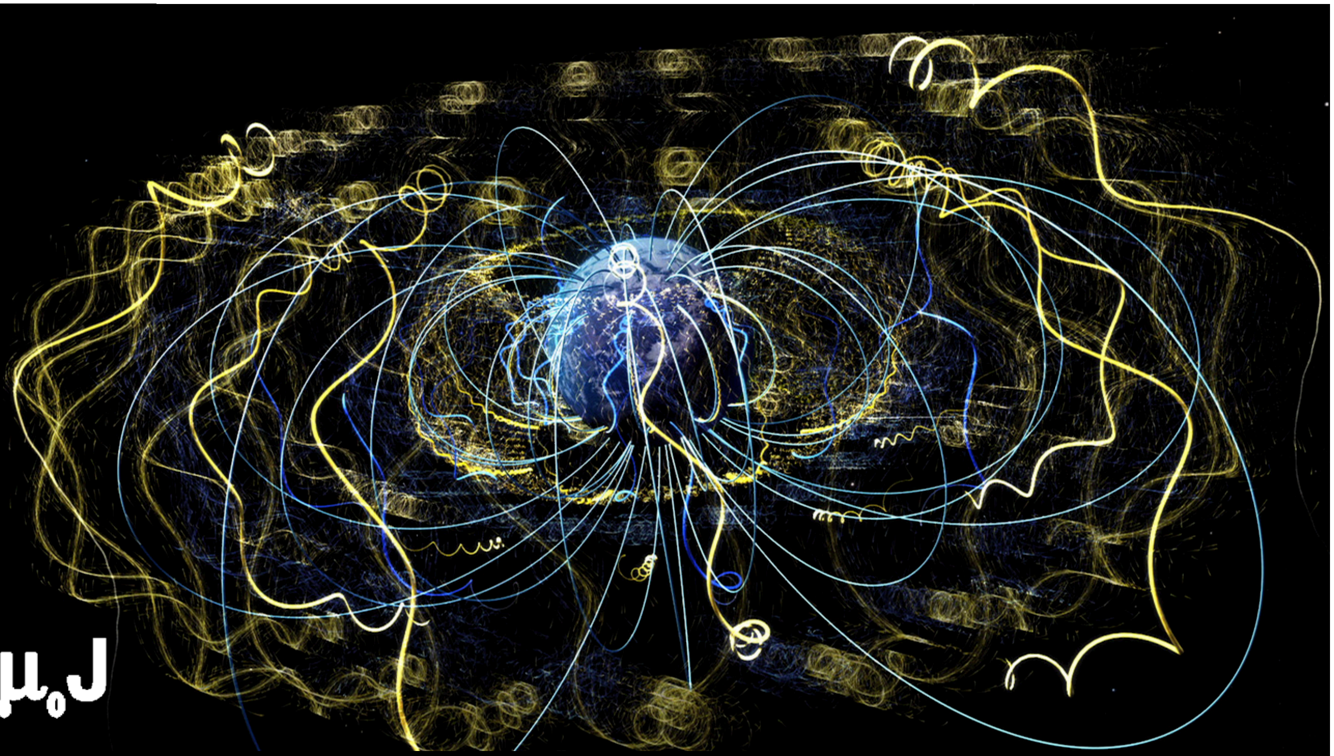
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas

Virtual by Zoom!

MWF 9:30-10:20 Lecture

Lienard-Wiechert Potentials

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(c\,\Delta r - \overrightarrow{\Delta r}\cdot\vec{v})}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(c\,\Delta r - \overrightarrow{\Delta r}\cdot\vec{v})} = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

All in terms of retarded variables…

What about v not small?

$$V(\vec{r},t) = \frac{qc}{4\pi\varepsilon_0} \frac{1}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

