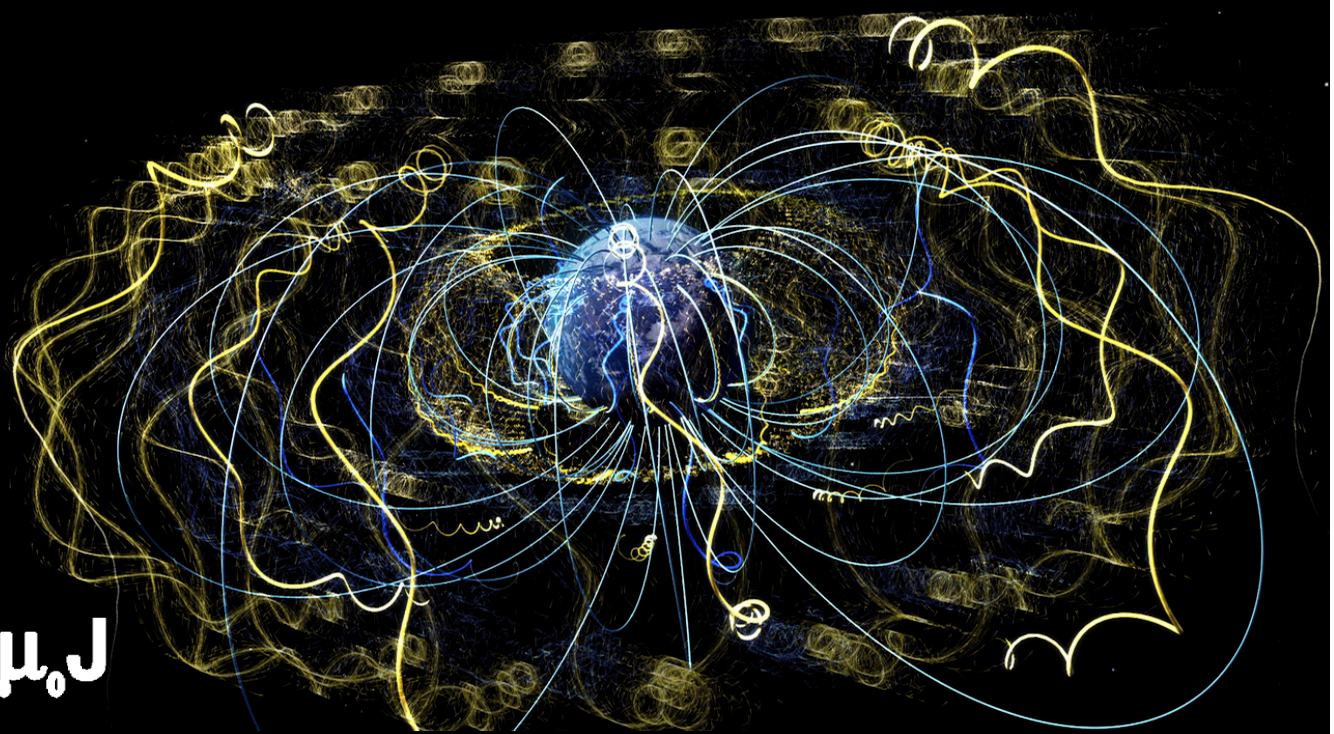
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

