Euclidean Relativistic Quantum Mechanics

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Motivation

• Construct relativistic quantum models of systems with a finite number of degrees of freedom

• Desirable features:

  • Models motivated by field theory

  • Cluster properties
Elements of relativistic quantum theory

- Hilbert space $\mathcal{H}$ (requirement for a quantum theory)

- Unitary representation of the Poincaré group (requirement for relativistic invariance of quantum probabilities).

- Cluster properties (required for tests of relativity on isolated subsystems)

- Spectral condition (required for stability of theory)
Input

Reflection-positive Euclidean Green function(s) or generating functional

Problem

Construct relativistic quantum mechanical models

(We want to avoid using analytic continuation!)
Field theory motivation

Euclidean generating functional or Green functions:

\[ Z[f] := \frac{\int D_e[\phi] e^{-A[\phi] + i\phi(f)}}{\int D_e[\phi] e^{-A[\phi]}} = \sum_{n} \frac{(i)^n}{n!} S_n(f, \cdots, f) \text{ } \text{ n times} \]

\[ A[\phi] = \text{Action}, \quad D_e[\phi] = \text{Euclidean “path measure”} \]

\[ f(\tau, x) = \text{Positive Euclidean-time support test functions} \]

\[ S_+ := \{ f(\tau, x) \in S | f(\tau, x) = 0, \quad \tau < 0 \} \]

Euclidean time reflection

\[ \theta f(\tau, x) := f(-\tau, x) \]
Reconstruction of Quantum Mechanics
Osterwalder and Schrader - C.M.P. 31(1973)83;42(1975)281

Vectors (dense set)

\[ B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \quad \quad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)} \]

\[ b_j, c_k \in \mathbb{C} \quad \quad f_j, g_k \in \mathcal{S}_+ \quad \quad N_b, N_c < \infty \]

Hilbert space inner product

\[ \langle B|C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j] \]
Remarks

\( B[\phi] \) "wave functionals"

\[
\langle B|C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]
\]

- The inner product is the **physical (Minkowski)** inner product!
- The generating functional and test functions are **Euclidean**!
- All integrals are over **Euclidean** space-time variables!
- No analytic continuation is used to calculate the Minkowski scalar product!
Reflection positivity

(Osterwalder-Schrader Positivity)

$$\langle B | B \rangle \geq 0$$

Property of $Z[f]$ or \{ $S_n(x_1, \cdots, x_n)$ \}

$$M_{ij} = Z[f_i - \theta f_j] \geq 0 \quad \forall \quad \{ f_1, \cdots, f_N \} \in S_+$$
Operator algebra

(for scattering asymptotic condition)

\[ BC[\phi] = B[\phi]C[\phi] = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k e^{i\phi(g_k+f_j)} = \sum_{n=1}^{N_c} d_n e^{i\phi(h_n)} \]

\[ N_d = N_b N_c \quad h_n = g_k + f_j \quad d_n = b_j c_k \]
Cluster properties

\[ g_a(\tau, x) := g(\tau, x - a) \]

\[
\lim_{|a| \to \infty} (Z[f + g_a] - Z[f]Z[g]) \to 0
\]

\[
\lim_{|a| \to \infty} S_{m+n}(f, \cdots, f, g_a, \cdots, g_a) = S_m(f, \cdots f)S_n(g, \cdots, g)
\]
Operators

\[ [T(\beta, a), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_j, \beta, a)} \]

\[ f_{n,\beta,a}(\tau, x) := f_n(\tau - \beta, x - a) \quad \beta > 0 \]

\[ [U(R), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_j, R)} \]

\[ f_{j,R}(\tau, x) := f_j(\tau, Rx) \quad f_j \in S_+ > 0 \]

\[ [W(\hat{n}, \psi), B][\phi] := \sum_{j} c_j e^{i\phi(f_k, \psi, \hat{n})} \]

\[ f_{j,\phi,\hat{n}}(\tau, x) := f_j(\tau', x') \quad f_f \in S_{\chi,+} \]

\[ S_{\chi,+} = \{ f \in S_+ | f(\tau, x) = 0 \quad \tan^{-1}(\frac{\tau}{|x|}) \geq \chi \} \]

\[ \psi < \pi / 2 - \chi \]

