Classically there are particles (matter)

waves (radiation)

Motivate Structure of QM by looking at EM waves

Two observational facts at odds with classical physics

1) black body spectrum
2) photoelectric effect

both solved by PHOTONS with energy $E = N\hbar\omega$

Let's see consequences of matching classical EM to quantum $E = n\hbar\omega$

<u>Classical</u> EM wave propagating in z direction

$$\vec{E}(\vec{r},t) = \begin{pmatrix} E_x(\vec{r},t) \\ E_y(\vec{r},t) \\ 0 \end{pmatrix} = \begin{pmatrix} E_x^0 \cos(kz - \omega t + \alpha_x) \\ E_y^0 \cos(kz - \omega t + \alpha_y) \\ 0 \end{pmatrix} \quad \text{with } k = \frac{2\pi}{\lambda}$$

$$E_x(\vec{r},t) = \overbrace{E_x^0 e^{i\alpha_x}}^{E_x} e^{i(kz-\omega t)}$$

$$E_y(\vec{r},t) = \underbrace{E_y^0 e^{i\alpha_y}}_{E_y} e^{i(kz-\omega t)}$$

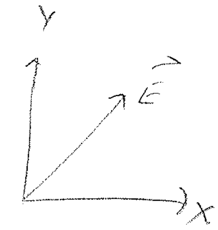
obey wave equation so $\mathbb{R}$ part does too

Polarization States

1) $E_y = 0$: x-polarized

2) $E_x = 0$ y-polarized

3) $E_x = E_y$ polarized at 45° :



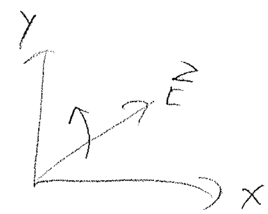
4) $E_y = e^{i\pi/2} E_x = i E_x$

y-component lags x-component by 90°

RIGHT CIRCULARLY POLARIZED

$\mathrm{Re}\, \hat{E}_y(\vec{r},t) \sim \cos(kz - \omega t + \pi/2)$

$\mathrm{Re}\, E_x(\vec{r},t) \sim \cos(kz - \omega t)$



⋮

Energy density $\mathcal{E}(\hat{r},t) = \epsilon_0 \left[|E_x|^2 \cos^2(kz - \omega t + \alpha_x) + |E_y|^2 \cos^2(kz - \omega t + \alpha_y) \right]$

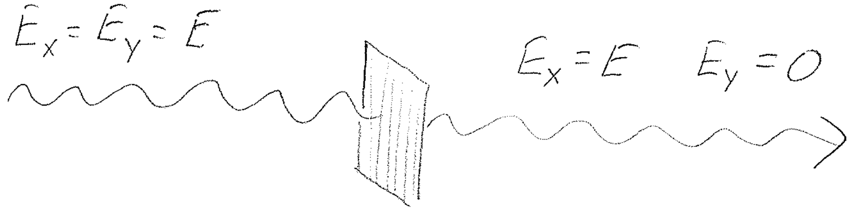
Integrate over $V \gg \lambda^3$ to get total energy

$$\mathcal{E}_{total} = \int_V d^3r\, \mathcal{E}(\hat{r},t) = \epsilon_0 \left(|E_x|^2 + |E_y|^2 \right) \cdot \underbrace{\tfrac{1}{2}}_{\text{avg. of } \cos^2()} \cdot \underbrace{V}_{\text{volume}}$$

$$= \tfrac{1}{2} \epsilon_0 |E|^2 \cdot V$$

1,3

Take a wave polarized 45° & pass through a filter that passes x-polarized



$\mathcal{E}_{tot} \propto |E_x|^2 + |E_y|^2$

$\propto 2|E|^2$

$\mathcal{E}_{tot} \propto |E_x|^2$

$\propto |E|^2$

Energy cut in half

<u>Quantum</u> $\mathcal{E}_{tot} = N\hbar\omega \qquad N = \text{integer}$

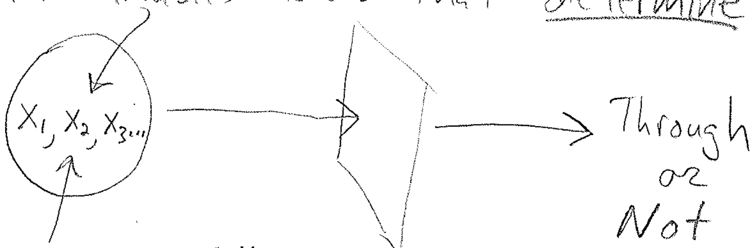
can't have ½ a photon so $n/2$ photons pass