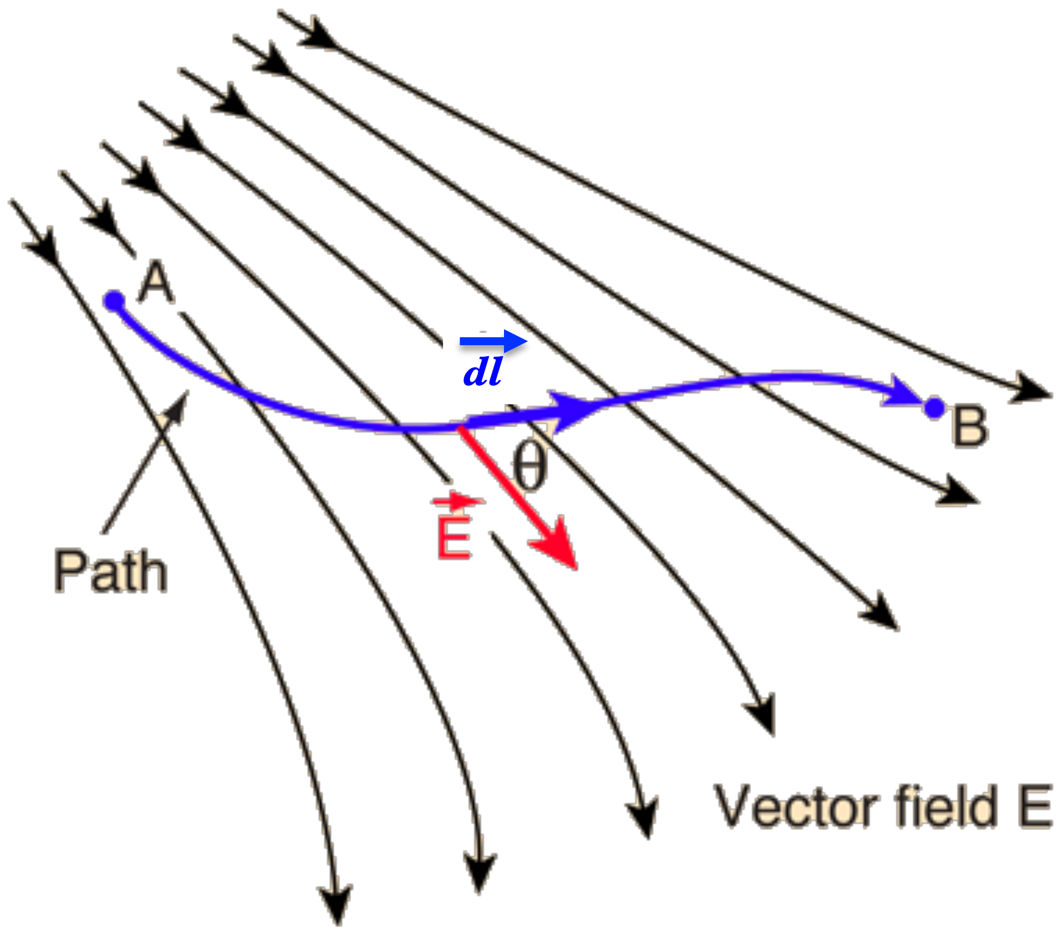
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Line Integrals



Line Integrals

$\int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{l}$

$d\vec{l} = [dx, dy, dz]$

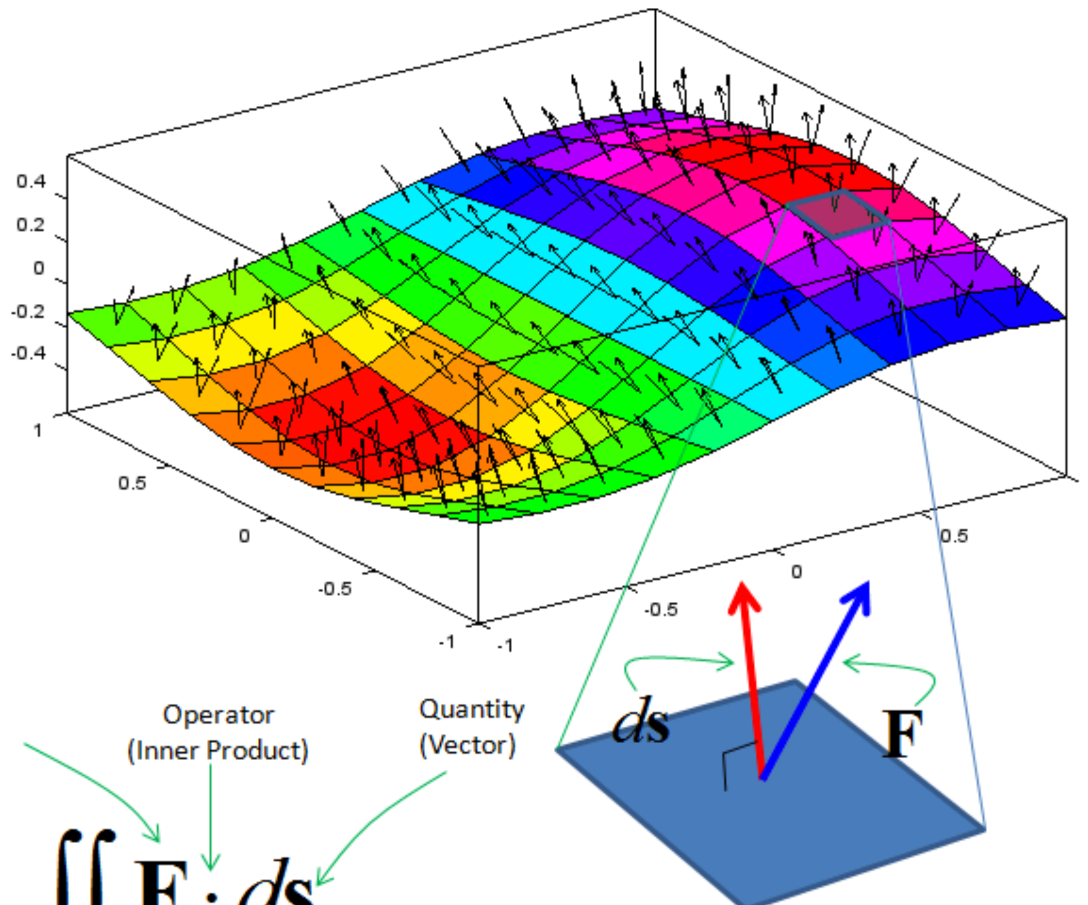
$\Rightarrow \int_{\vec{a}}^{\vec{b}} (A_x dx + A_y dy + A_z dz)$