\[ \tau' = \tau \cos(\psi) - x_{\hat{n}} \sin(\psi) \quad x'_{\hat{n}} = x_{\hat{n}} \cos(\psi) + \tau \sin(\psi) \]
Poincaré generators

\[ [H, B][\phi] = -\frac{\partial}{\partial \beta} \left( T(\beta, 0) B \right) [\phi]_{\beta=0} \]

\[ [P, B][\phi] = -i \frac{\partial}{\partial a} \left( T(0, a) B \right) [\phi]_{a=0} \]

\[ [(K \cdot \hat{n}), B][\phi]) = -\frac{\partial}{\partial \psi} \left( W(\hat{n}, \psi) B \right) [\phi]_{\psi=0} \]

\[ [(J \cdot \hat{n}), B][\phi]) = -i \frac{\partial}{\partial \psi} \left( R(\hat{n}, \psi) B \right) [\phi]_{\psi=0} \]

\[ [M^2, B][\phi] := \left( \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial a^2} \right) \left( T(\beta, a) B \right) [\phi]_{\beta=0, a=0} \]
One parameter groups and semigroups

\[ T(\beta, a) = e^{-\beta H + ia \cdot P} \]

\[ U(R(\hat{n}, \psi)) = e^{iJ \cdot \hat{n} \psi} \]

\[ W(\hat{n}, \psi) = e^{K \cdot \hat{n} \psi} \]
Domains for local symmetric semigroups

\{ H, P, J, K \}

**Self-adjoint (on physical Hilbert space)**

\[ H \geq 0 \quad \text{(Follows from reflection positivity)} \]

**Satisfy Poincaré commutation relations**

**No analytic continuation used!**
Given a reflection positive Euclidean Green function or generating function we have:

- Hilbert space scalar product.
- A dense set of normalizable vectors.
- A representation of the Poincaré Lie algebra in terms of self-adjoint operators.
Comments:

- While analytic continuation is not used, reflection positivity ensures the existence of an analytic continuation.

- We can exploit the ability to calculate matrix elements of all operators in a dense set of normalizable states.

- $e^{-\beta H}$ and $H$ have the same eigenstates.
Particles: mass eigenstates

Dense set + Gram-Schmidt

\[ \downarrow \]

Orthonormal basis of “wave functionals”

\[ B_n[\phi] \quad \langle B_n|B_m \rangle = \delta_{mn} \]

Solve for eigenstates in point spectrum of \( M^2 \)

\[ (M^2 B_\lambda)[\phi] = \lambda^2 B_\lambda[\phi] \]

\[ B_\lambda[\phi] = \sum_n b_n B_n[\phi] \]

\[ \sum_n \langle B_m|M^2|B_n \rangle b_n = \lambda^2 b_m \]
Particles: mass-momentum eigenstates
(use translations and Fourier transforms)

\[ B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)} \quad \text{(mass eigenfunctional)} \]

\[ B_\lambda(p)[\phi] = \int \frac{d^3a}{(2\pi)^{3/2}} e^{-ip\cdot a} [T(0, a), B_\lambda][\phi] \]

\[ \langle C | B_\lambda(p) \rangle := \int \frac{d^3a}{(2\pi)^{3/2}} e^{-ip\cdot a} \sum_{j=1}^{N_c} \sum_n b_n c_j^* Z[f_n,a - \theta g_j] \]
Particles: Mass-momentum-spin eigenstates
(project on $SU(2)$ irreducible representations)

Normalize $B_\lambda(p)$

$$B_\lambda(p)[\phi] \quad \langle B_\lambda(p')|B_\lambda(p)\rangle = \delta(p' - p)$$

$$B_{\lambda,j}(p, \mu)[\phi] := \int_{SU(2)} dR[U(R), B_\lambda(R^{-1}p)]\phi]D^{j*}_{\mu j}(R) \quad \langle C|B_{\lambda,j}(p, \mu)\rangle :=$$

$$\int \frac{d^3 a dR}{(2\pi)^{3/2}} e^{-iR^{-1}p \cdot a} \sum_{j=1}^{N_c} \sum_n c_j^{*} b_n Z[f_{n,a,R} - \theta g_j]D^{j*}_{\mu j}(R)$$

$$f_{n,a,R}(\tau, x) = f_n(\tau, Rx - a)$$
Finite Poincaré transformations of one-particle states \((\lambda \in \sigma_{pp})\)

