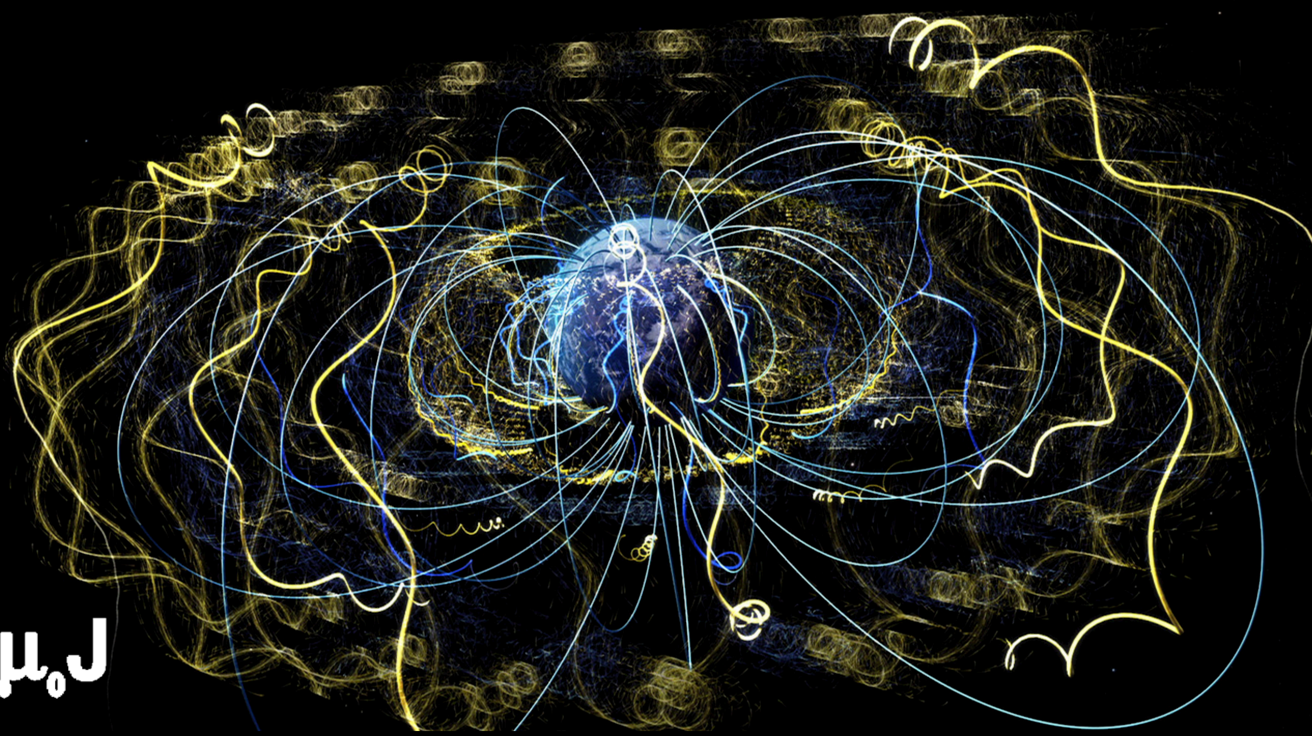
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas
Virtual by Zoom!
MWF 9:30-10:20 Lecture

Lorentz Transformation of 4-Vectors

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{bmatrix}$$

$$\beta = \frac{\mathrm{v}}{\mathrm{c}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}}$$

Proper Velocity

dx^μ/dt is not a 4-vector because dt is not invariant

$dx^\mu/d\tau$ is a 4-vector because $d\tau = \sqrt{1 - v^2/c^2}\, dt$ is an invariant

$$\eta^\mu = dx^\mu/d\tau$$

$$= \begin{pmatrix} c/\sqrt{1-v^2/c^2} \\ v_x/\sqrt{1-v^2/c^2} \\ v_y/\sqrt{1-v^2/c^2} \\ v_z/\sqrt{1-v^2/c^2} \end{pmatrix}$$

often written as

$$\eta^\mu = \begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix} \quad \text{w/ } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Proper Velocity & Energy-Momentum 4-Vectors

$$\boldsymbol{\eta}^{\mu} = \frac{\partial x_{\mu}}{\partial \tau} = \gamma \frac{\partial x_{\mu}}{\partial t} = \gamma \frac{\partial}{\partial t}(ct, \vec{x}) = \gamma (c, \frac{\partial \vec{x}}{\partial t})$$

$$\boldsymbol{p}^{\mu} = m\boldsymbol{\eta}^{\mu} = \left(\frac{E}{c}, p_x, p_y, p_z \right)$$