- Looks like a simple scalar integral

- But $\vec{A}$ and $d\vec{l}$ depend on the path

- So $\int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{l}$ in general depends on the path between $\vec{a}$ and $\vec{b}$

Surface Integrals



Quantity (Vector)

Operator (Inner Product)

Quantity (Vector)

$$\iint \mathbf{F} \cdot ds$$

Surface Integrals

$$\int_S \vec{A} \cdot d\vec{a} = \int_S \vec{A} \cdot \hat{n} \, da$$

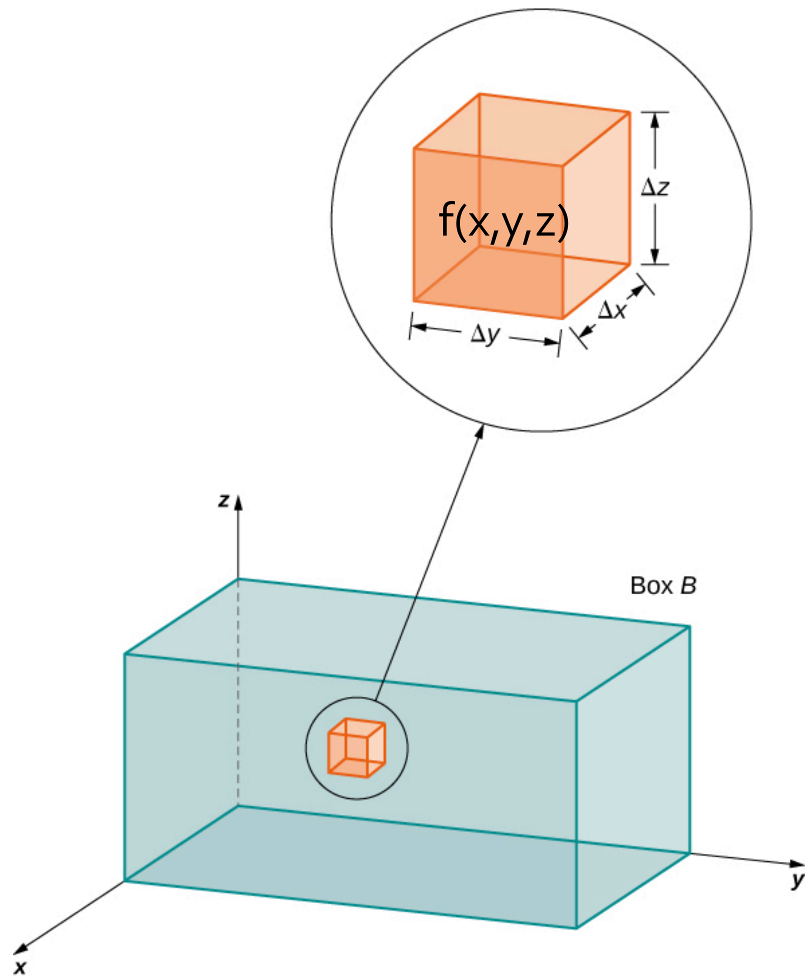
w/ $\hat{n}$ = normal to surface $d\vec{a}$

