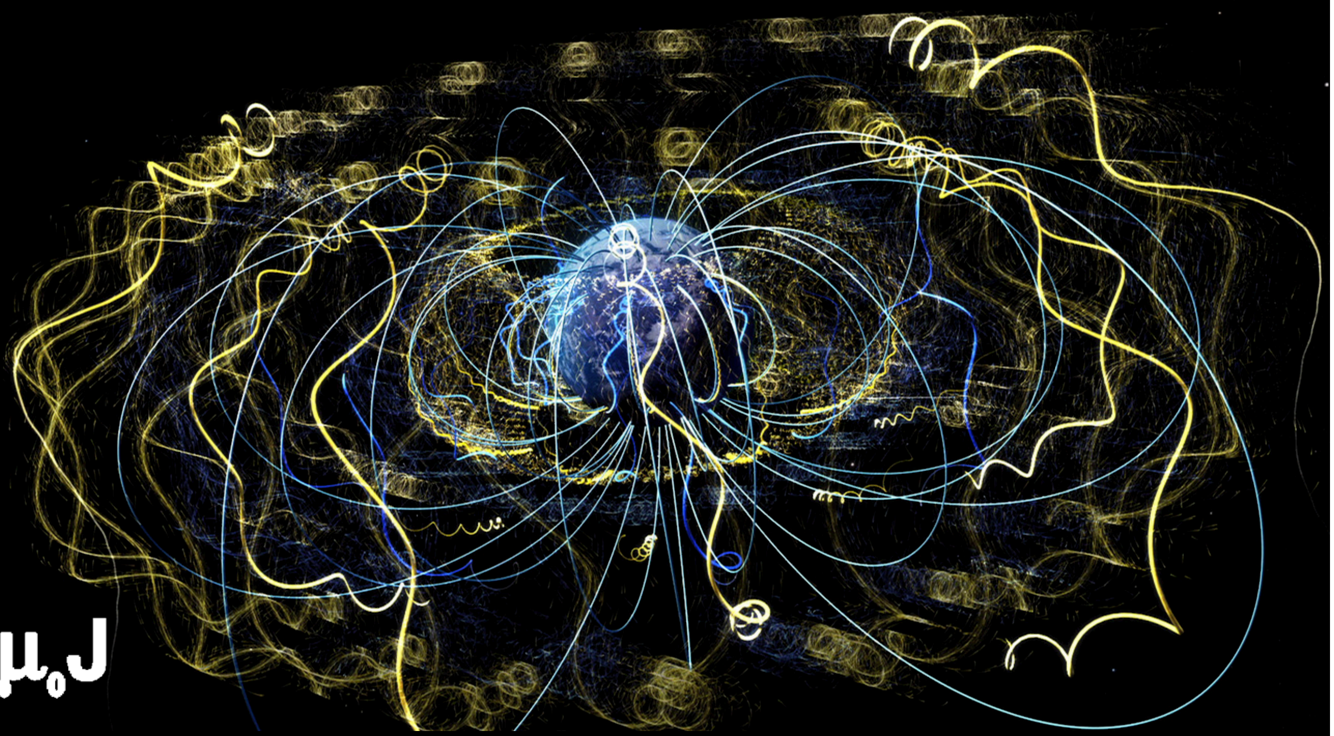
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