for constant $\vec{v}$

Rewrite denominator

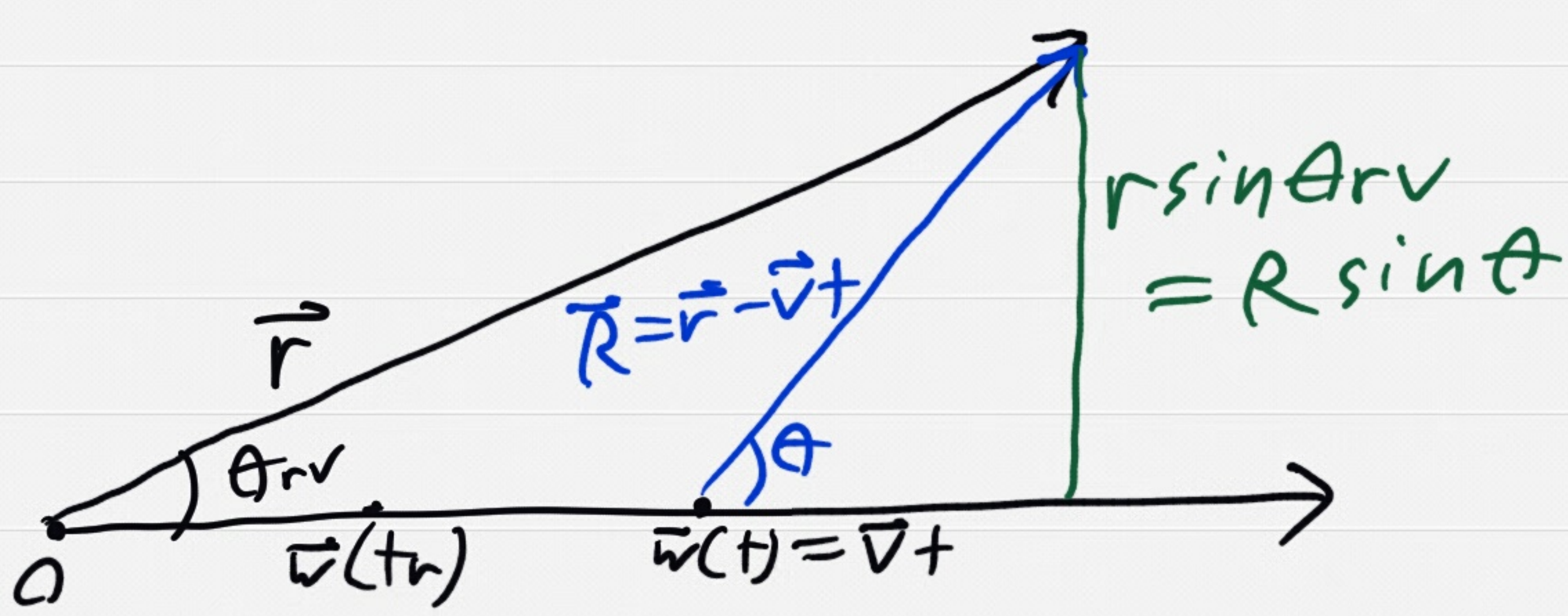
$$\sqrt{c^4t^2 + (\vec{r}\cdot\vec{v})^2 - 2c^2t\,\vec{r}\cdot\vec{v} + c^2r^2 - v^2r^2 - c^4t^2 + v^2c^2t^2}$$

$$= \sqrt{c^2(r^2 - 2t\,\vec{r}\cdot\vec{v} + v^2t^2) + (\vec{r}\cdot\vec{v})^2 - r^2v^2}$$