can't let ½ of photon through before knowing how many <u>will</u> be. (requires predicting future)

$\Rightarrow$ $\boxed{\text{photons must pass through polarizer probabilistically.}}$

Note: there is a conceivable loophole

hidden variables . . . that <u>determine</u> if photon gets through



depending on $X_1, X_2, \ldots$

determined by emitter in complicated way that SEEMS random.

1,4 If hidden variables exist...

correlations between photons from single emitter

predicted experimental effect. (Bell's theorem)

experiments violate the prediction

Quantum Mechanics is Probabilistic. Period.

(details of Bell's theorem later)

Probability of passing = $\dfrac{\text{Energy through}}{\text{Energy incident}}$

$$\frac{|E_x|^2}{|E_x|^2 + |E_y|^2} = \frac{|E_x|^2}{|E|^2} = \cos^2\theta$$

or if $\vec{E} = \vec{E}_{RCP} + \vec{E}_{LCP}$

$$\text{Prob} = \frac{|\vec{E}_{RCP}|^2}{|\vec{E}_{RCP}|^2 + |\vec{E}_{LCP}|^2}$$

1.5

Classical $\Leftrightarrow$ Quantum

$$\mathcal{E}_{tot} = \frac{\epsilon_0}{2} |E|^2 V = N\hbar\omega$$

consider one photon $N = 1$:

$$|E|^2 \cdot \frac{\epsilon_0 V}{2\hbar\omega} = 1$$

DEFINE $|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$ w/ $\psi_x = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}}\, E_x$, $\psi_y = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}}\, E_y$

Dirac Notation

"KET" shorthand for which is a 2-D $\mathbb{C}$ vector

Because $\frac{|E|^2 \epsilon_0 V}{2\hbar\omega} = 1 \rightarrow |\psi_x|^2 + |\psi_y|^2 = 1$ independent of V

e.g. $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\alpha} \\ e^{i\alpha} \end{pmatrix}$ is polarized at 45° to X axis

$|X\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ X polarized

$|Y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Y polarized

$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ R circularly polarized

$|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ L circularly polarized

1.6

Define corresponding row vector

$$\underbrace{\langle\psi|}_{} = (\psi_x^* \ \ \psi_y^*)$$

"BRA" (Bra-ket)

$$\& \ \langle\phi|\psi\rangle = \begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix}^{\dagger} \begin{pmatrix}\psi_x \\ \psi_y\end{pmatrix} \equiv \phi_x^* \psi_x + \phi_y^* \psi_y = \langle\psi|\phi\rangle^*$$

because $|\psi_x|^2 + |\psi_y|^2 = 1$

$$\langle\psi|\psi\rangle = 1 \qquad \langle x|x\rangle = \langle y|y\rangle = 1 = \langle R|R\rangle = \langle L|L\rangle$$

Last time...

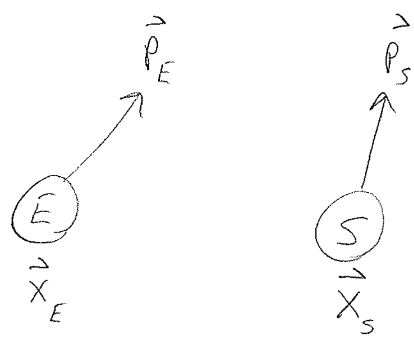
$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

$\uparrow$ ket $\qquad$ $\uparrow$ ψ_x, ψ_y complex numbers

$$|\psi\rangle = \psi_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \psi_x |x\rangle + \psi_y |y\rangle$$

Kets or state vectors are the fundamental objects in quantum mechanics rather than particle coordinates.



$$\Rightarrow \quad |\vec{x}_E, \vec{p}_E, \vec{x}_S, \vec{p}_S, M_E, M_S\rangle$$

coordinates & momenta evolve according to a differential equation

?

