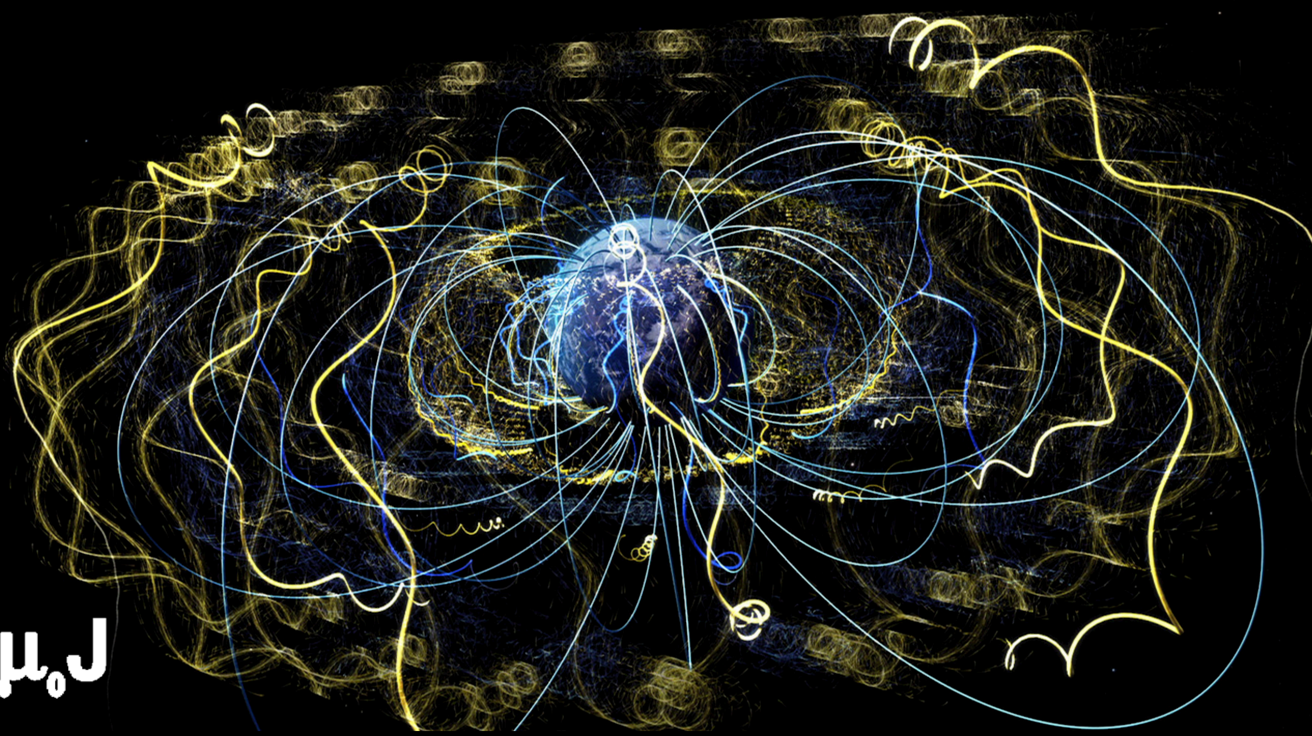
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301