$$= \sqrt{c^2|\vec{r} - \vec{v}t|^2 + (\vec{r}\cdot\vec{v})^2 - r^2v^2}$$

$$= \sqrt{c^2|\vec{r} - \vec{v}t|^2 + r^2v^2\cos^2\theta_{rv} - r^2v^2}$$

$$= \sqrt{c^2R^2 - r^2v^2\sin^2\theta_{rv}}$$

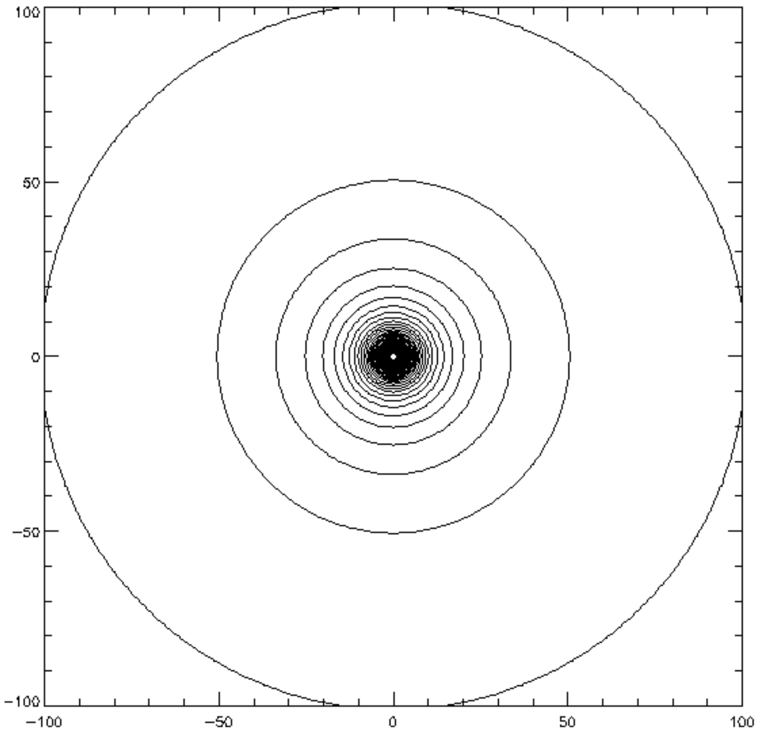


$$\boxed{V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 R} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}}}$$

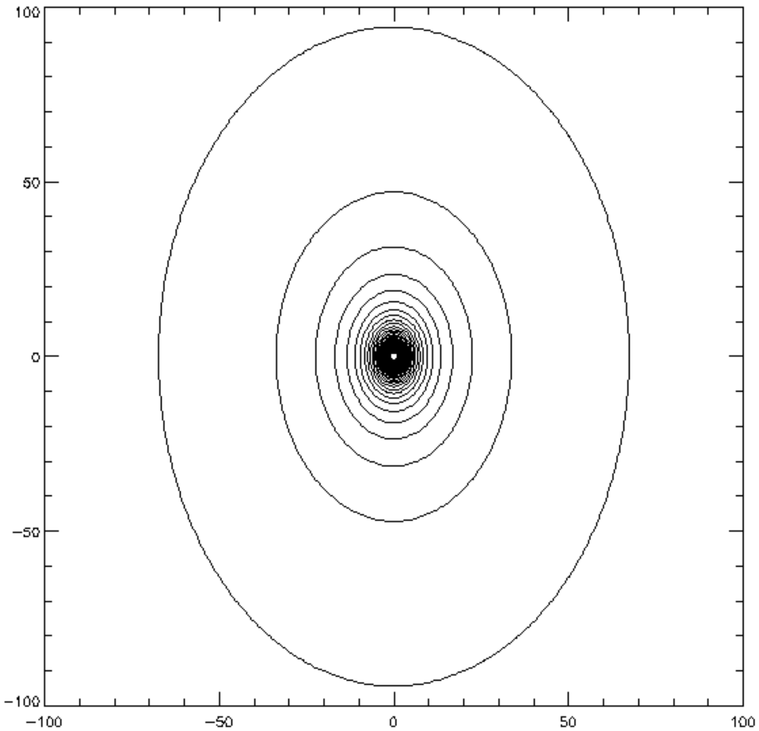
Useful form for const. velocity motion, expressed in terms of current (not retarded) position and position-velocity angle