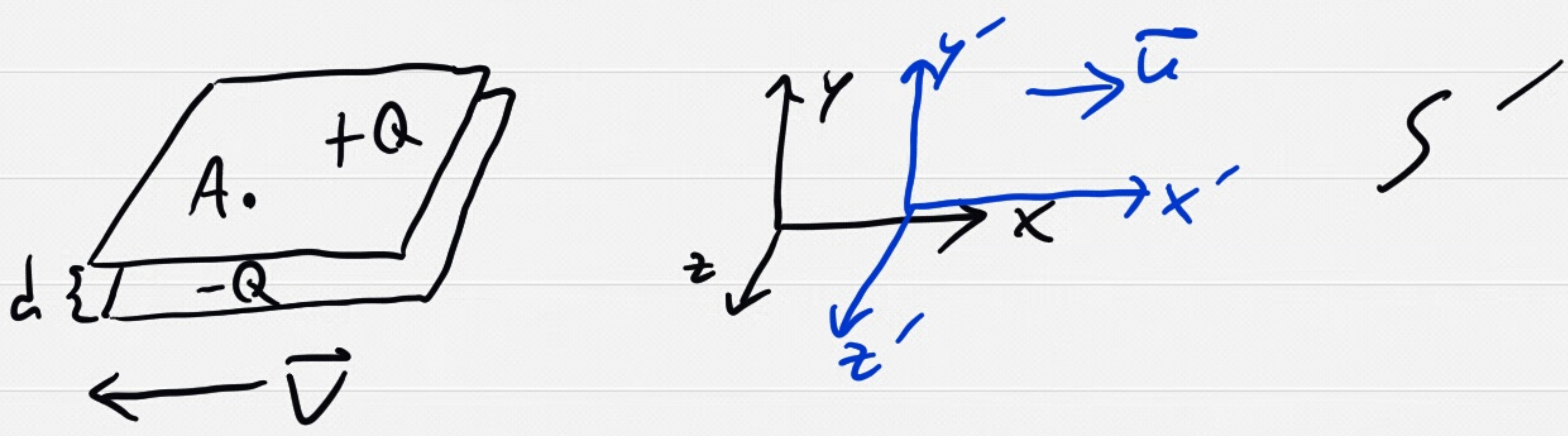
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas
Virtual by Zoom!
MWF 9:30-10:20 Lecture

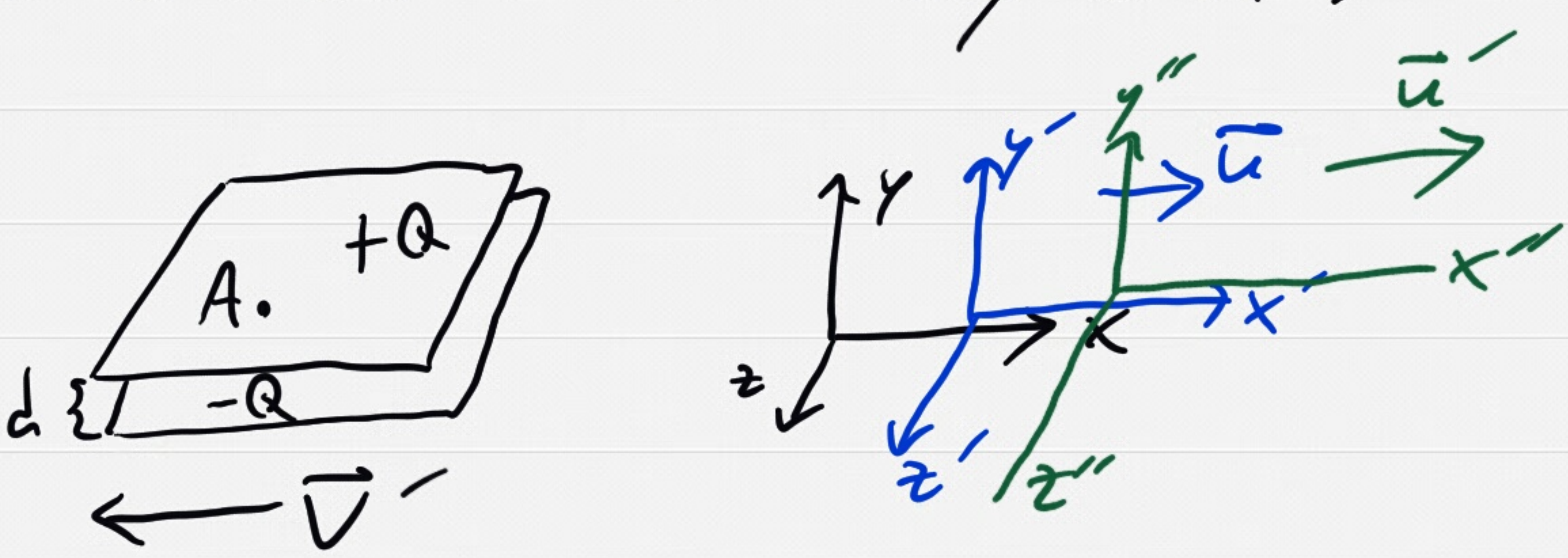
– Now start w/ both $\vec{E}$, $\vec{B}$
(start in S')