$|x\rangle$ & $|y\rangle$ are orthoganal : $\langle x|y\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\dagger}\begin{pmatrix}0\\1\end{pmatrix} = 0$

said to form an orthonormal basis

$$|\psi\rangle = \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} = \psi_x\,|x\rangle + \psi_y\,|y\rangle$$

similarly for $|R\rangle$ & $|L\rangle$

$$|\psi\rangle = \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} = \frac{\psi_x - i\psi_y}{\sqrt{2}}\,|R\rangle + \frac{\psi_x + i\psi_y}{\sqrt{2}}\,|L\rangle$$

take scalar product w/ $\langle x|$

$$\langle x|\psi\rangle = \psi_x \underbrace{\langle x|x\rangle}_{1} + \psi_y \underbrace{\langle x|y\rangle}_{0} = \psi_x$$

1.8

from classical EM

$$\vec{L} = \frac{1}{\mu_0 c^2} \int d^3r \, \vec{r} \times \underbrace{(\vec{E} \times \vec{B})}_{\vec{P} = \frac{1}{c^2}\vec{S}} \quad \left[\text{Jackson, prob. } 7.27\right]$$

$(\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \vec{B}/\mu_0)$

Jackson eq. 6.123

- plane wave in z-direction
- finite width beam → $\vec{E} \times \vec{B}$ has $\hat{z}$ component near edge of beam!

$$L_z = \frac{V}{2\omega} \frac{1}{\mu_0 c^2} \left(\underbrace{\left| \frac{E_x - iE_y}{\sqrt{2}} \right|^2}_{E_{RCP}} - \underbrace{\left| \frac{E_x + iE_y}{\sqrt{2}} \right|^2}_{E_{LCP}} \right)$$

That was classical, For quantum,

plug in $E_{x,y} = \sqrt{\frac{2\hbar\omega}{V\epsilon_0}} \, \psi_{x,y}$ ($\hat{E}$ for one photon)

$$L_z = \hbar \left(\left| \frac{\psi_x - i\psi_y}{\sqrt{2}} \right|^2 - \left| \frac{\psi_x + i\psi_y}{\sqrt{2}} \right|^2 \right)$$

$$= \hbar \, |\langle R | \psi \rangle|^2 - \hbar \, |\langle L | \psi \rangle|^2$$

$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$|R\rangle$ has $L_z = +\hbar$ $\qquad$ $|L\rangle$ has $L_z = -\hbar$

Expt: absorption changes L_z by $\pm\hbar$

NOT by something between $-\hbar$ & $+\hbar$

1.9

So, we interpret L_z for $|\psi\rangle = \alpha|R\rangle + \beta|L\rangle$

as a average value over many expts

photons in eigenstates $|R\rangle$ & $|L\rangle$ have definite

$$L_z = \pm\hbar$$

$$(\text{avg of } L_z) = \hbar \underbrace{|\langle R|\psi\rangle|^2}_{\substack{\text{prob. that} \\ |\psi\rangle \text{ has } L_z = +\hbar}} - \hbar \underbrace{|\langle L|\psi\rangle|^2}_{\substack{\text{prob that} \\ |\psi\rangle \text{ has } L_z = -\hbar}}$$

For a given $|\psi\rangle$ the avg value of L_z is given by the EXPECTATION VALUE

$$(\text{avg of } L_z) = \hbar\left(|\langle R|\psi\rangle|^2 - |\langle L|\psi\rangle|^2\right)$$

$$= \langle\psi|\,\hbar\left(|R\rangle\langle R|\psi\rangle - |L\rangle\langle L|\psi\rangle\right)$$

$$= \langle\psi|\,\hbar \underset{\sim}{S}\left(|R\rangle\langle R|\psi\rangle + |L\rangle\langle L|\psi\rangle\right)$$

$$= \langle\psi|\,\hbar \underset{\sim}{S}\,|\psi\rangle$$

we've ASSUMED some $\underset{\sim}{S}$ such that $\underset{\sim}{S}|R\rangle = |R\rangle$