Lienard-Wiechert Potentials for Uniformly Moving Point Charge

Contour levels not same on two plots - for visualization of contour shape only

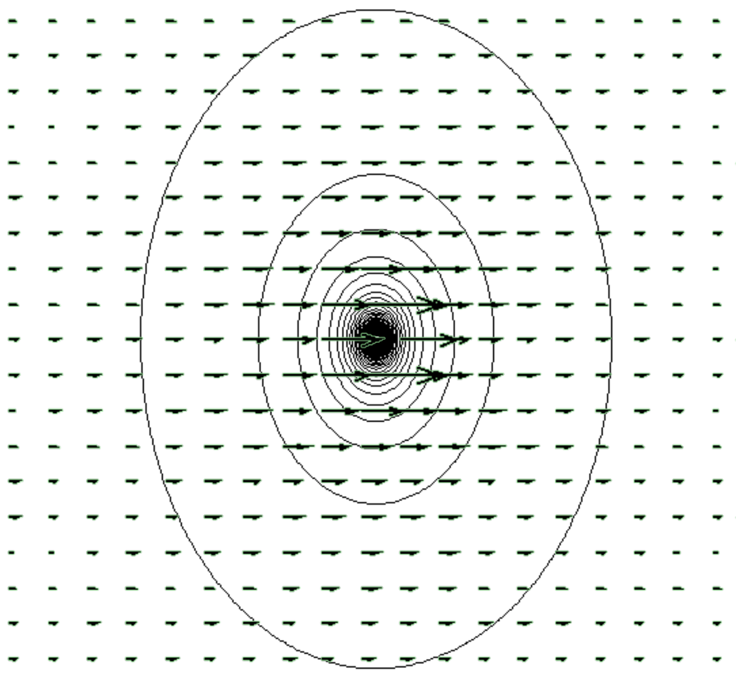


v/c = 0



v/c = 0.7

Lienard-Wiechert Potentials for Uniformly Moving Point Charge



- In general $\vec{E} = -\nabla V - \partial \vec{A}/\partial t$

$\vec{B} = \nabla \times \vec{A}$

- But R, θ depend on $\vec{r}, t$ so messy to calculate even for special case of const. velocity

- Look @ $\theta = 0$, $x > vt$

$$V(x,t) = \frac{q}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0 |\vec{r} - \vec{v}t|}$$

$$= \frac{q}{4\pi\varepsilon_0 (x - vt)}$$

$$\vec{A}(x,t) = \frac{\vec{v}}{c^2} V = \frac{q v \hat{x}}{4\pi\varepsilon_0 c^2 (x - vt)}$$

$$\vec{E} = -\nabla V - \partial \vec{A}/\partial t$$

$$= \frac{q \hat{x}}{4\pi\varepsilon_0 (x - vt)^2} + \frac{q v \hat{x}}{4\pi\varepsilon_0 c^2} \frac{-v}{(x - vt)^2}$$

$$= \frac{q \hat{x}}{4\pi\varepsilon_0 R^2} \left(1 - \frac{v^2}{c^2}\right)$$

- Looks like $\vec{E}$ for stationary particle but reduced by $1 - v^2/c^2$

Look @ $\theta = \pi/2$, $t = 0$

$$V(y,t) = \frac{q}{4\pi\varepsilon_0 R} \frac{1}{\sqrt{1 - v^2/c^2}}$$

at $t = 0$, $R = y$

$$V(y,t) = \frac{q}{4\pi\varepsilon_0 y} \frac{1}{\sqrt{1 - v^2/c^2}}$$

$E_y = -\partial V/\partial y$ (no $\vec{A}$ term since $\vec{A}$ along $\hat{x}$)

$$\vec{E} = \frac{q\hat{y}}{4\pi\varepsilon_0 y^2} \frac{1}{\sqrt{1 - v^2/c^2}}$$

Increased in perpendicular direction compared to stationary charge

General Eq. for $\vec{E}$ and $\vec{B}$ of point charge

$$V(\vec{r},t) = \frac{qc}{4\pi\varepsilon_0} \frac{1}{c\Delta r - \vec{\Delta r}\cdot\vec{v}}$$

$$\vec{A}(\vec{r},t) = \frac{q\vec{v}}{4\pi\varepsilon_0 c} \frac{1}{c\Delta r - \vec{\Delta r}\cdot\vec{v}}$$

Heinous Algebra

$$\Downarrow$$

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \frac{q}{4\pi\varepsilon_0} \frac{\Delta r}{(\vec{\Delta r}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\Delta r}\times(\vec{u}\times\vec{a})\right] \\ \vec{B}(\vec{r},t) &= \frac{1}{c}\,\hat{\Delta r}\times\vec{E}(\vec{r},t) \end{aligned}}$$

$\vec{w}(t)$ = trajectory of q

with
$$\vec{v} = d\vec{w}/dt\,\big|_{tr}$$
$$\vec{a} = d\vec{v}/dt\,\big|_{tr}$$
$$\vec{\Delta r} = \vec{r} - \vec{w}(tr)$$
$$\vec{u} = c\,\hat{\Delta r} - \vec{v}$$

Special Case: Uniform Motion

$$\vec{a} = 0 \quad , \quad \vec{w} = \vec{v} t$$

$$\vec{u} = c\,\hat{\Delta r} - \vec{v}$$

$$= \frac{c\,\vec{\Delta r}}{\Delta r} - \vec{v}$$

$$= \frac{c\,(\vec{r} - \vec{v} t_r)}{c\,(t - t_r)} - \vec{v}$$

Meanwhile $\vec{\Delta r} \cdot \vec{u} = c\Delta r - \vec{\Delta r} \cdot \vec{v}$
which we've looked at before