MWF 9:30-10:20 Lecture

Sample Problem

$$\nabla(\Delta r) = ?$$

$$\Delta r = |\Delta \vec{r}|$$

$$= \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

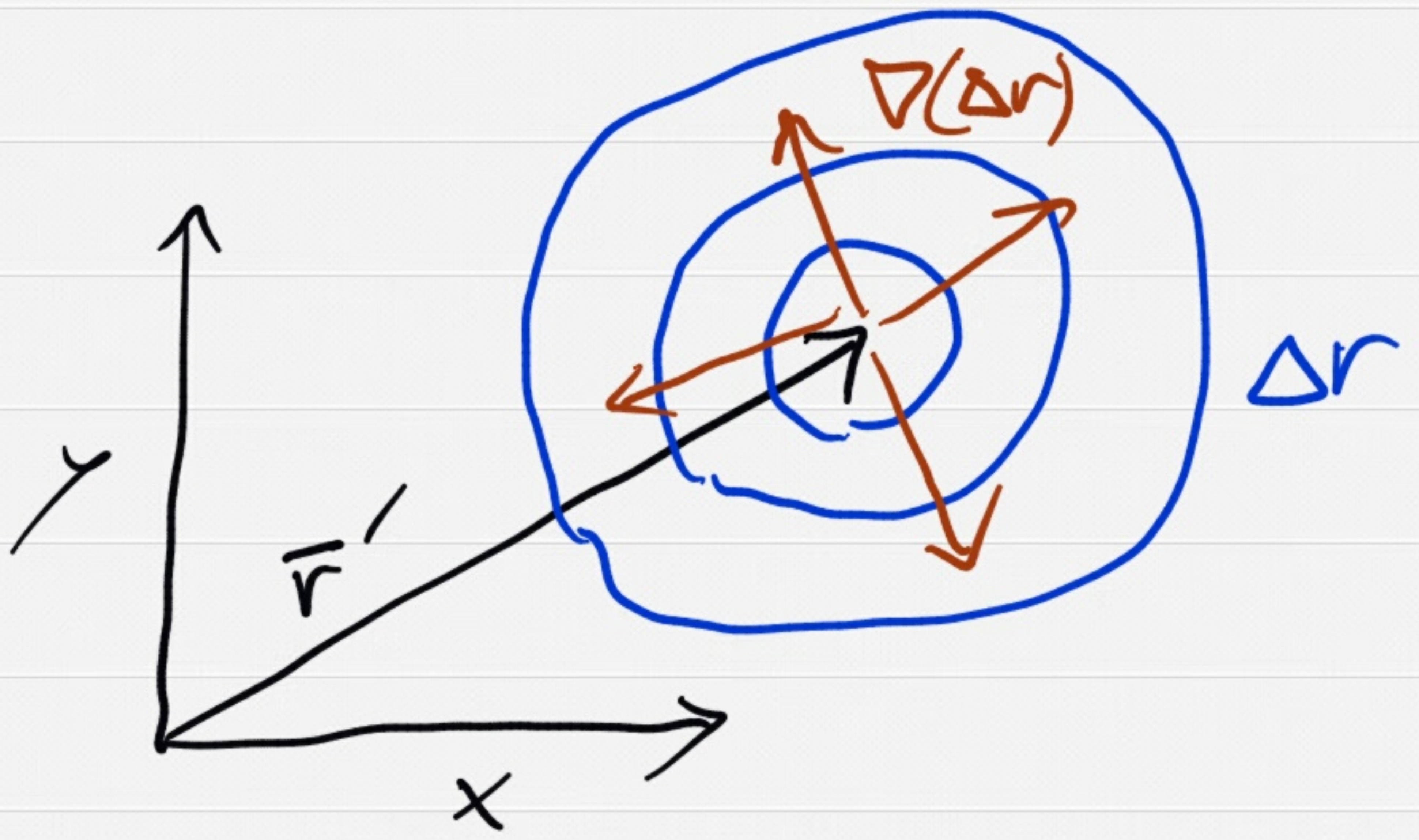
$$= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\partial \Delta r / \partial x = \frac{1/2 \cdot 2(x-x')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \Delta x / \Delta r$$

$$\nabla(\Delta r) = \frac{\Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}}{\Delta r}$$

$$= \frac{\Delta \vec{r}}{\Delta r} = \Delta \hat{r}$$



∇ operator

$$\nabla f = \partial f/\partial x \,\hat{x} + \partial f/\partial y \,\hat{y} + \partial f/\partial z \,\hat{z}$$

looks like $[\partial/\partial x, \partial/\partial y, \partial/\partial z]\, f$

looks like vector × scalar

$$\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$$

looks like a vector

Actually ∇ is an "operator"