$\underset{\sim}{S}|L\rangle = -|L\rangle$

1.10

Outer Product

$$|\psi\rangle\langle\phi| \equiv \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix}(\phi_x^* \ \phi_y^*) = \begin{pmatrix}\psi_x\phi_x^* & \psi_x\phi_y^*\\ \psi_y\phi_x^* & \psi_y\phi_y^*\end{pmatrix}$$

e.g. $|X\rangle\langle X| = \begin{pmatrix}1\\0\end{pmatrix}\otimes(1\ 0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$$|Y\rangle\langle Y| = \begin{pmatrix}0\\1\end{pmatrix}\otimes(0\ 1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$|X\rangle\langle Y| = \begin{pmatrix}1\\0\end{pmatrix}\otimes(0\ 1) = \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$|Y\rangle\langle X| = \begin{pmatrix}0\\1\end{pmatrix}\otimes(1\ 0) = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

we've been writing things like $\underbrace{|R\rangle}_{\text{ket}}\underbrace{\langle R|\psi\rangle}_{\text{number}}$

which we interpretted as $|R\rangle(\langle R|\psi\rangle)$
but outer product shows us that associativity doesn't matter; $\underbrace{(|R\rangle\langle R|)}_{\text{matrix}}|\psi\rangle$

$$|X\rangle\langle X| + |Y\rangle\langle Y| = \underline{\mathbb{1}}$$

Completeness relation. True for any orthonormal basis

1.11

$$\mathbb{1}|\psi\rangle = \big(|x\rangle\langle x| + |y\rangle\langle y|\big)|\psi\rangle$$

$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

expansion we had before!

what about $S \cdot \mathbb{1} = S\big(|R\rangle\langle R| + |L\rangle\langle L|\big)$

$$= S|R\rangle\langle R| + S|L\rangle\langle L|$$

$$= |R\rangle\langle R| - |L\rangle\langle L|$$

Born Rule

this gives us S in the basis of $|R\rangle$ & $|L\rangle$

we've seen

$|\langle R|\psi\rangle|^2$ = Prob (photon in state $|\psi\rangle$ tranfers $L_z = +\hbar$ when absorbed)

= Prob (photon in state $|\psi\rangle$ passes filter that lets through R)

therefore

= Prob (photon behaves as in state $|R\rangle$)

$$|\psi\rangle = |R\rangle\underbrace{\langle R|\psi\rangle}_{\text{amplitude to be R}} + |L\rangle\underbrace{\langle L|\psi\rangle}_{\text{amplitude to be L}}$$

1.12

Classically, particles are in particular states

e.g. at $\vec{r}$ moving at $\vec{v}$

Quantum: particle in state $|\psi\rangle$ always
has probability of being in state $|\phi\rangle$
unless $|\psi\rangle$ & $|\phi\rangle$ are orthogonal

"Prob. of Being in state $|\phi\rangle$" $\equiv$ "behaving in an expt as if it were in a state $|\phi\rangle$"

Using the usual rules for composition of probabilities...

Prob. of x-pol photon going through y-polaroid =

$$\underbrace{|\langle R|X\rangle|^2}_{\text{Prob of X-pol photon being R-pol}} \; \underbrace{|\langle Y|R\rangle|^2}_{\text{Prob of R-pol photon passing through Y-polaroid}} + \underbrace{|\langle L|X\rangle|^2}_{\text{Prob. of X-photon being L}} \cdot \underbrace{|\langle Y|L\rangle|^2}_{\text{prob of L-photo going through Y-polaroid}}$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$ Wrong