Same algebra

$\Downarrow$

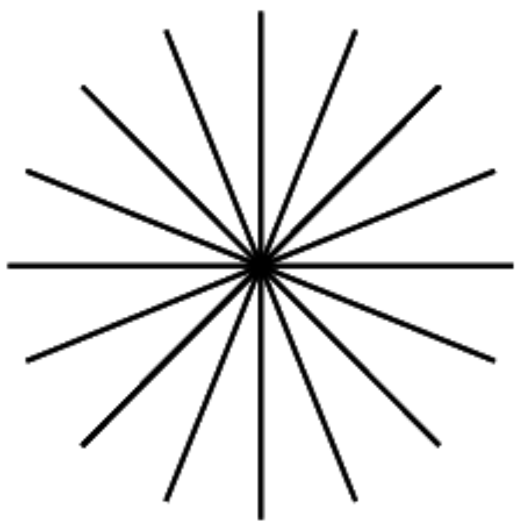
$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2}}$$

$\theta = 0, \pi/2 \Rightarrow$ special cases we derived

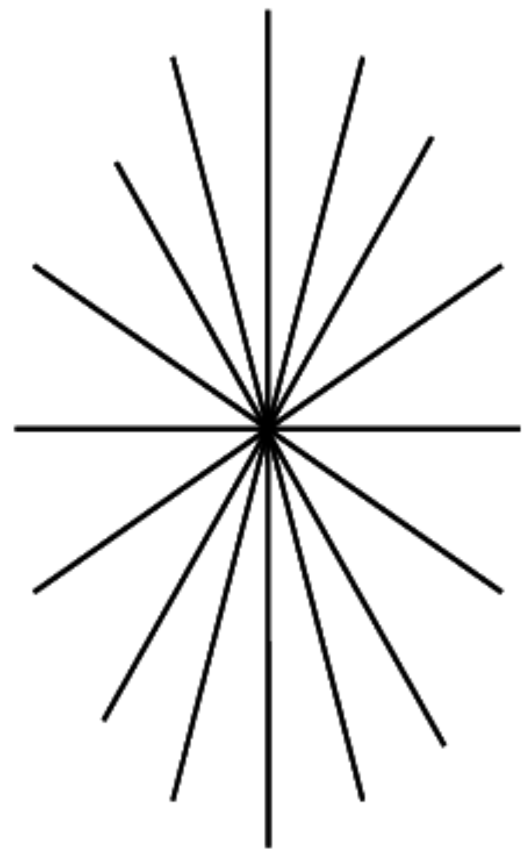
- Though V steeper in direction of motion, $\vec{A}$ overcomes this effect and gives weaker $\vec{E}$ along motion