∇f makes sense, $f\nabla$ does not

What about ∇ applied to vectors?

$$\nabla \cdot \vec{A} = [\partial/\partial x, \partial/\partial y, \partial/\partial z] \cdot [A_x, A_y, A_z]$$

$$= \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z$$

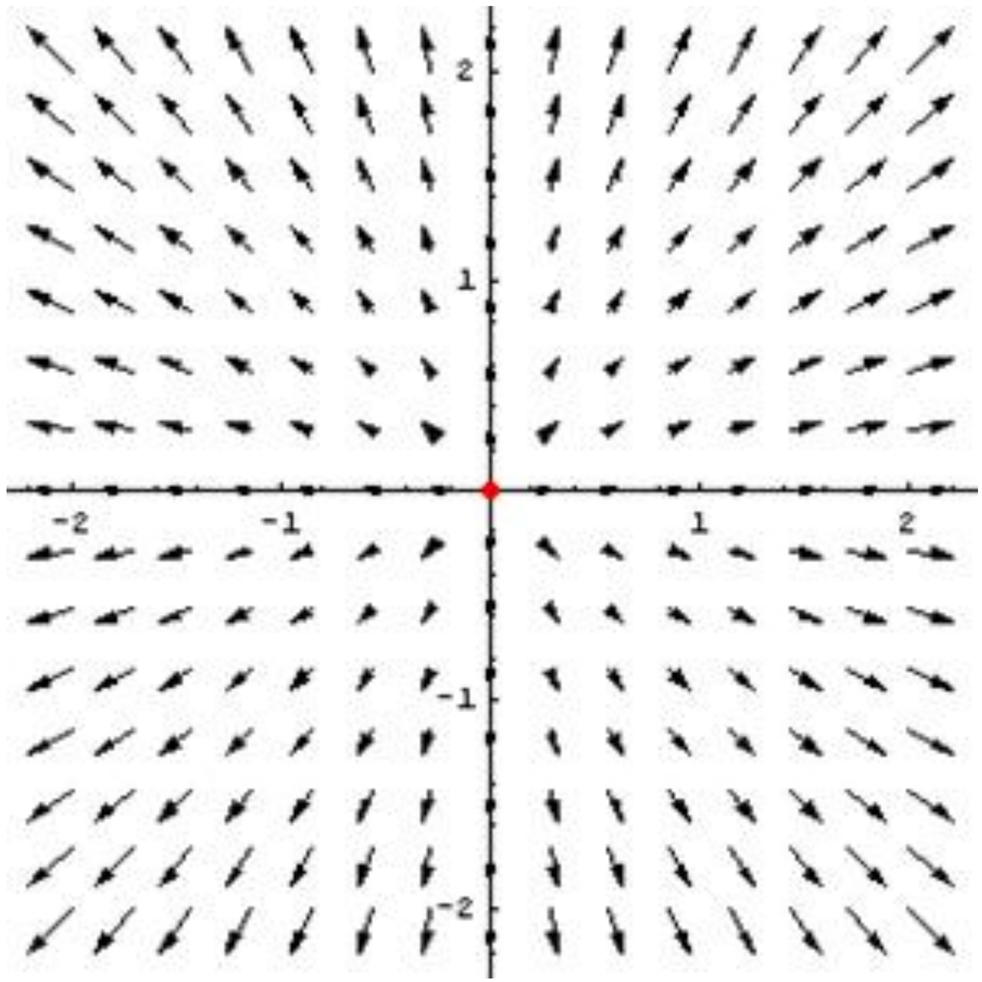
= Divergence of $\vec{A}$

Derivative parallel = flux outward

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$ = Curl of $\vec{A}$

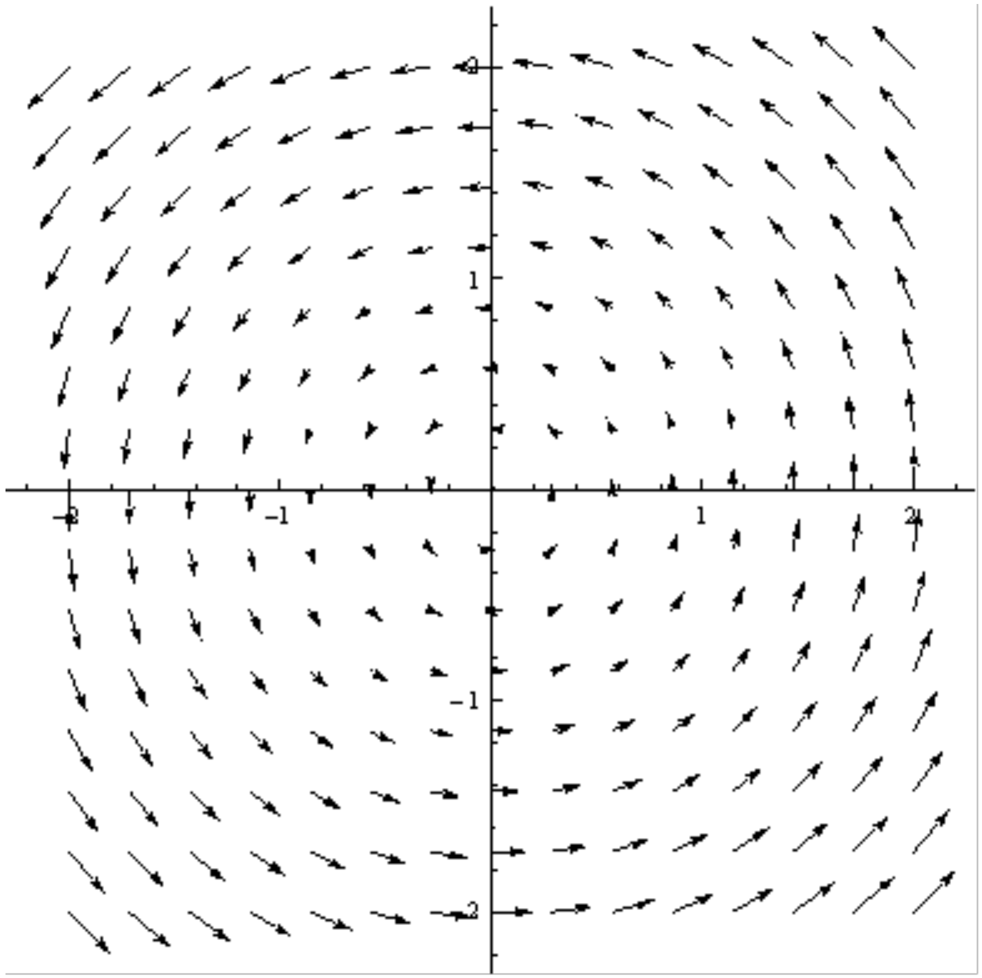
Derivative perpendicular = circulation

Example Vector Field



A(x,y,z) = [x,y, 0]

Example Vector Field



A(x,y,z) = [-y,x, 0]

Sample Fields

$$\vec{A}(x,y,z) = [x, y, 0]$$

$$\nabla \cdot \vec{A} = \partial x/\partial x + \partial y/\partial y + \partial/\partial z \, 0$$

$$= \boxed{2}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix}$$

$$= \boxed{0}$$

$$\vec{A}(x,y,z) = [-y, x, 0]$$

$$\nabla \cdot \vec{A} = \partial/\partial x(-y) + \partial/\partial y(x) + \partial/\partial z(0)$$