Should be 0

1.13 Prob. of x-pol photon going through

y-polaroid $= |\langle y|x\rangle|^2 = 0$ ✓

But we could have used R, L basis

$$|x\rangle = |R\rangle\langle R|x\rangle + |L\rangle\langle L|x\rangle$$

Prob. AMP. of x-pol photon passing through y-polaroid

$$= \langle y|\left[\,|R\rangle\langle R|x\rangle + |L\rangle\langle L|x\rangle\,\right]$$

$$P(x \text{ thru } y) = |\quad|^2 = \underbrace{|\langle y|R\rangle|^2\,|\langle R|x\rangle|^2 + |\langle y|L\rangle|^2\,|\langle L|x\rangle|^2}_{\text{naive classical result}} +$$

Cancelled by:

$$+ \left(\langle y|R\rangle\langle R|x\rangle\right)^* \langle y|L\rangle\langle L|x\rangle + \langle y|R\rangle\langle R|x\rangle\left(\langle y|L\rangle\langle L|x\rangle\right)^*$$

$$\langle x|R\rangle = \begin{pmatrix}1\\0\end{pmatrix}^\dagger \frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt2} \qquad \langle y|R\rangle = \begin{pmatrix}0\\1\end{pmatrix}^\dagger \frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix} = \frac{i}{\sqrt2}$$

$$\langle x|L\rangle = \begin{pmatrix}1\\0\end{pmatrix}^\dagger \frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt2} \qquad \langle y|L\rangle = \begin{pmatrix}0\\1\end{pmatrix}^\dagger \frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{-i}{\sqrt2}$$

$$P(x \text{ thru } y) = \frac12\cdot\frac12 + \frac12\cdot\frac12 + \left(\frac{i}{\sqrt2}\frac{1}{\sqrt2}\right)^* \frac{-i}{\sqrt2}\frac{1}{\sqrt2} + \frac{i}{\sqrt2}\frac{1}{\sqrt2}\left(\frac{-i}{\sqrt2}\frac{1}{\sqrt2}\right)^*$$

$$= \frac14 + \frac14 - \frac14 \qquad - \frac14$$

$$= 0 \checkmark$$

1.14

operator maps $|\alpha\rangle \rightarrow |\beta\rangle = \underset{\sim}{O}|\alpha\rangle$

~~ad~~ operators can be added:

$$\underset{\sim}{X} + \underset{\sim}{Y} = \underset{\sim}{Y} + \underset{\sim}{X}$$

$$\underset{\sim}{X} + (\underset{\sim}{Y} + \underset{\sim}{Z}) = (\underset{\sim}{X} + \underset{\sim}{Y}) + \underset{\sim}{Z}$$

Linear operator: $\underset{\sim}{O}\left[c_\alpha|\alpha\rangle + c_\beta|\beta\rangle\right] = c_\alpha \underset{\sim}{O}|\alpha\rangle + c_\beta \underset{\sim}{O}|\beta\rangle$

acting on bra: $\langle\alpha| O$

Hermitian adjoint: $\left(\underset{\sim}{X}|\alpha\rangle\right)^\dagger = \langle\alpha|\underset{\sim}{X}^\dagger$

Hermitian operator: $\underset{\sim}{X} = \underset{\sim}{X}^\dagger$

Multiplication in general $\underset{\sim}{X}\underset{\sim}{Y} \neq \underset{\sim}{Y}\underset{\sim}{X}$

but it is true that $(\underset{\sim}{X}\underset{\sim}{Y})\underset{\sim}{Z} = \underset{\sim}{X}(\underset{\sim}{Y}\underset{\sim}{Z})$

$$(\underset{\sim}{X}\underset{\sim}{Y})^\dagger = \underset{\sim}{Y}^\dagger \underset{\sim}{X}^\dagger$$

