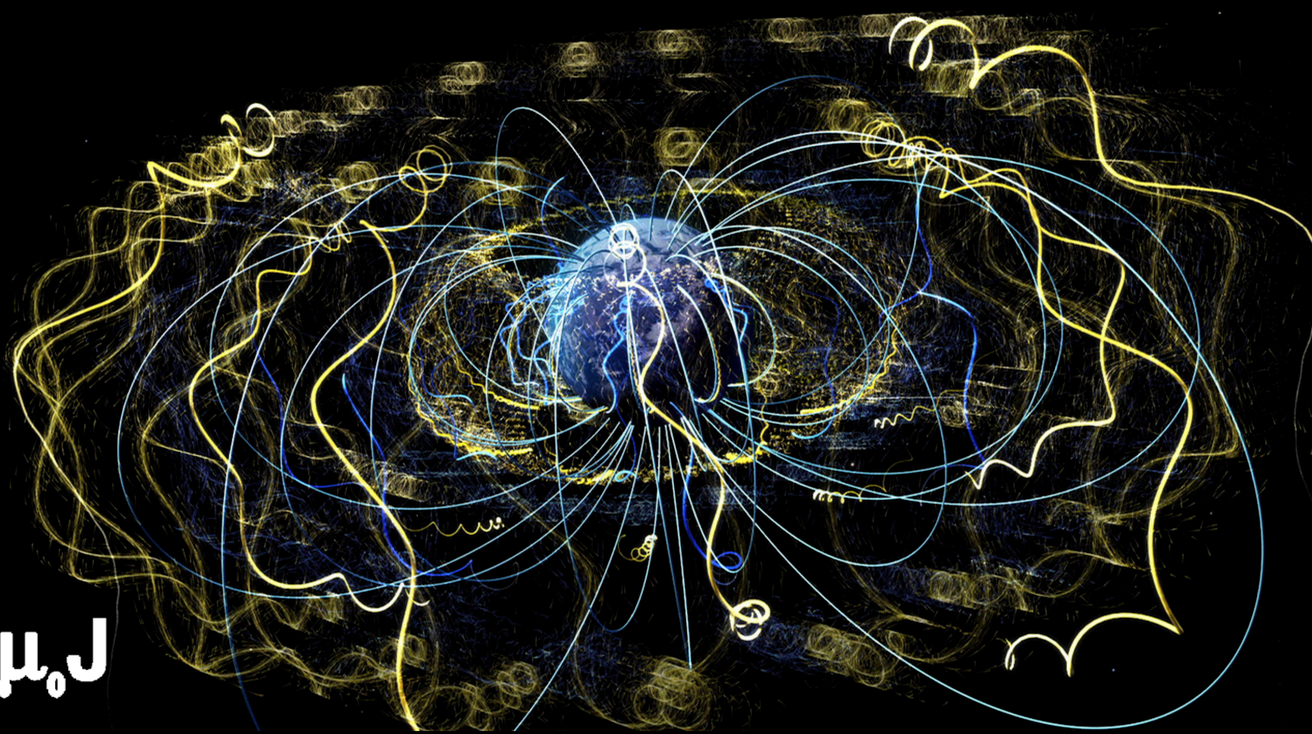
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Laplace's Equation in 2-D

$$V(x,y) = \frac{1}{2\pi R} \oint V \, dl$$

around a circle centered @ (x,y)

V has no local extrema (except at boundaries)