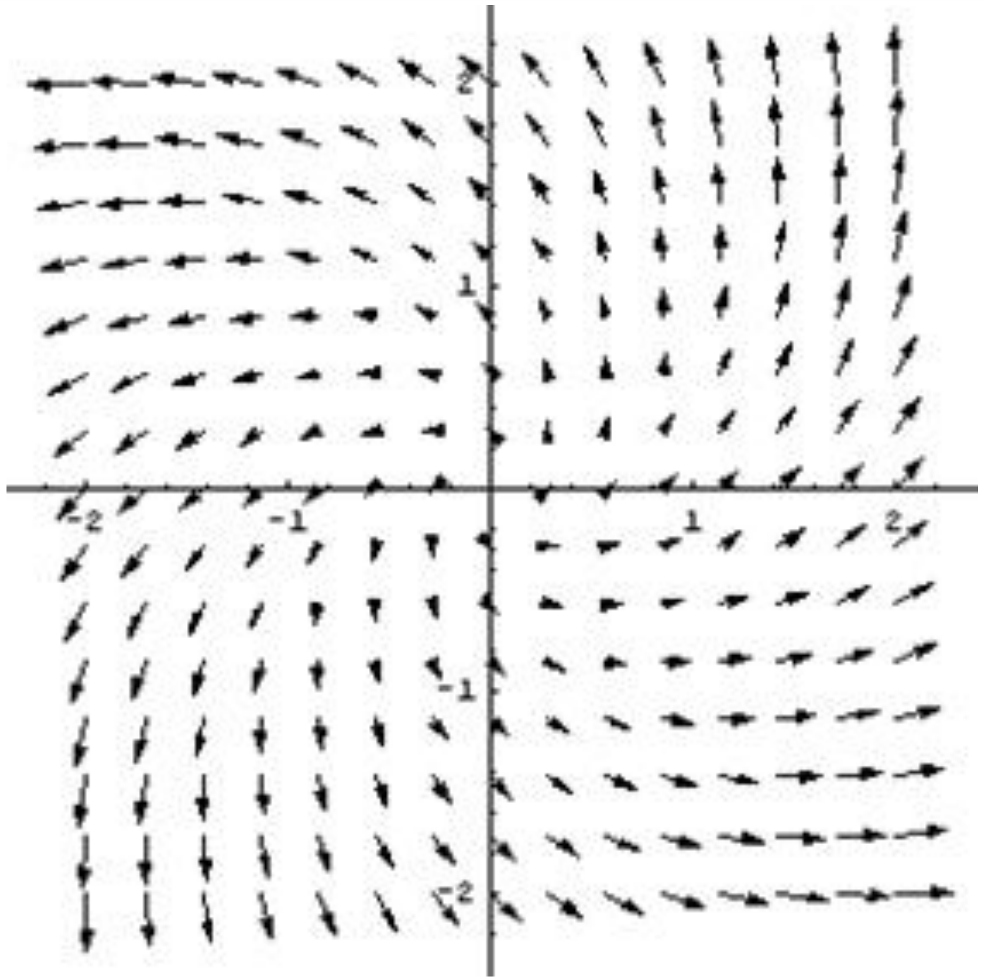
$$= \boxed{0}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$$= 0\hat{x} + 0\hat{y} + (\partial x/\partial x - \partial/\partial y(-y))\hat{z}$$