One-particle subspaces are irreducible subspaces with respect to the Poincaré group

\[ \downarrow \]

\[ \langle C | U[\Lambda, a] | B_\lambda, j(p, \mu) \rangle = \]

\[ \sum_{\mu' = -j}^j \int dp' \langle C | B_\lambda, j(p', \mu') \rangle D_{\rho' \mu', \rho \mu}^\lambda j [\Lambda, a] \]

\[ D_{\rho' \mu', \rho \mu}^\lambda j [\Lambda, a] = \]

\[ \delta(\Lambda \rho - p') \sqrt{\omega_\lambda (p')} e^{-i \omega_\lambda (p') a^0 - ip' \cdot a} D_{\mu' \mu}^j [\Lambda_c^{-1}(\frac{p'}{\lambda}) \Lambda \Lambda_c(\frac{p}{\lambda})] \]

\[ \omega_\lambda (p) = \sqrt{\lambda^2 + p \cdot p} \quad (p')^j = \Lambda^j_0 \omega_\lambda (p) + \Lambda^j_k p^k \]

Here \(\Lambda_c(\frac{p}{\lambda})\) is a rotationless Lorentz boost.
Scattering

- Time-dependent scattering has been used successfully to treat few-body problems (Kröger, Phys. Reports, 210(1992)46) in non-relativistic quantum mechanics.

- Calculations use wave packets and normalizable states.

- Haag-Ruelle scattering is a natural field theoretic generalization of non-relativistic time-dependent scattering theory (unlike LSZ it uses strong limits).
Construct

\[ J : \otimes \mathcal{H}_{\lambda_i j_i} = \mathcal{H}_f \to \mathcal{H} \quad U_f[\Lambda, a] = \otimes U_{\lambda_i j_i}[\Lambda, a] \]

The strong limit exists

\[ |\psi_\pm \rangle = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_f t} |\psi_{f \pm} \rangle = \Omega_\pm |\psi_{f \pm} \rangle \]

The wave operators satisfy (Ruelle H.P.A. 35(1962)34)

\[ U[\Lambda, a]\Omega_\pm = \Omega_\pm U_f[\Lambda, a] \]
Structure of $J$

$$B_{\lambda,j}(p, \mu)|0\rangle = |(\lambda, j)p, \mu\rangle$$

Creates one-particle state out of the vacuum

$$\int J_i(p_i, \mu_i)f(p, \mu) dp =$$

$$\int \left( -i\omega_{\lambda}(p)B_{\lambda,j}(p, \mu) - i[H, B_{\lambda,j}(p, \mu)] \right)f(p, \mu) dp$$

(\cdots) selects “creation part” of $B_{\lambda,j}(p, \mu)$

$$J(p_1, \mu_1, \cdots, p_n, \mu_n) = \prod J_i(p_i, \mu_i)|0\rangle$$
Wave functional representation

\[ B_{\lambda,j}(p, \mu) \rightarrow B_{\lambda,j}(p, \mu)[\phi] \]

\[ \int J_i(p_i, \mu_i)[\phi] f(p, \mu) dp = \]

\[ \int (-i\omega_{\lambda}(p)B_{\lambda,j}(p, \mu)[\phi] - i[H, B_{\lambda,j}(p, \mu)][\phi]) f(p, \mu) dp \]

\[ J[\phi] = \prod J_i(p_i, \mu_i)[\phi] \]
Two-space Scattering

\[
\lim_{t \to \pm \infty} \left\| (e^{-iHt} |\psi_\pm\rangle - J e^{-iH_f t} |\psi_f \pm\rangle) \right\| = 0
\]

\[
|\psi_\pm\rangle = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_f t} |\psi_f \pm\rangle = \Omega_\pm (H, J, H_f) |\psi_f\rangle
\]

\[
S_{fi} = \langle \psi_+ | \psi_- \rangle = \langle \psi_f + | \Omega_+^{\dagger} (H, J, H_f) \Omega_- (H, J, H_f) |\psi_f -\rangle
\]