Laplace's Equation in 3-D

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint V \, da$$

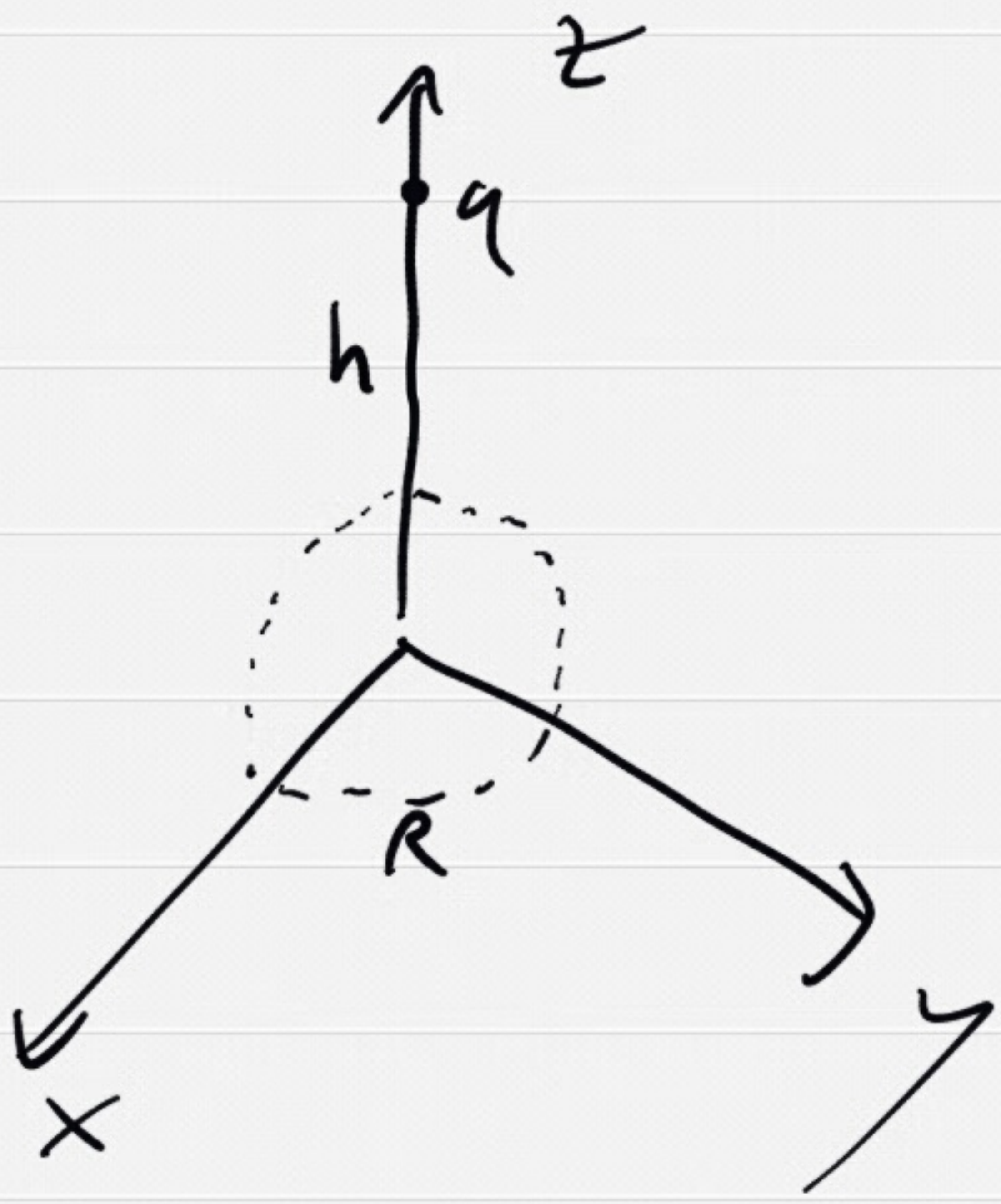
over a sphere centered at $\vec{r}$

V has no local extrema (except at boundaries)

Earnshaw's Theorem

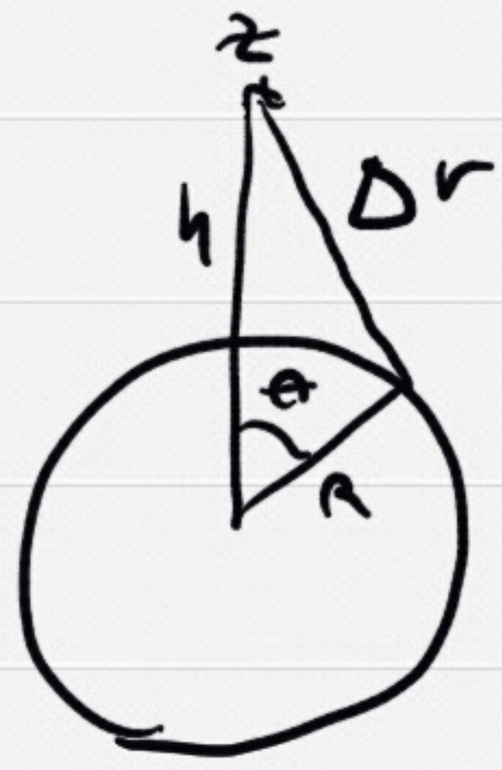
No local extrema

$\Rightarrow$ no stable trapping w/ electrostatic fields



Prove for point charge, then use superposition to extend to all cases

$$V = \frac{1}{4\pi\varepsilon_0} \cdot q/\Delta r$$



$$\Delta r^2 = h^2 + R^2 - 2Rh\cos\theta$$
(law of cosines)