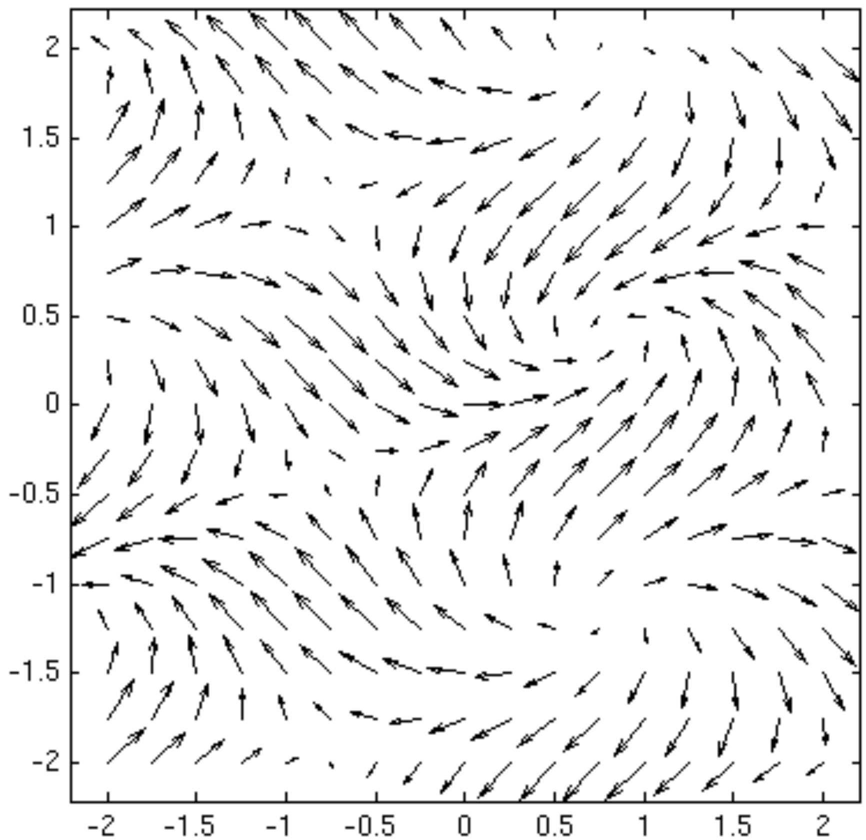
$$= \boxed{2\hat{z}}$$

Example Vector Field



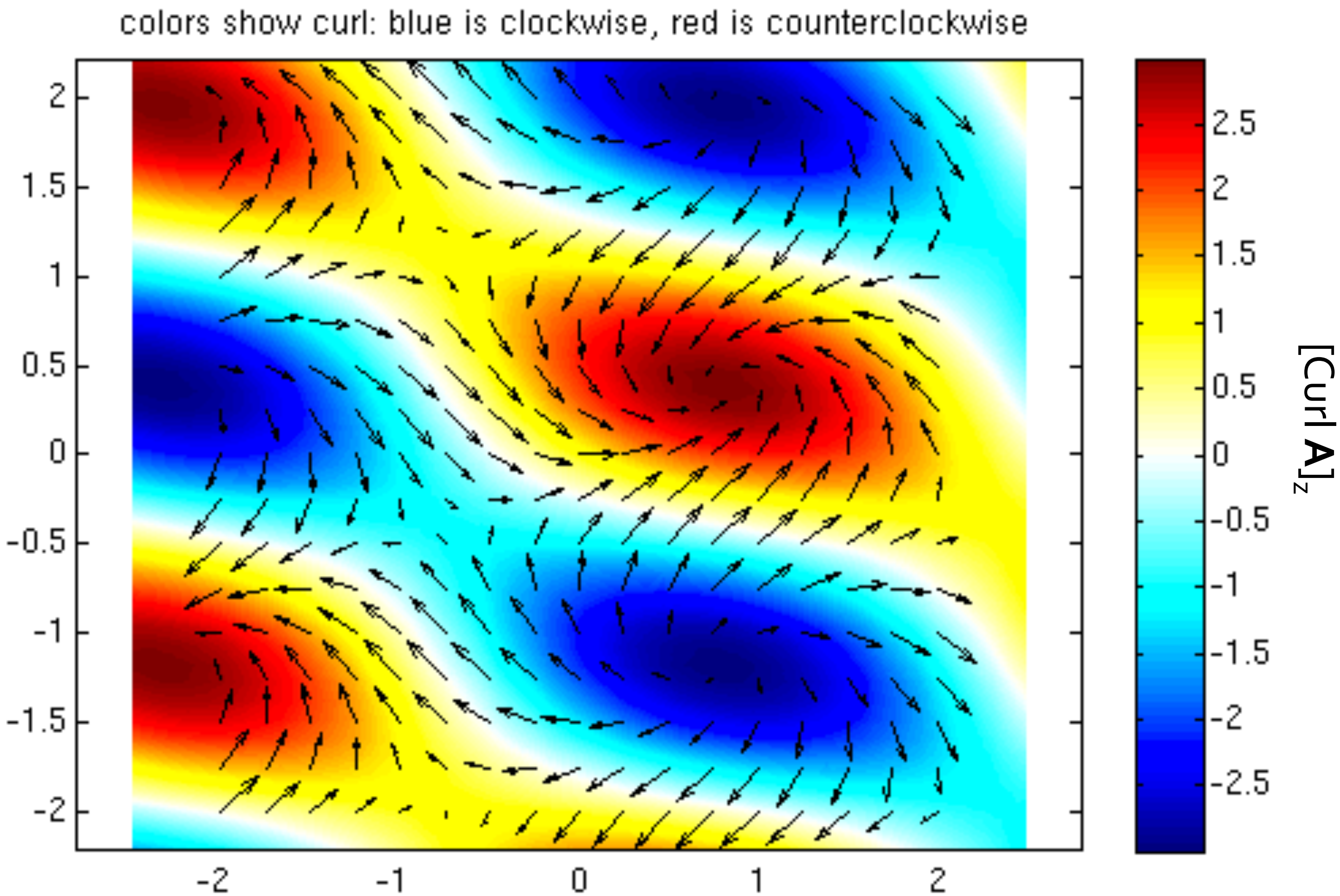
A(x,y,z) = [x-y,x+y, 0]