Energy-Momentum 4-Vector

$$p^\mu = m\eta^\mu = \begin{pmatrix} mc/\sqrt{1-v^2/c^2} \\ mv_x/\sqrt{1-v^2/c^2} \\ mv_y/\sqrt{1-v^2/c^2} \\ mv_z/\sqrt{1-v^2/c^2} \end{pmatrix}$$

is also a 4-vector

$p^\mu p_\mu$ is invariant

$$p^\mu p_\mu = \frac{-(mc)^2 + (mv_x)^2 + (mv_y)^2 + (mv_z)^2}{1-v^2/c^2}$$

$$= \frac{-m^2c^2 + m^2v^2}{1 - v^2/c^2}$$

$$= -m^2c^2$$

$$-m^2c^2 = \left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{mc}{\sqrt{1-v^2/c^2}}\right)^2$$

Identify $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1-v^2/c^2}}$

$E = mc^2/\sqrt{1-v^2/c^2}$

$$\Rightarrow -m^2c^2 = p^2 - (E/c)^2$$

$$\Rightarrow \boxed{E^2 = p^2c^2 + m^2c^4}$$

Energy - Momentum

E, $\vec{p}$ conserved in closed systems, just as in nonrelativistic case

Low-Velocity Limit

$$\vec{p} = m\vec{v}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\cdots\right)$$

$$\sim m\vec{v} \quad \text{if } v \ll c$$

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

$$\sim \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic energy}} \quad \text{if } v \ll c$$

Force and Work

$\vec{F} = d\vec{p}/dt$

$W = \int \vec{F} \cdot d\vec{l} = E_f - E_i$

w/ all quantities relativistic

Proof:

$$W = \int \vec{F} \cdot d\vec{l}$$

$$= \int \frac{d\vec{p}}{dt} \cdot d\vec{l} = \int \frac{d\vec{p}}{dt} \cdot \frac{d\vec{l}}{dt} dt$$

$$= \int d\vec{p}/dt \cdot \vec{v}\, dt$$

$$= \int d/dt \left(\frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \right) \cdot \vec{v}\, dt$$

$$= \int \left(\frac{m\, d\vec{v}/dt}{\sqrt{1 - v^2/c^2}} + \frac{m\vec{v} \cdot \vec{v}/c^2 \cdot d\vec{v}/dt}{(1 - v^2/c^2)^{3/2}} \right) \cdot \vec{v}\, dt$$

$$= \int \frac{(1 - v^2/c^2 + v^2/c^2)\, m\, d\vec{v}/dt}{(1 - v^2/c^2)^{3/2}} \cdot \vec{v}\, dt$$

$$= \int \frac{m\vec{v} \cdot d\vec{v}/dt}{(1 - v^2/c^2)^{3/2}} dt$$

$$= \int d/dt \left(\frac{mc^2}{\sqrt{1 - v^2/c^2}} \right) dt$$

$$= E_f - E_i$$

Minkowski Force

$\vec{F} = d\vec{p}/dt$ is not a 4-vector

$K^\mu = d\vec{p}/d\tau$ = Minkowski Force

$K^0 = \frac{1}{c}\, dE/d\tau$

$\vec{K} = d\vec{p}/d\tau = \vec{F}/\sqrt{1-v^2/c^2}$

Lorentz Force is an ordinary force

Field Tensor

- $\vec{E}$ and $\vec{B}$ are clearly not 4-vectors
- They're part of a tensor

Recall $\quad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

4-vector transformation

$$a^{\mu'} = \Lambda^{\mu}_{\ \nu} a^{\nu} \quad (1 \text{ summation})$$

Tensor transformation

$$t^{\mu\nu'} = \Lambda^{\mu}_{\ \lambda} \Lambda^{\nu}_{\ \sigma} t^{\lambda\sigma} \quad (2 \text{ sums})$$

w/ $t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}$

Field Tensor

"Field Tensor"

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

"Dual Tensor"

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Invariants

Note: $F_{\mu\nu}$ same as $F^{\mu\nu}$
but w/ values in
0-th row & column
opposite

$$F^{\mu\nu}F_{\mu\nu} = \left(-E^2/c^2 + B^2\right) \cdot 2$$

$$G^{\mu\nu}G_{\mu\nu} = \left(-B^2 + E^2/c^2\right) \cdot 2$$

$\Rightarrow E^2 - c^2 B^2$ invariant

$$F^{\mu\nu}G_{\mu\nu} = \frac{-\vec{E}\cdot\vec{B}}{c} \cdot 4$$

$\Rightarrow \vec{E}\cdot\vec{B}$ invariant