- Amazingly, $\vec{E}$ points from present (not retarded) position in this case

E-field of Uniformly Moving Charge



$v = 0$



$v = 0.8\,c$

$$\vec{B} = \frac{1}{c} \hat{\Delta r} \times \vec{E}$$

w/ $$\hat{\Delta r} = \frac{\vec{r} - \vec{v} t_r}{c(t - t_r)}$$

$$= \frac{\vec{r} - \vec{v} t + \vec{v}(t - t_r)}{c(t - t_r)}$$

$$= \frac{\vec{R}}{c(t - t_r)} + \frac{\vec{v}}{c}$$

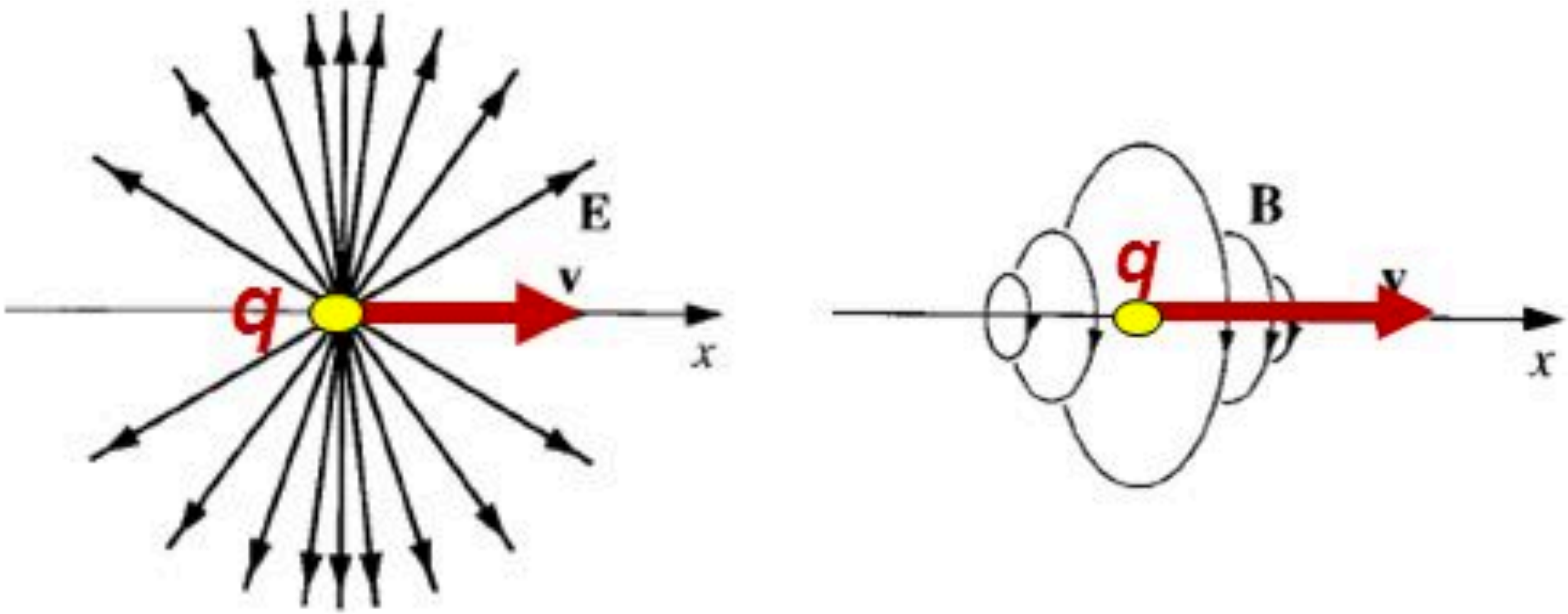
$$\Rightarrow \vec{B} = \frac{1}{c} \hat{\Delta r} \times \vec{E}$$

$$= \frac{\vec{v}}{c^2} \times \vec{E} \quad \text{since } \vec{R} \times \hat{R} = 0$$

$$\Rightarrow \boxed{\vec{B} = \frac{1}{c^2} \vec{v} \times \vec{E}}$$

perpendicular to both $\vec{v}$ & $\vec{R}$
for uniform motion

E & B of Uniformly Moving Point Charge



Check Your Understanding

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(c\,\Delta r - \overrightarrow{\Delta r}\cdot\vec{v})} \qquad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(c\,\Delta r - \overrightarrow{\Delta r}\cdot\vec{v})} = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

- The full form of the Lienard-Wiechert potentials for a moving point charge is as shown, written in terms of the retarded position.

- For the special case of a particle traveling with constant velocity v along the x-axis, starting at the origin at t = 0, show that along the x-axis these reduce to (in terms of the current position, for an appropriate normalization B):

$$V(x,t) = \frac{B}{x - vt} \qquad \vec{A}(x,t) = \frac{Bv}{c^2(x-vt)}\hat{x}$$

Q1:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{c\Delta r - \vec{\Delta r}\cdot\vec{v}}$$

w/ $\vec{\Delta r} = \vec{r} - \vec{v}t_r = (x - vt_r)\hat{x}$

$\Delta r = c(t - t_r) = x - vt_r$

$$V(x,t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{c(x-vt_r) - v(x-vt_r)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qc}{(c-v)(x-vt_r)}$$

but $c(t-t_r) = x - vt_r$

$\Rightarrow (v-c)t_r = x - ct$

or $t_r = (x-ct)/v-c$

$$x - vt_r = \frac{(v-c)x - v(x-ct)}{v-c}$$

$$= \frac{c(vt - x)}{v-c}$$

$$= \frac{c(x-vt)}{c-v}$$

$$\Rightarrow \boxed{V(x,t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{x-vt}}$$

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