Example Vector Field



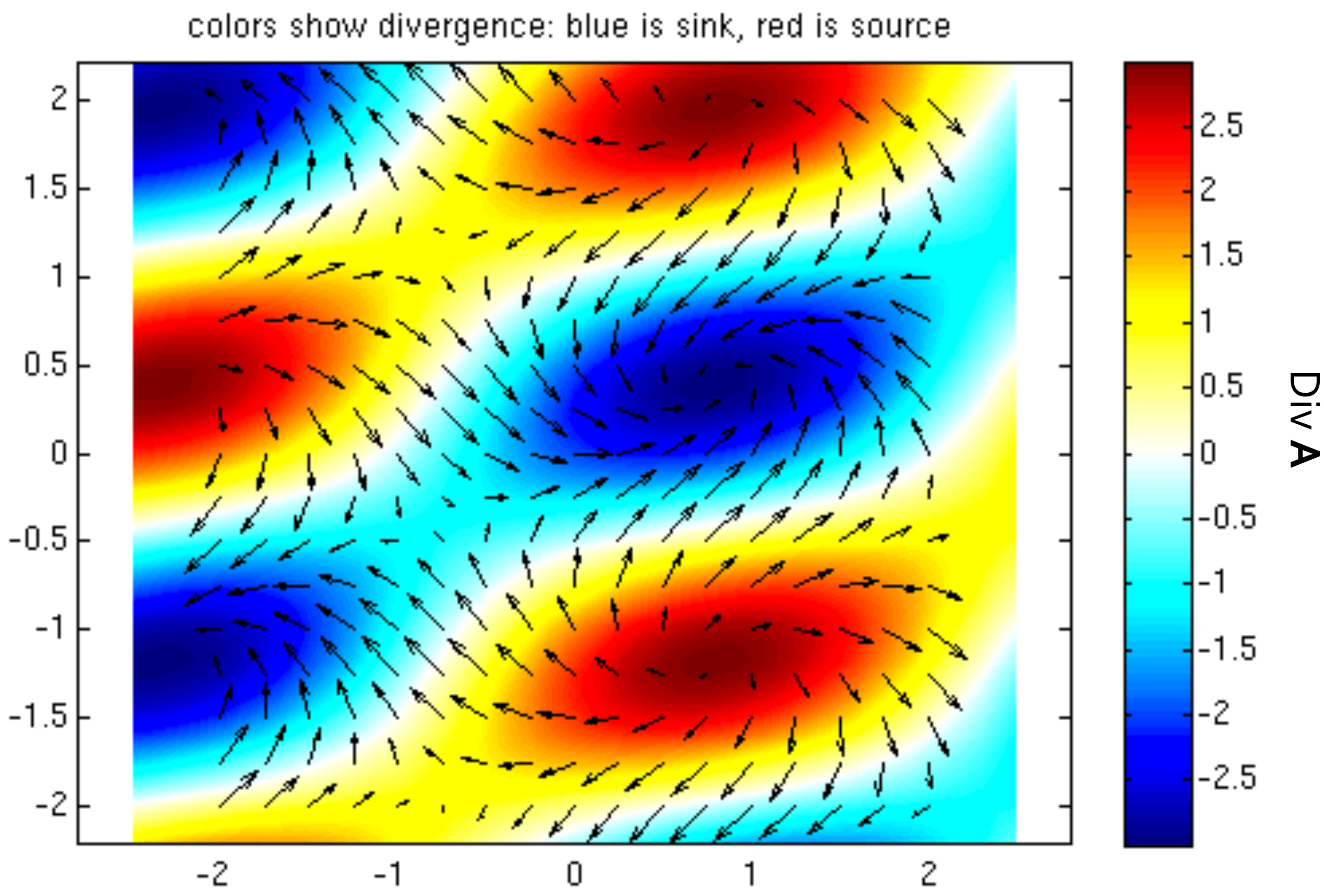
$\mathbf{A}(x,y,z) = [\cos(x+2y), \sin(x-2y), 0]$