$$\langle V \rangle = \frac{1}{4\pi R^2} \cdot \frac{q}{4\pi\varepsilon_0} \int (h^2 + R^2 - 2hR\cos\theta)^{-\frac{1}{2}} R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi R^2} \cdot \frac{q}{4\pi\varepsilon_0} \cdot \int_0^{2\pi} \int_0^{\pi} \frac{R^2}{\sqrt{(h^2 + R^2 - 2hR\cos\theta)}} \sin\theta \, d\theta \, d\phi$$

$$= \frac{2\pi R^2}{4\pi R^2} \cdot \frac{q}{4\pi\varepsilon_0} \int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{h^2 + R^2 - 2hR\cos\theta}}$$

$$= \frac{1}{2} \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{hR} \cdot \sqrt{h^2 + R^2 - 2hR\cos\theta} \Big|_0^{\pi}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{2hR} \left[\sqrt{h^2 + R^2 + 2hR} - \sqrt{h^2 + R^2 - 2hR} \right]$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{2hR} \left[(h + R) - (h - R) \right]$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0 h} = V(0,0,0)} \; //$$

Boundary Conditions & Uniqueness Theorems

① Solution to Laplace's Eq. fully determined by V on boundary

Proof: Assume two solutions V_1, V_2

$$\nabla^2 V_1 = 0 \quad , \quad \nabla^2 V_2 = 0$$

Consider $V_3 = V_1 - V_2$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

- But $V_3 = 0$ on all boundaries - since V on boundary fixed
- Since all extrema must be on boundary, $V_3 = 0$ everywhere

$$\Rightarrow V_1 = V_2$$

$\Rightarrow$ only one solution //

Corollary to 1

- $\rho(\vec{r})$ + V on boundary fully specifies $V(\vec{r})$

$$\nabla^2 V_1 = -\rho/\varepsilon_0$$
$$\nabla^2 V_2 = -\rho/\varepsilon_0$$

$$V_3 = V_1 - V_2$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2$$
$$= -\rho/\varepsilon_0 + \rho/\varepsilon_0$$
$$= 0$$

$V_3 = 0$ on boundary

$\Rightarrow$ $V_3 = 0$ everywhere

$\Rightarrow$ $V_1 = V_2$

$\Rightarrow$ only one solution

② In region surrounded by conductors, $\vec{E}$ uniquely determined if Q_i on each conductor specified

Proof: suppose two solutions $\vec{E}_1$, $\vec{E}_2$

$$\nabla \cdot \vec{E}_1 = \rho/\varepsilon_0$$
$$\nabla \cdot \vec{E}_2 = \rho/\varepsilon_0$$