$$\vec{E}' = -\frac{Q}{\varepsilon_0 A_0} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}\,\hat{y}$$

$$\vec{B}' = \frac{\mu_0 Q}{A_0} \cdot \frac{v}{\sqrt{1 - v^2/c^2}}\,\hat{z}$$

Transform again!



$$v' = \frac{u + u'}{1 + \frac{uu'}{c^2}}$$ to left in S''

$$\vec{E}'' = -\frac{Q}{\varepsilon_0 A_0} \cdot \frac{1}{\sqrt{1 - v'^2/c^2}}\,\hat{y}$$

$$= \vec{E}' \cdot \frac{\sqrt{1 - v^2/c^2}}{\sqrt{1 - v'^2/c^2}}\,\hat{y}$$

$$\sqrt{\frac{1-v^2/c^2}{1-v'^2/c^2}} = \sqrt{\frac{1-u^2/c^2}{1-\frac{(u+u')^2}{c^2(1+uu'/c^2)^2}}}$$

$$= \sqrt{\frac{(1-u^2/c^2)\left(1+\frac{uu'}{c^2}\right)^2}{\left(1+\frac{uu'}{c^2}\right)^2-(u+u')^2/c^2}}$$

$$= \left(1+\frac{uu'}{c^2}\right)\sqrt{\frac{1-u^2/c^2}{1+\frac{2uu'}{c^2}+\frac{u^2u'^2}{c^4}-\frac{u^2}{c^2}-\frac{u'^2}{c^2}-\frac{2uu'}{c^2}}}$$

$$= \left(1+\frac{uu'}{c^2}\right)\sqrt{\frac{1-u^2/c^2}{(1-u^2/c^2)(1-u'^2/c^2)}}$$

$$= \left(1+\frac{uu'}{c^2}\right)\cdot\frac{1}{\sqrt{1-u'^2/c^2}}$$

$$= \left(1+\frac{uu'}{c^2}\right)\gamma'$$

$$\Rightarrow \vec{E}'' = \gamma'\left(1+\frac{uu'}{c^2}\right)\vec{E}'$$

or $\quad E_y'' = \gamma'\left(1+\frac{uu'}{c^2}\right)E_y'$

But $\quad \frac{uE_y'}{c^2} = -B_z'$

$$\Rightarrow E_y'' = \gamma'(E_y' - u'B_z')$$

Similarly $\quad E_z'' = \gamma'(E_z' + u'B_y')$

Next:

$$\vec{B}'' = \frac{1}{c^2}\vec{v}' \times \vec{E}''$$

$$B_z'' = -\frac{v'}{c^2} E_y''$$

$$= -\frac{u+u'}{1+uu'/c^2} \cdot \gamma' \cdot \left(1+\frac{uu'}{c^2}\right) E_y' / c^2$$

$$= -\gamma'\left(u E_y' + u' E_y'\right)/c^2$$

$$= -\gamma'\left(-c^2 B_z' + u' E_y'\right)/c^2$$

$$= \gamma'\left(B_z' - \frac{u'}{c^2} E_y'\right)$$

similarly $B_y'' = \gamma'\left(B_y' + \frac{u'}{c^2} E_z'\right)$

Finally (dropping all extra primes)

$$\boxed{\begin{aligned} &E_x' = E_x, \quad E_y' = \gamma(E_y - uB_z) \\ &E_z' = \gamma(E_z + uB_y) \\ &B_x' = B_x, \quad B_y' = \gamma\left(B_y + \frac{u}{c^2}E_z\right) \\ &B_z' = \gamma\left(B_z - \frac{u}{c^2}E_y\right) \end{aligned}}$$