- often explicitly written

$$\iint \vec{A} \cdot d\vec{a}$$

- $\vec{A}$, $d\vec{a}$ both depend on surface orientation
- In general $\int \vec{A} \cdot d\vec{a}$ depends on shape of surface S

Volume Integrals



Volume Integrals

$$\int_V f\, d\tau \quad \text{or} \quad \int_V f\, dV$$

w/ $d\tau = dx\, dy\, dz$

- Can also explicitly write
  $\iiint f\, d\tau$

- Easier than line or surface integrals since no dot products

- still tricky for complex shapes

Fundamental Theorem of Calculus

$$\int_a^b \left(\frac{df}{dx}\right) dx = \int_a^b df = f(b) - f(a)$$

Fundamental Theorem : Gradients

$$\int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{l} = \int_{\vec{a}}^{\vec{b}} df = f(\vec{b}) - f(\vec{a})$$

- Independent of path $\vec{a} \to \vec{b}$
- Total elevation gain independent of path

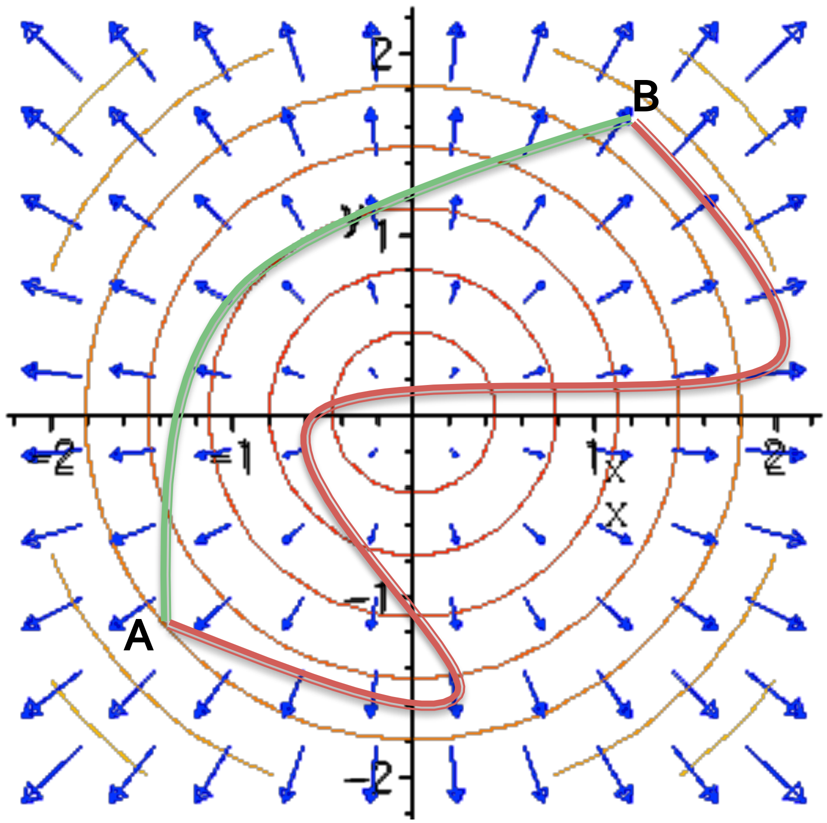
Divergence Theorem

$$\int_V (\nabla \cdot \vec{A})\, d\tau = \oint_S \vec{A} \cdot d\vec{a}$$

w/ S = closed surface that bounds volume V

- Total divergence in volume equal to flux through surface

Gradient Theorem

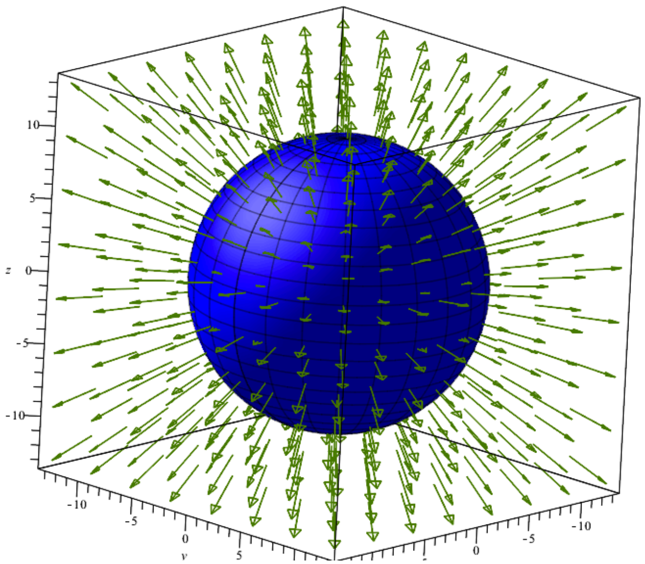


$$\int_C \nabla f \cdot d\mathbf{s} = f(B) - f(A)$$

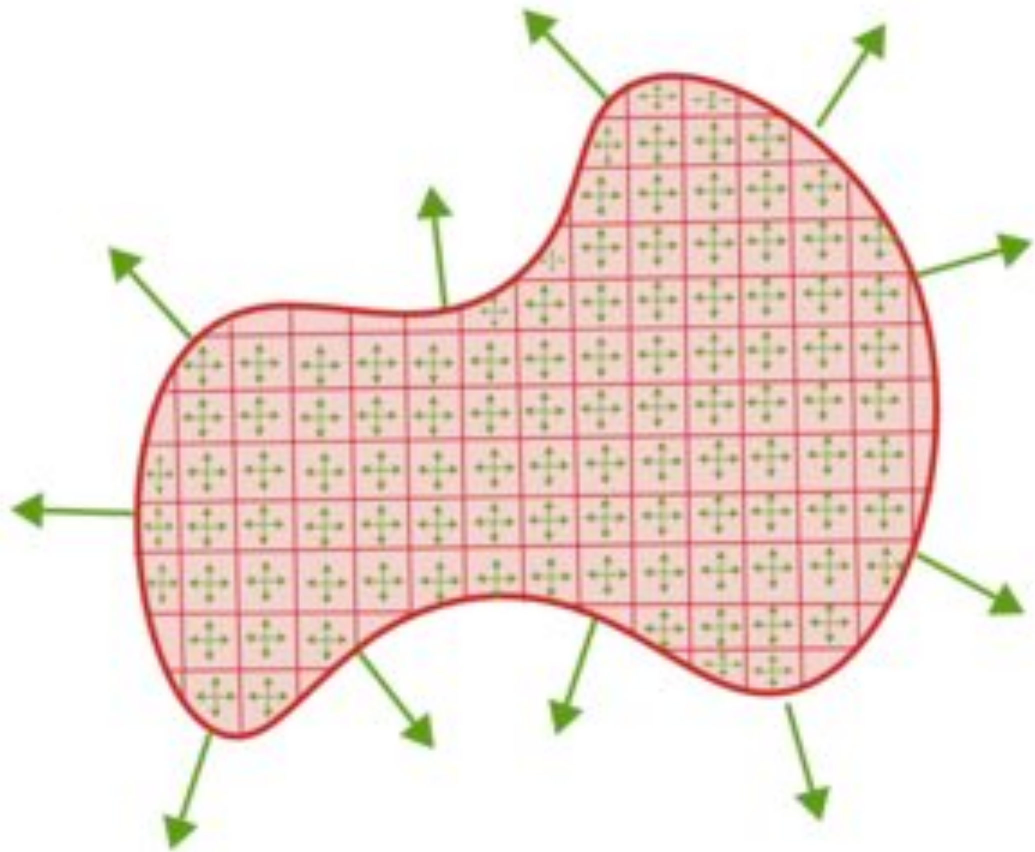
Total of slope*distance = total elevation change

Net elevation gain does not depend on path!

Divergence Theorem



$$\iint \nabla \cdot \mathbf{F} \, dA$$
$$= \int_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds$$