\[
= \lim_{t \to \infty} \langle \psi_f + | e^{iH_f t} J^{\dagger} e^{-2iHt} J e^{iH_f t} |\psi_f -\rangle
\]
Kato-Birman invariance principle

\[
\lim_{t \to \pm\infty} \| (e^{-iHt} |\Psi\pm\rangle - Je^{-iHf} t |\Psi_f\pm\rangle) \| = 0
\]

\[\Downarrow\]

\[
\lim_{n \to \pm\infty} \| (e^{ine^{-\beta H}} |\Psi\pm\rangle - Je^{ine^{-\beta Hf}} |\Psi_f\pm\rangle) \| = 0
\]

Provides a possible computational strategy

\[
\langle \Psi_f | S | \Psi_f \rangle \approx \langle \Psi_f | e^{-ine^{-\beta Hf}} J^\dagger e^{2ine^{-\beta H}} Je^{-ine^{-\beta Hf}} | \Psi_f \rangle
\]

\[
e^{2inx} \approx \sum_{m=0}^{N(n)} c_m(n) x^m \quad \rightarrow \quad e^{2ine^{-\beta H}} \approx \sum_{m=0}^{N(n)} c_m(n) e^{-m\beta H}
\]

Convergence is \textbf{uniform} for each fixed \( n \)!
Matrix elements

\[ \langle B | e^{-m\beta H} | C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,m\beta,0} - \theta f_j] \]

\[ C[\phi] = J e^{-ine^{-\beta H_f}} |\Psi_f\rangle[\phi] \]

\[ B[\phi] = J e^{ine^{-\beta H_f}} |\Psi_f\rangle[\phi] \]

Computable by quadrature in terms of \( Z[f] \) or \( \{S_n\} \)
Scattering in Euclidean space

- The results are standard Minkowski space results expressed in a representation where the Minkowski scalar product is evaluated in terms of a Euclidean generating functional.

- The time limits in the scattering theory are strong limits (compared with the weak limits used in LSZ scattering).

- The Haag-Ruelle scattering theory does not distinguish elementary and composite asymptotic states.

- Explicit representations of the Poincaré group exist for the bound and scattering states.

\[ U[\Lambda, a] \Omega_\pm = \Omega_\pm U_f[\Lambda, a] \]
Maiani-Testa No-Go Theorem (P.L.B. 245(1990)585)

\[ \langle 0|\phi_\pi(p_1)\phi_\pi(p_2)J(0)|0 \rangle \]

- Use LSZ interpolating fields for pions. Field creates more than 1-pion states from vacuum.
- Uses \( \beta_1 \gg \beta_2 \gg 0 \)
- Uses Euclidean correlation functions.

This approach

- Uses Haag-Ruelle fields. Fields create only 1 pion states from vacuum - products approach exact scattering states in strong limit.
- \( \beta \) is a fixed adjustable parameter \( (H \leftrightarrow e^{-\beta H}) \).
- Uses Minkowski scalar product.
- Calculations require wave packets, one-body solutions; no singularities, no analytic continuation.
Summary of formal results
Given a reflection positive generating functional

\[ \Downarrow \]

- Hilbert-space scalar product, \( \{ H, P, J, K \} \)
- Single-particle states
- Scattering states, \( S \)-matrix elements.
- Finite Poincaré transformations on single-particle states and scattering states.
Comments

- Constructing a reflection positive Euclidean invariant generating functional is almost equivalent to constructing a non-trivial field theory (this must be relaxed for model applications).

- Practical calculations use a weakened form of reflection positivity (limited permutation symmetry).

- Full permutation symmetry = locality.

- Osterwalder-Schrader reconstruction of relativistic quantum mechanics does not require locality.
Green function approach - limited reflection positivity

\[ Z[f] = \sum_n \frac{i^n}{n!} S_n(f, \cdots, f) \]

\[ x := (\tau, x) \quad \theta x = (-\tau, x) \quad f(x_1, \cdots, x_n) \in S_+ \]

\[ \int d^4x_1 \cdots d^4x_4 f^*_2(\theta x_2, \theta x_1) S_4(x_1, x_2; x_3, x_4) f_2(x_3, x_4) \geq 0 \]