Example Curl



A(x,y,z) = [cos(x+2y), sin(x-2y), 0]

Example Divergence



A(x,y,z) = [cos(x+2y), sin(x-2y), 0]

Product Rules

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$
$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$
$$\frac{d}{dx}(f\cdot g) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Similarly:

$$\nabla(f+g) = \nabla f + \nabla g$$
$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$
$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$$

$$\nabla(kf) = k\nabla f$$
$$\nabla\cdot(k\vec{A}) = k\,\nabla\cdot\vec{A}$$
$$\nabla\times(k\vec{A}) = k\,\nabla\times\vec{A}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$$

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$$
$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times\nabla f$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

Note: $(\vec{A}\cdot\nabla)\vec{B} \neq \vec{A}(\nabla\cdot\vec{B})$

$$\vec{A}(\nabla\cdot\vec{B}) = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot (\partial B_x/\partial x + \partial B_y/\partial y + \partial B_z/\partial z)$$

$$= \text{scalar} \times \vec{A}$$

$$(\vec{A}\cdot\nabla)\vec{B} = (A_x\,\partial/\partial x + A_y\,\partial/\partial y + A_z\,\partial/\partial z) \cdot (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

= 9 terms including cross derivatives

Second Derivatives

$\nabla \times (\nabla f) = 0$

$\nabla \cdot (\nabla \times \vec{A}) = 0$

$\nabla (\nabla \cdot \vec{A})$ not often used

$\nabla \cdot (\nabla f) = \nabla^2 f$

$= \partial^2 f / \partial x^2 + \partial^2 f / \partial y^2 + \partial^2 f / \partial z^2$

$=$ Laplacian of f

$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$(\nabla^2 \vec{A} = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z})$

Example:

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$= \left(\partial/\partial y \, \partial f/\partial z - \partial/\partial z \, \partial f/\partial y \right) \hat{x}$

$+ \left(\partial/\partial z \, \partial f/\partial x - \partial/\partial x \, \partial f/\partial z \right) \hat{y}$

$+ \left(\partial/\partial x \, \partial f/\partial y - \partial/\partial y \, \partial f/\partial x \right) \hat{z}$

$= 0$ by equality of mixed derivatives