Total divergence in volume = total outward flux through bounding surface

Stokes' Theorem

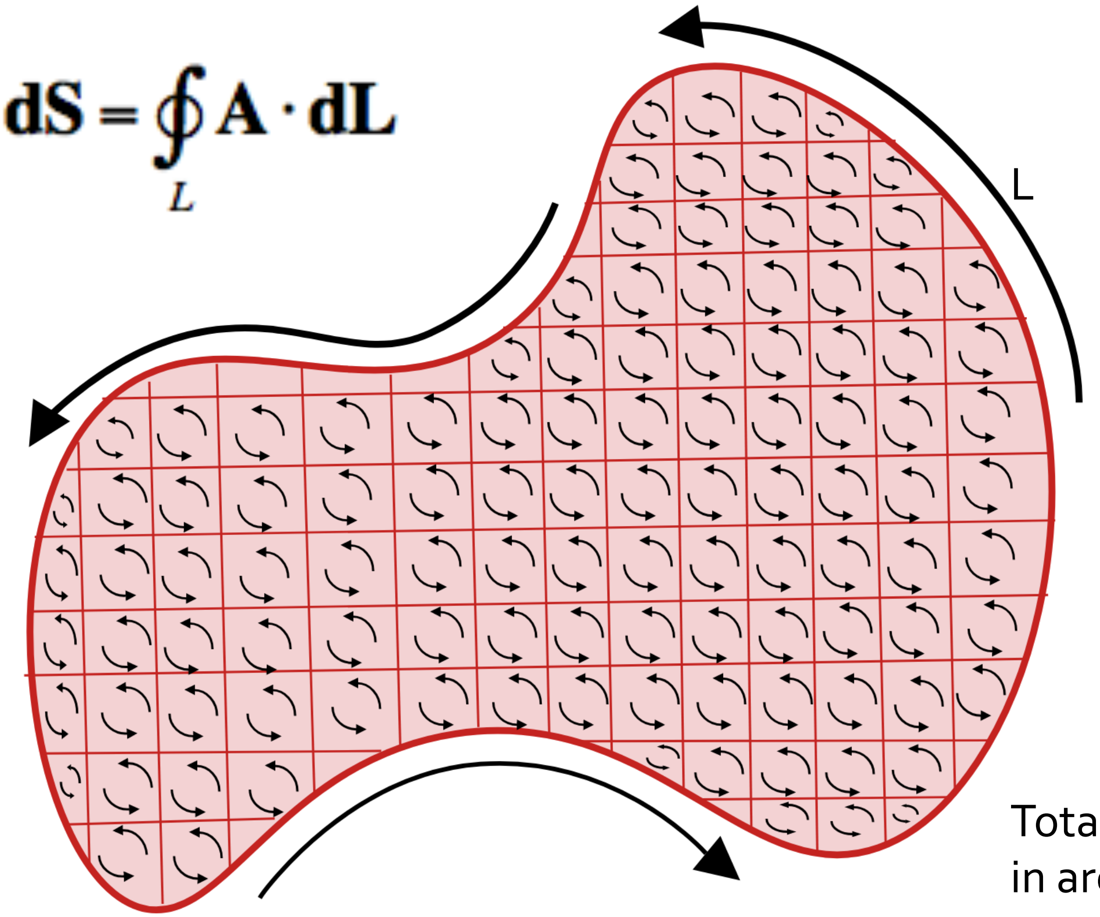
$$\int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \oint_P \vec{A} \cdot d\vec{l}$$

w/ P the closed loop around surface S

- Total circulation in area equal to flow around boundary
- Independent of shape of S