\[ \int d^4x_1 d^4x_2 f^*_1(\theta x_1) S_2(x_1; x_2) f_1(x_2) \geq 0. \]

\[ S_4(x_1, x_2; x_3, x_4) = S_4(x_2, x_1; x_3, x_4) = S_4(x_1, x_2; x_4, x_3) \]
Green function representation

\[ Z \rightarrow S = \begin{pmatrix}
S_2(x; y) & S_3(x; y_1, y_2) & \cdots \\
S_3(x_1, x_2; y) & S_4(x_1, x_2; y_1, y_2) & \vdots \\
\vdots & \vdots & \ddots
\end{pmatrix} \]

\[ B[\phi] \rightarrow \begin{pmatrix}
f_1(x_{11}) \\
f_2(x_{21}, x_{22}) \\
\vdots
\end{pmatrix} \]

\[ \langle B|C \rangle = (\theta f_B, Sf_C)_e \]
\(\langle x|H|f \rangle := \{0, \frac{\partial}{\partial x_{11}^0} f_1(x_{11}), \left( \frac{\partial}{\partial x_{21}^0} + \frac{\partial}{\partial x_{22}^0} \right) f_2(x_{21}, x_{22}), \cdots \}\)

\(\langle x|P|f \rangle := \{0, -i \frac{\partial}{\partial x_{11}} f_1(x_{11}), -i \left( \frac{\partial}{\partial \vec{x}_{21}} + \frac{\partial}{\partial \vec{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \}\)

\(\langle x|J|f \rangle := \{0, -i \vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} f_1(x_{11}), -i \left( \vec{x}_{21} \times \frac{\partial}{\partial \vec{x}_{21}} + \vec{x}_{22} \times \frac{\partial}{\partial \vec{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \}\)

\(\langle x|K|f \rangle := \{0, \left( \vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11} \frac{\partial}{\partial \vec{x}_{11}} \right) f_1(x_{11}), \left( \vec{x}_{21} \frac{\partial}{\partial x_{21}^0} - x_{21} \frac{\partial}{\partial \vec{x}_{21}} + \vec{x}_{22} \frac{\partial}{\partial x_{22}^0} - x_{22} \frac{\partial}{\partial \vec{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \}\).
Modifications for spin

\[ J : \left( -i \vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left( -i \vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} + \vec{\Sigma} \right) \]

\[ K : \left( \frac{\vec{x}_{11}}{\partial x^0_{11}} - x^0_{11} \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left( \frac{\vec{x}_{11}}{\partial x^0_{11}} - x^0_{11} \frac{\partial}{\partial \vec{x}_{11}} + \vec{B} \right) \]

where

\[ \vec{\Sigma} = i \vec{\nabla}_\phi D(e^{-i \frac{\vec{\sigma}}{2} \cdot \vec{\phi}}, e^{i \frac{\vec{\tau}}{2} \cdot \vec{\phi}})_{aa'} \]

and

\[ \vec{B} = \vec{\nabla}_\rho D(e^{-i \frac{\vec{\sigma}}{2} \cdot \vec{\rho}}, e^{i \frac{\vec{\tau}}{2} \cdot \vec{\rho}})_{aa'} \]

and \( D(g_1, g_2) \) is a representation of \( SU(2) \times SU(2) \).
Two-body scattering

\[ S_2(x; y) \quad K(x_1, x_2; y_2, y_1) \]

\[ S_0 = S_2(x_1; y_1)S_2(x_2; y_2) \]

\[ S_4 = S_0 + S_0KS_4 \]

\[ S = \begin{pmatrix} S_2(x; y) & 0 \\ 0 & S_4(x_1, x_2; y_1, y_2) \end{pmatrix} \]

\[ \langle C|B \rangle = (\theta g_1, S_2f_1)_e + (\theta g_2, S_4f_2)_e \]
Particles - scalar
(case of free $S_2$)

\[(f, \theta S_2 f)_e\]

\[= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{ip \cdot (\theta x - y)}}{p^2 + m^2} f(y)\]

\[= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{-ip_0 \cdot (x_0 + y_0) + i\vec{p} \cdot (\vec{x} - \vec{y})}}{(p^0 + i\omega_m(\vec{p}))(p^0 - i\omega_m(\vec{p}))} f(y)\]

\[= \int d^3p \frac{|g(\vec{p})|^2}{2\omega_m(\vec{p})} \geq 0\]

where

\[g(\vec{p}) := \frac{1}{(2\pi)^{3/2}} \int d^4y f(y) e^{-\omega_m(\vec{p})y_0 - i\vec{p} \cdot \vec{y}}.\]
Particles - fermions