W $\rightarrow$

1.15

Projection operator

$$\underset{\sim}{P}_\alpha = |\alpha\rangle\langle\alpha| \qquad \underset{\sim}{P}_\alpha |\psi\rangle = |\alpha\rangle\langle\alpha|\psi\rangle$$

projects out the $|\alpha\rangle$ component

Just like the $\hat{a}$ component of $\vec{V}$ is

$$\vec{V}_a = \hat{a}\,(\hat{a}\cdot\vec{V})$$

Identity operator

if $|\alpha\rangle$'s form a complete orthonormal set

$$\underset{\sim}{\mathbb{1}} = \sum_\alpha |\alpha\rangle\langle\alpha| \quad \& \quad \underset{\sim}{\mathbb{1}}\,|\psi\rangle = |\psi\rangle \quad \forall\, |\psi\rangle$$

Eigenstates

Operator $\underset{\sim}{O}$ maps $|\alpha\rangle \rightarrow |\beta\rangle = \underset{\sim}{O}\,|\alpha\rangle$

eigenstate: $\underset{\sim}{O}\,|a_i\rangle = a_i\,|a_i\rangle$

label with eigenvalue

eigenvalue

Hermitian operators are special: they represent observables

<u>Theorem</u> eigenvalues of $\underset{\sim}{A}^{\dagger} = \underset{\sim}{A}$ are $\mathbb{R}$ & eigenkets corresponding to different eigenvalues are orthogonal

$$\langle \alpha_2 | \{ \quad \underset{\sim}{A} |\alpha_1\rangle = \alpha_1 |\alpha_1\rangle \}$$

$$\{ \langle \alpha_2 | \underset{\sim}{A} = \alpha_2^* \langle \alpha_2 | \} |\alpha_1\rangle$$

$$\langle \alpha_2 | \underset{\sim}{A} | \alpha_1 \rangle = \alpha_1 \langle \alpha_2 | \alpha_1 \rangle$$

$$\langle \alpha_2 | \underset{\sim}{A} | \alpha_1 \rangle = \alpha_2^* \langle \alpha_2 | \alpha_1 \rangle$$

$$0 = (\alpha_1 - \alpha_2^*) \langle \alpha_2 | \alpha_1 \rangle$$

α_1 & α_2 the same $\rightarrow \langle \alpha_2|\alpha_1\rangle \neq 0 \rightarrow \quad \alpha_1 - \alpha_2^* = 0 \qquad \alpha_1 = \alpha_2 \in \mathbb{R}$

α_1 & α_2 different $\rightarrow \alpha_1 - \alpha_2^* \neq 0 \rightarrow \quad \langle \alpha_2|\alpha_1\rangle = 0$

The eigenstates of a Hermitian operator form a complete set

$$|\psi\rangle = \sum_{\alpha} c_\alpha |\alpha\rangle$$

arbitrary

complex numbers

eigenkets of $\underset{\sim}{A} = \underset{\sim}{A}^{\dagger}$

Note: we use normalized states $\langle \psi|\psi\rangle = 1$

1.17

Abstract bra-kets, concrete matrices

$$|\psi\rangle = \sum_j c_j |a_j\rangle = \sum_j |a_j\rangle\langle a_j|\psi\rangle$$

act with some arbitrary operator $\underset{\sim}{O}|\psi\rangle$

$$\underset{\sim}{O}|\psi\rangle = \sum_i c_i \underset{\sim}{O}|a_i\rangle$$

Insert $\underset{\sim}{1}$:

$$\underset{\sim}{O}|\psi\rangle = \underset{\sim}{1}\,\underset{\sim}{O}\,\underset{\sim}{1}|\psi\rangle = \left(\sum_i |a_i\rangle\langle a_i|\right)\underset{\sim}{O}\left(\sum_j |a_j\rangle\langle a_j|\right)|\psi\rangle$$

$$= \sum_{i,j} |a_i\rangle \underbrace{\underbrace{\langle a_i|\underset{\sim}{O}|a_j\rangle}_{O_{ij}\,\leftarrow\,\text{"matrix element"}} \underbrace{\langle a_j|\psi\rangle}_{c_j}}$$

$$= \sum_i |a_i\rangle \cdot c'_i$$

c'_i ← coefficients determine $|\psi\rangle$ in the $|a_i\rangle$ basis

$$\underbrace{\begin{pmatrix} c'_1 \\ c'_2 \\ \vdots \end{pmatrix}}_{c'_i} = \underbrace{\begin{pmatrix} \langle a_1|\underset{\sim}{O}|a_1\rangle & \langle a_1|\underset{\sim}{O}|a_2\rangle & \cdots \\ \langle a_2|\underset{\sim}{O}|a_1\rangle & \langle a_2|\underset{\sim}{O}|a_2\rangle & \\ \vdots & & \ddots \end{pmatrix}}_{O_{ij}} \underbrace{\begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \\ \vdots \end{pmatrix}}_{c_j}$$