Special Cases

$\boxed{\vec{B} = 0}$

$$\vec{B}' = \left[0, \frac{\gamma u}{c^2} E_z, -\frac{\gamma u}{c^2} E_y\right]$$

$$= \left[0, \frac{u}{c^2} E_z', -\frac{u}{c^2} E_y'\right]$$

$$= -\frac{1}{c^2} \vec{u} \times \vec{E}'$$

$$= \frac{1}{c^2} \vec{v} \times \vec{E}' \quad \text{if } \vec{v} = -\vec{u}$$

so $\vec{B} = \dfrac{\vec{v} \times \vec{E}}{c^2}$ for moving object

$\boxed{E = 0}$

$$\vec{E}' = [0, -\gamma u B_z, \gamma u B_y]$$

$$= [0, -u B_z', u B_y']$$

$$= \vec{u} \times \vec{B}'$$

$$= -\vec{v} \times \vec{B}' \quad \text{for moving object}$$

This is often called the motional electric field and is very important in space physics

Transformations

If S′ is moving with speed v in the positive x direction relative to S, then the coordinates of the same event in the two frames are related by:

Galilean transformation
(classical)

$$x' = x - ut$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

Lorentz transformation
(relativistic)

$$x' = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{u}{c^2}x)$$

Note: This assumes (0,0,0,0) is the same event in both frames.

Lorentz Transformation of 4-Vectors

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{bmatrix}$$

$$\beta = \frac{\mathrm{v}}{\mathrm{c}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}}$$

Quick Intro to Four-Vectors

Lorentz Transformation

$$x' = \gamma(x - ut)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - u/c^2\, x)$$

Rewrite: $$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$

Can represent as:

$$x^{\mu'} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

w/ repeated summation convention that:

$$\Lambda^{\mu}{}_{\nu}\, x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

Repeated Summation Convention

Einstein Summation Convention:

The convention that repeated indices are implicitly summed over. This can greatly simplify and shorten equations involving tensors. For example, using Einstein summation,

$$a_i\, a_i \equiv \sum_i a_i\, a_i$$

And

$$a_{i\,k}\, a_{i\,j} = \sum_i a_{i\,k}\, a_{i\,j}.$$

The convention was introduced by Einstein (1916, sec. 5), who later jested to a friend, "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..." (Kollros 1956; Pais 1982, p. 216).

- Any set of four quantities that transforms w/ this same matrix is a four-vector

$$x^\mu = (ct, x, y, z)$$
$$x_\mu = (-ct, x, y, z)$$

- The quantity $x^\mu x_\mu$
$$= -(ct)^2 + x^2 + y^2 + z^2$$
is invariant

Proof: $x'^\mu x'_\mu = -(ct')^2 + x'^2 + y'^2 + z'^2$
$$= -(\gamma(ct - \beta x))^2 + (\gamma(x - \beta ct))^2 + y^2 + z^2$$
$$= -\gamma^2((ct)^2 + (\beta x)^2 - 2\beta x ct)$$
$$+ \gamma^2(x^2 + (\beta ct)^2 - 2\beta x ct) + y^2 + z^2$$
$$= \gamma^2(1-\beta^2)(x^2 - (ct)^2) + y^2 + z^2$$
$$= -(ct)^2 + x^2 + y^2 + z^2$$
since $\gamma^2(1-\beta^2) = 1$ //

- The same is true for any 4-vector

Interval

$$\Delta x^\mu = x_A^\mu - x_B^\mu$$

$$I = \Delta x^\mu \Delta x_\mu$$

$$= -c^2 \Delta t^2 + \Delta r^2$$

$$= \text{const.}$$

$$= \text{Invariant Interval}$$

Δt & Δr depend on reference frame, but not I