(Euclidean time reversal has a spinor component)

\[(f, \theta\gamma^0 S_2 f)_e\]

\[
= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) e^{ip \cdot (\theta x - y)} \gamma^0 \frac{m - p \cdot \gamma^e}{p^2 + m^2} f(y)
\]

\[
= \int g^\dagger(\vec{p}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{p}) d^3p
\]

where

\[
\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0}{2\omega_m(\vec{p})}
\]
Scattering in Euclidean space

**Approximation 1:** Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.

\[
\langle \psi_{f+} | S \psi_{f-} \rangle = \langle \psi_{f+} | \psi_{f-} \rangle - 2\pi i \langle \psi_{f+} | \delta(E_+ - E_-) T | \psi_{f-} \rangle
\]

\[
\langle p'_1, \mu'_1, p'_2, \mu'_2 | T | p_1, \mu_1, p_2, \mu_2 \rangle \approx \frac{\langle \psi_{f+} | S | \psi_{f-} \rangle - \delta_{ab} \langle \psi_{f+} | \psi_{f-} \rangle}{2\pi i \langle \psi_{f+} | \delta(E_+ - E_-) | \psi_{f-} \rangle}
\]
Scattering injection operators (N=2)

**Approximation 2:** Calculate $\psi_\lambda(x, \tau; p, \mu)$

- $\psi_\lambda(x, \tau; p, \mu)$: eigenstate of $M^2, P, j^2, j_z$ in one-body Hilbert space generated by $S_2$.

$$J_i(x, \tau; p, \mu) := (-i\omega_\lambda(p) + i\frac{\partial}{\partial \tau})\psi_\lambda(x, \tau; p, \mu)$$

$$J(x_1, \tau_1, x_2, \tau_2; p_1, \mu_1, p_2, \mu_2) = J_1(x_1, \tau_1; p_1, \mu_1) J_2(x_2, \tau_2; p_2, \mu_2)$$
Scattering in Euclidean space

Use time-dependent scattering to calculate $S$ matrix elements in normalizable states.

Use Kato-Birman invariance principle to express $S$ in terms of $e^{-\beta H}$.

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle$$

$$= \lim_{t \to \infty} \langle \Psi_{f+} | e^{iH_f t} J^\dagger e^{-2iH t} e^{iH_f t} J | \Psi_{f-} \rangle$$

$$= \lim_{n \to \infty} \langle \Psi_{f+} | e^{-in e^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$
Scattering in Euclidean space

**Approximation 3:** Replace $n$ by large fixed $n$.

\[
\langle \Psi_{f+} | S | \Psi_{f-} \rangle \\
\approx \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} Je^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle
\]
Approximation 4: Uniform polynomial approximation

\[ e^{2in\varepsilon - \beta H} \approx \sum c_m(n)(e^{-\beta mH}) \]

**Note**: \( \sigma(e^{-\beta H}) \in [0,1] \) (compact)

\[ e^{2in\varepsilon x} \approx \sum c_m(n)x^m \quad x \rightarrow e^{-\beta H} \]

\[ |e^{2in\varepsilon x} - \sum c_m(n)x^m| < \epsilon(n) \quad \forall x \in [0,1] \]

\[ \Downarrow \]

\[ \| [e^{2in\varepsilon - \beta H} - \sum c_m(n)(e^{-\beta mH})] \psi \| < \epsilon(n)\|\psi\| \]
\[ f(x) \approx \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^{N} c_k T_k(x) \]

\[ c_j = \frac{2}{N + 1} \sum_{k=1}^{N} f(\cos(\frac{2k - 1}{N + 1} \pi)) \cos(j \frac{2k - 1}{N + 1} \pi) \]

\[ f(e^{-\beta H}) \approx \frac{1}{2} c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H}) \]

\[ f(x) = e^{2inx} \]

\[ |e^{2inx} - P_N(x)| < 2 \frac{n^{N+1}}{(N + 1)!} \]
\[ \langle \psi_{f+} | S | \psi_{f-} \rangle \approx \]

\[ = \sum c_m(n) \langle \psi_{f+} | e^{-ine^{-\beta H_f}} J \dagger (e^{-\beta mH}) J e^{-ine^{-\beta H_f}} | \psi_{f-} \rangle \]