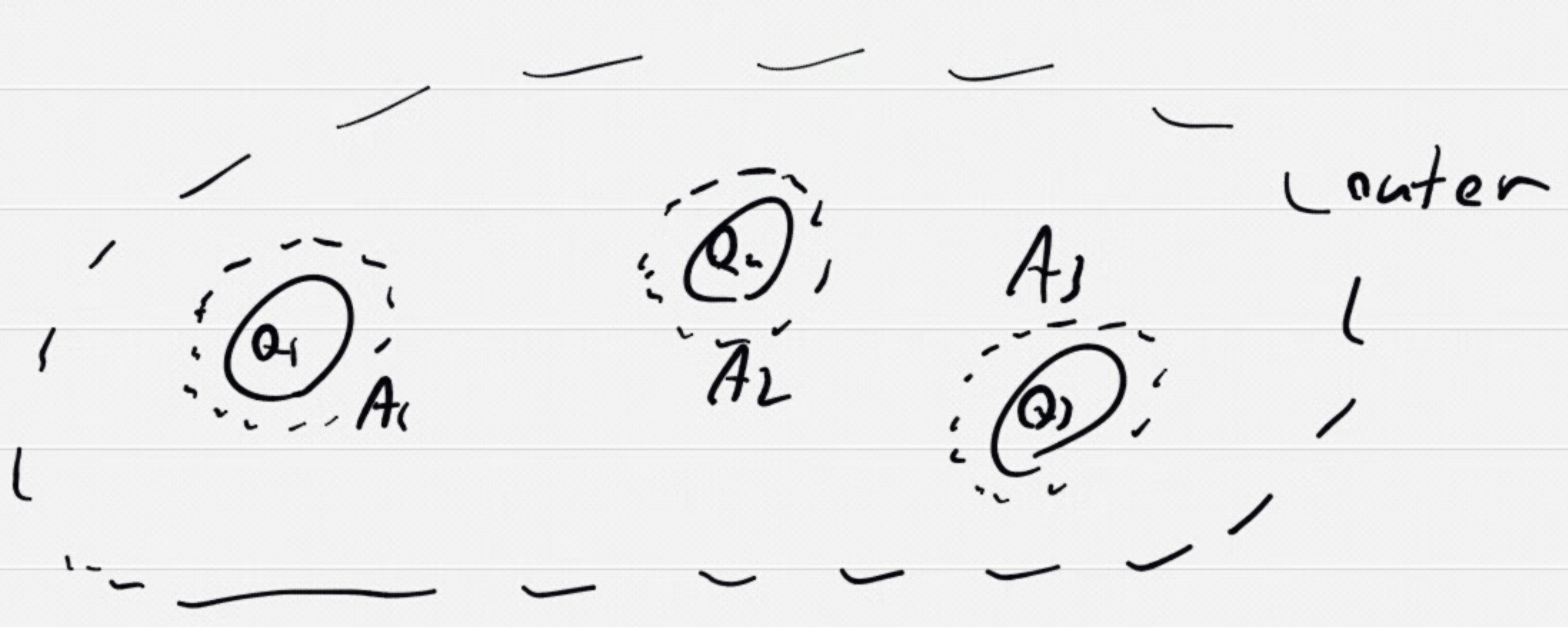
$$\oint_{A_i} \vec{E}_1 \cdot d\vec{a} = Q_i/\varepsilon_0, \quad \oint_{A_i} \vec{E}_2 \cdot d\vec{a} = Q_i/\varepsilon_0$$

w/ A_i a Gaussian surface surrounding conductor i

Also
$$\oint_{outer} \vec{E}_1 \cdot d\vec{a} = Q_{tot}/\varepsilon_0$$
$$\oint_{outer} \vec{E}_2 \cdot d\vec{a} = Q_{tot}/\varepsilon_0$$

Put $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\nabla \cdot \vec{E}_3 = 0$$
$$\oint \vec{E}_3 \cdot d\vec{a} = 0 \quad \text{over each surface}$$



- Each conductor is equipotential

$\Rightarrow V_3 = K_i$ on surface of conductor i

Trick: $\nabla \cdot (f\vec{A}) = f\nabla\cdot\vec{A} + \vec{A}\cdot\nabla f$

$$\Rightarrow \nabla\cdot(V_3\vec{E}_3) = V_3\nabla\cdot\vec{E}_3 + \vec{E}_3\cdot\nabla V_3$$
$$= \vec{E}_3\cdot -\vec{E}_3$$
$$= -E_3^2$$

$$\Rightarrow \int_V \nabla\cdot(V_3\vec{E}_3)\,d\tau = \oint_S V_3\vec{E}_3\cdot d\vec{a} = -\int_V E_3^2\,d\tau$$

$$\oint_S V_3\vec{E}_3\cdot d\vec{a} = \sum_{\text{boundaries}} \oint K_i\vec{E}_3\cdot d\vec{a} = 0$$

$$\Rightarrow \int_V E_3^2\,d\tau = 0$$

since $E_3^2 \geq 0$

$\Rightarrow E_3^2 = 0$ everywhere

$\Rightarrow \vec{E}_3 = 0$

$\Rightarrow \vec{E}_1 = \vec{E}_2$

$\Rightarrow$ only one solution //