[ \Downarrow \]

\[ \int \phi(x) P(x) dx \approx P(\langle x^n \rangle_\phi) \]

**exact for** \( P(x) \) **a polynomial of degree 2!**
Moving singularities

\[ J_{mn} := \int \frac{\phi_m(x)\phi_n(y)}{k - x - y} \, dx \, dy = \]

\[ \int \frac{\Phi(x)}{k - x - m - n} \, dx \]

\[ \Phi(x) := \int \phi(y)\phi(x - y) \, dy \]

\[ D\Phi(x) = \sum_l r_l T^l \Phi(x) \quad \int \Phi(x) \, dx = 1 \]
Autocorrelation function

\[ \Phi(x) \]

\[ \begin{align*}
0 & \quad 2 & \quad 4 & \quad 6 & \quad 8 & \quad 10 \\
\Phi(x) & \quad -0.2 & \quad 0 & \quad 0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1
\end{align*} \]
\[ f(x) = g(x) + \int \frac{K(x,y)}{y - i0^+} f(y) \, dy \]

\[ f \approx \sum f_n \phi_{mn}(x) \]

\[ f_l = g(x_l) + \sum_{nk} K(x_l, y_n) \Gamma_{kn}^l J_k w f_n \]

\[ \mathcal{W} \Rightarrow \text{solve} \Rightarrow \mathcal{W}^{-1} \Rightarrow \]

\[ f(x) = g(x) + \sum_{nk} K(x, y_n) \Gamma_{kn}^l J_k w f_n \]

\[ J_n := 2^{m/2} \int dx \frac{\phi_{mn}(x)}{x - i0^+} \]
Required input

a. **Kernel at one point quadrature points**

b. **Driving at one point quadrature points**

c. **Integrals** $\Gamma_{mn}^l$ and $J_k$ on one scale
\textbf{K=3, E=10 MeV, J=-7}

<table>
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<th>$\epsilon$</th>
<th>percent</th>
<th>on-shell value</th>
<th>on-shell error</th>
<th>mean-square error</th>
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<td>0</td>
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Quantum fields $\Phi(x)$

$$\Phi(x) = \sum_{n=-\infty}^{\infty} \Phi(m, n)\phi_{m,n}(x) + \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \Psi(m, n)\psi_{m,n}(x)$$

$$\Phi(m, n) := \int \phi_{m,n}(x)\Phi(x)dx$$

$$\Psi(m, n) := \int \psi_{m,n}(x)\Phi(x)dx$$
Volume Cutoff $N$ (both sums)

\[ \sum_{n=-\infty}^{\infty} \rightarrow \sum_{n=-N}^{N} \]

Resolution Cutoff $M$ (wavelet field sum)

\[ \sum_{k=-\infty}^{m-1} \rightarrow \sum_{k=M}^{m-1} \]
\[
\frac{d\Phi}{dx}(x) = \sum_{n=-\infty}^{\infty} \Phi(m, n) \frac{d\phi_{m,n}}{dx}(x) + \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \Psi(m, n) \frac{d\psi_{m,n}}{dx}(x)
\]

**Derivative of the wavelet or scaling function can be expanded in basis of wavelets and scaling functions**

\[
\frac{d\phi_{m,n}}{dx}(x) = \\
\sum_{n=-\infty}^{\infty} \left( \frac{d\phi_{m,n}}{dx}, \phi_{m,n}(x) \right) \phi_{m,n}(x) + \\
\sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \left( \frac{d\phi_{m,n}}{dx}, \psi_{k,n}(x) \right) \psi_{k,n}(x)
\]

:::
Operator products

\[ \Phi(x)^2 = \]

\[ \sum_{n_1, n_2 = -\infty}^{\infty} \Phi(m, n_1)\Phi(m, n_2)\phi_{m,n_1}(x)\phi_{m,n_2}(x) + \]

\[ \sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k = -\infty}^{m-1} \Phi(m, n_1)\Psi(k, n_2)\phi_{m,n_1}(x)\psi_{k,n_2}(x) + \]

\[ \sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k = -\infty}^{m-1} \Psi(k, n_1)\Phi(m, n_2)\psi_{k,n_1}(x)\phi_{m,n_2}(x) + \]

\[ \sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k_1, k_2 = -\infty}^{m-1} \Psi(k_1, n_1)\Psi(k_2, n_2)\phi_{m,n_1}(x)\psi_{m,n_2}(x) \]
\[ \phi_{m,n_1}(x) \phi_{m,n_2}(x) = \]

\[ \sum_{n_3=-\infty}^{\infty} \phi_{m,n_3}(x) \int \phi_{m,n_3}(y) \phi_{m,n_1}(y) \phi_{m,n_2}(y) dy \]

\[ \sum_{n_3=-\infty}^{\infty} \sum_{k=-\infty}^{m-1} \psi_{k,n_3}(x) \int \psi_{k,n_3}(y) \phi_{m,n_1}(x) \phi_{m,n_2}(y) dy \]

\[ \vdots \]
Change of scale

\[ \Phi(m, n) = W_{nl}^{m1} \Phi(m - 1, l) + W_{nl}^{m2} \Psi(m - 1, l) \]

\( W_{nl}^{m1} \) and \( W_{nl}^{m2} \) exactly computable

Use resolution cutoff after computing all derivatives and operator products. Cutoff gives projection on subspace of given resolution.
\[ [G_i(x), G_j(y)] = ig_{ijk} \delta(x - y) G_k(x) \]

\[ \sum_n \psi_n(x) = 1 \quad \text{(partition of unity)} \]

\[ \sum_{mn} [G_i(n), G_j(m)] = ig_{ijk} \sum_n G_j(n) \]

\[ G_i(n) = \int \phi_n(x) G_i(x) dx \]
• Generalization of block-spin approximation. Has more smoothness.

• Map connecting different resolutions is invertible.

• Scale invariance means that the dependence on the $\Psi$ fields vanishes.

• Local operator products and derivatives of fields are described in terms of analytically computable overlap integrals.

• $\Psi$ fields describe physics lost on changing scales.

• Natural projections on given scale.
Open Questions?

- Reflection Positivity?
- Exact local gauge invariance?
- Poincare invariance?
Conclusion

- Wavelet bases have all of the advantages of spline bases with the additional properties:
  a. orthonormal basis.
  b. sparse matrix (even in momentum space).
  c. wavelet transform automatically finds structure.
  d. application to scattering with Malfliet-Tjon potential more accurate than what we can obtain with spline basis.

- Tested on two-body scattering using partial wave and direct three-dimensional integration.

- Interesting possibilities for discrete field theories.