Example: spin-1/2

another 2-state system like photon polarization

states are $|+\rangle$ & $|-\rangle$ (or $|\uparrow\rangle$ & $|\downarrow\rangle$, spin up or down wrt a magnetic field $\vec{B}$)

orthonormal: $\langle +|+\rangle = \langle -|-\rangle = 1 \quad \langle +|-\rangle = \langle -|+\rangle = 0$

like photon polarization, these states have particular angular momentum (but $\pm\hbar/2$) along $\hat{z}$ axis, the direction of $\vec{B}$.

eigenstates of $\underset{\sim}{S}_z$: $\quad \underset{\sim}{S}_z |\pm\rangle = \pm \hbar/2 \, |\pm\rangle$

$$\underset{\sim}{S}_z = \frac{\hbar}{2} \underbrace{|+\rangle\langle +|}_{\text{project } |+\rangle} - \frac{\hbar}{2} \underbrace{|-\rangle\langle -|}_{\text{project } |-\rangle} \qquad \underset{\sim}{1} = |+\rangle\langle +| + |-\rangle\langle -|$$

eigenvalue for that state

Other operators:

$$\underset{\sim}{S}_+ = \hbar\, |+\rangle\langle -| \quad \& \quad \underset{\sim}{S}_- = \hbar\, |-\rangle\langle +|$$

$$\left.\begin{aligned} \underset{\sim}{S}_+ |-\rangle &= \hbar\, |+\rangle\langle -|-\rangle = \hbar\, |+\rangle \\ \underset{\sim}{S}_- |+\rangle &= \hbar\, |-\rangle\langle +|+\rangle = \hbar\, |-\rangle \end{aligned}\right\} \text{these raise or lower the spin (\& multiply by } \hbar\text{)}$$

1.19 Matrix forms:

$$\underset{\sim}{S}_z = \begin{pmatrix} \langle + | \underset{\sim}{S}_z | + \rangle & \langle + | \underset{\sim}{S}_z | - \rangle \\ \langle - | \underset{\sim}{S}_z | + \rangle & \langle - | \underset{\sim}{S}_z | - \rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\underset{\sim}{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underset{\sim}{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

NON-Hermitian

Measurement

photon angular momentum can be $\pm\hbar$
electron spin angular momentum can be $\pm\hbar/2$

QM can only give probabilities of these values being measured

Once measured, you CAN'T get something else by measuring a 2nd time because it would violate conservation laws!

Observable $\underset{\sim}{A}$ with eigenvalues a_n corresponds to a measurement:

$$\underbrace{|\psi\rangle}_{\text{arbitrary state}} = \sum_n |a_n\rangle\langle a_n|\psi\rangle \xrightarrow{\text{measurement}} \underbrace{|a_m\rangle}_{\text{one state results with probability } |\langle a_m|\psi\rangle|^2 \text{ resulting in observed value } a_m}$$

$$\underset{\sim}{A} = \sum_n \underset{\substack{\uparrow \\ \text{observed value}}}{a_n} \overbrace{|a_n\rangle\langle a_n|}^{\text{projection}}$$

This just projects to states for which $\underset{\sim}{A}$ "is a number"

$$\langle\psi|\underset{\sim}{A}|\psi\rangle = \sum_n a_n \underbrace{\langle\psi|a_n\rangle\langle a_n|\psi\rangle}_{\text{probability of measuring this value}}$$

Born rule (a postulate)

measurement results in a_n w/ probability $|\langle a_n|\psi\rangle|^2$