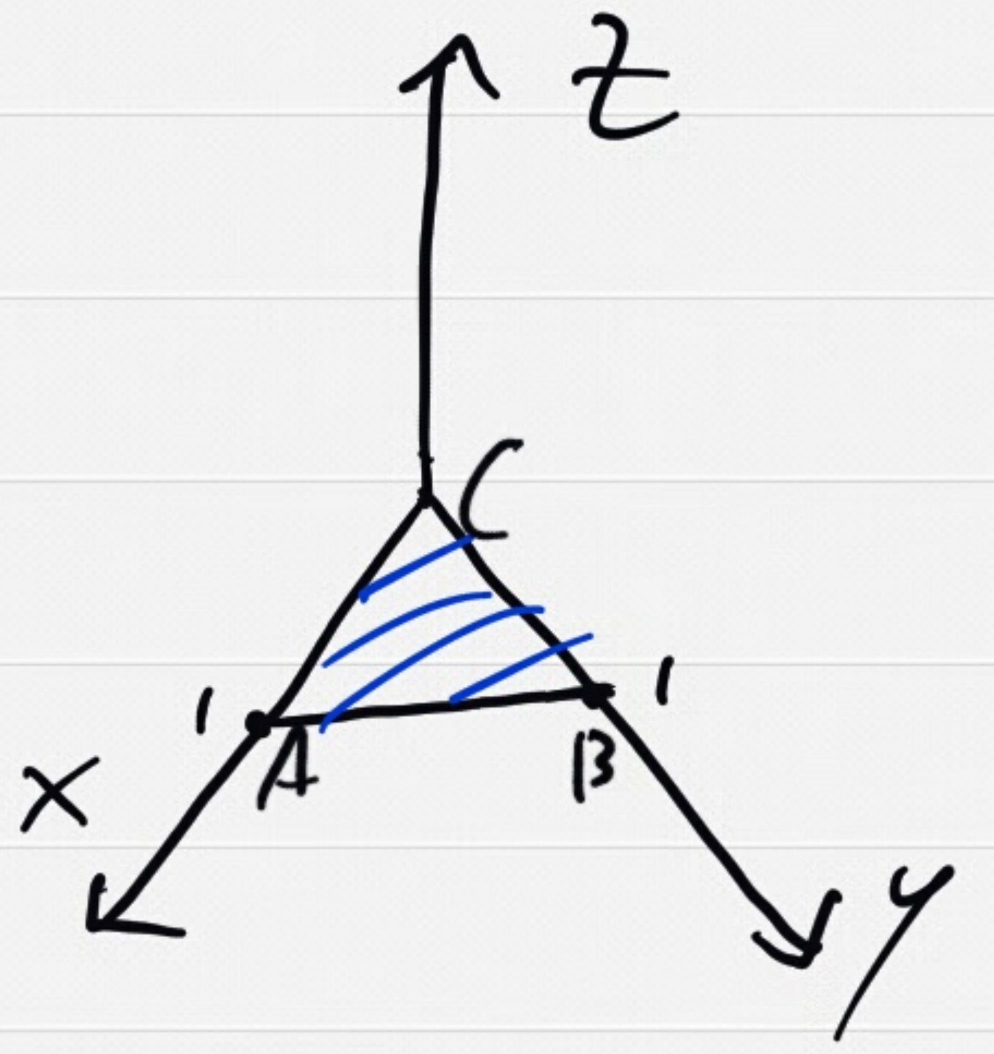
Stokes' Theorem

$$\int (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dL}$$



Total circulation
in area =
total circulation
around perimeter

Example



Check Stokes' Theorem for path ABC w/ $\vec{A}(x,y,z) = xy\hat{x}$

1. $$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 0 & 0 \end{vmatrix}$$

$$= 0\hat{x} + 0\hat{y} - \partial/\partial y(xy)\,\hat{z}$$

$$= -x\hat{z}$$

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{a}$$

$$= \int_S (-x\hat{z}) \cdot d\vec{a}$$

$$= \int_S (-x)\, da \quad \text{since } d\vec{a} = da\,\hat{z}$$

for path ABC by right hand rule

$$= \int_S (-x)\, dx\, dy$$

$$= \int_0^1 \left(\int_0^{1-x} (-x)\, dy \right) dx$$

$$= \int_0^1 \left(-xy \Big|_0^{1-x} \right) dx = \int_0^1 -x(1-x)\, dx$$

$$= \left(-x^2/2 + x^3/3 \right) \Big|_0^1 = -1/6$$

$$\int \vec{A}\cdot d\vec{l} = \int_{AB}\vec{A}\cdot d\vec{l} + \int_{BC}\vec{A}\cdot d\vec{l} + \int_{CA}\vec{A}\cdot d\vec{l}$$

$$\int_{AB}\vec{A}\cdot d\vec{l} = \int_{AB}(xy\hat{x})\cdot(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)$$

$$= \int_{AB} xy\,dx$$

Use dummy variable t

on path AB, $x = 1-t$, $y = t$

$$\Rightarrow dx = -dt$$

so $\int_{AB}\vec{A}\cdot d\vec{l}$

$$= \int_0^1 (1-t)\cdot t\cdot -dt$$

$$= \int_0^1 (t^2 - t)\,dt$$

$$= \left(t^3/3 - t^2/2\right)\Big|_0^1$$

$$= -1/6$$

$\int_{BC}\vec{A}\cdot d\vec{l} = 0$ since $\vec{A} = 0$ on path BC

$\int_{CA}\vec{A}\cdot d\vec{l} = 0$ since $\vec{A} = 0$ on path CA

$$\Rightarrow \int_{ABC}\vec{A}\cdot d\vec{l} = -1/6$$

//