Each approximation converges - the order of the approximations is important (1) → (2) → (3) → (4).
Three and four-body phenomenology and cluster properties

\[
S^{-1}(123) = \\
S^{-1}(1)S^{-1}(2)S^{-1}(3) - K(12)S^{-1}(3) - \\
K(23)S^{-1}(1) - K(31)S^{-1}(2) - K(123)
\]

\[
S^{-1}(1234) = \\
S^{-1}(1)S^{-1}(2)S^{-1}(3)S^{-1}(4) + K(12)K(34) + K(13)K(24) + \\
K(14)K(23) - K(12)S^{-1}(3)S^{-1}(4) - K(13)S^{-1}(2)S^{-1}(4) + \\
-K(14)S^{-1}(2)S^{-1}(3) - K(23)S^{-1}(1)S^{-1}(4) + \\
-K(24)S^{-1}(1)S^{-1}(3) - K(34)S^{-1}(1)S^{-1}(2) + \cdots
\]
Test of method: non-relativistic separable potential
(solvable so all approximations can be tested)

\[ H = \frac{k^2}{m} - \langle g | \lambda | g \rangle \]

\[ \langle k | g \rangle = \frac{1}{m^2 + k^2} \]

calculate \( \langle k' | T(k^+) | k \rangle \) using matrix elements of \( e^{-\beta H} \) in normalizable states.
Approximation 3:

Imaginary $S$ vs n-limit - 1 GeV
Approximation 3:

Real $S-1$ vs n-limit - 1 GeV

![Graph showing the real part of $S-1$ vs n-limit for 1 GeV. The x-axis represents "time steps" n, ranging from 0 to 500, while the y-axis represents the real part of $S-1$, ranging from -0.005 to 0.005. The graph shows multiple curves that converge to a horizontal line at 0 as n increases.]
Approximation 3:

Im S vs n-limit - 2.1 GeV
Approximation 4:

Degree 300 polynomial compared to $e^{-inx}$, $n = 220$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\Delta \cos(nx)$</th>
<th>$\Delta \sin(nx)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4.44089 \times 10^{-16}$</td>
<td>$8.32667 \times 10^{-15}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$2.35367 \times 10^{-14}$</td>
<td>$1.46966 \times 10^{-14}$</td>
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<tr>
<td>0.2</td>
<td>$5.55112 \times 10^{-16}$</td>
<td>$3.6797 \times 10^{-14}$</td>
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<tr>
<td>0.3</td>
<td>$3.84137 \times 10^{-14}$</td>
<td>$1.80689 \times 10^{-14}$</td>
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<td>0.4</td>
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<td>$1.32672 \times 10^{-14}$</td>
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<td>0.5</td>
<td>$2.77556 \times 10^{-15}$</td>
<td>$2.93793 \times 10^{-14}$</td>
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<td>0.6</td>
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<td>0.9</td>
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</tr>
<tr>
<td>1</td>
<td>$4.88498 \times 10^{-15}$</td>
<td>$6.61415 \times 10^{-14}$</td>
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</table>
Real part of $\langle k | T(k) | k \rangle$ (exact - black, polynomial - red)
Im part of $\langle k|\mathbf{T}(k)|k\rangle$ (exact - black, polynomial - red)
Conclusions - Outlook

- Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.
- Analytic continuation is not necessary.
- The Poincaré invariant S-matrix. Cluster properties are easily satisfied for fixed $N$.
- Models can be motivated by field-theory based phenomenology.
- A test using an exactly solvable model suggests that GeV scale scattering calculations are possible in this framework.
Future directions

• Euclidean BS free $S_2$.

• Euclidean BS $S_2$ with continuous Lehmann weight.

• Nakanishi representation and reflection positivity.

• Current matrix elements.
Thanks!

- Workshop organizers
- Nuclear Theory Center
- U.S. D.O.E. Office of Science