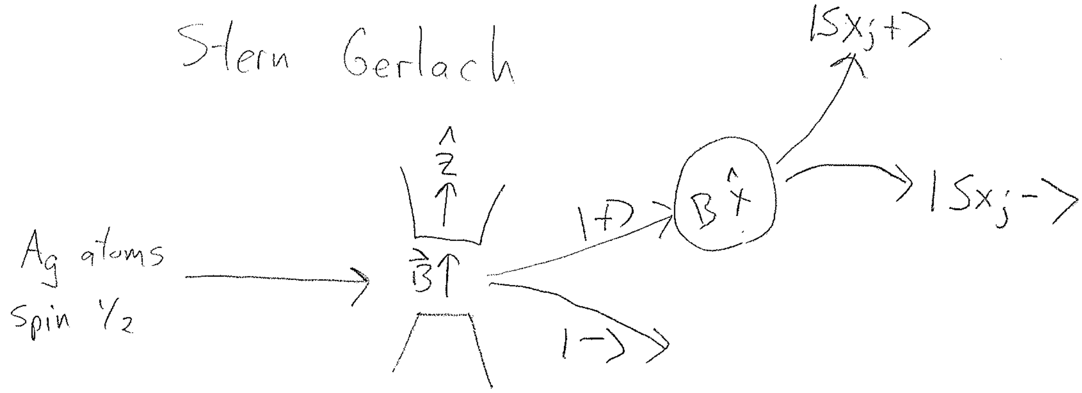
2.1 Stern Gerlach



$|\pm\rangle$ spin $\pm\hbar/2$ wrt $\hat{z}$ axis

$|S_x;\pm\rangle$ spin $\pm\hbar/2$ wrt $\hat{x}$ axis

atom in $|+\rangle$ beam has 50/50 probability of being $|S_x;+\rangle$ or $|S_x;-\rangle$

$$\Rightarrow \quad |\langle +|S_x;+\rangle| = |\langle -|S_x;+\rangle| = \frac{1}{\sqrt{2}}$$

probability $= \frac{1}{2} \rightarrow$ |prob. amplitude| $= \frac{1}{\sqrt{2}}$

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)e^{i\theta_+}$$

this satisfies $|\langle +|S_x;+\rangle| = \frac{1}{\sqrt{2}}$

$$\langle +|\frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)e^{i\theta_+} = \frac{1}{\sqrt{2}}\langle x|x\rangle e^{i\theta_+}$$

$|S_x;-\rangle$ has to look similar, but be $\perp$:

$$|S_x;-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \overset{\star}{-} e^{i\delta_1}|-\rangle\right)e^{i\theta_-}$$

rea

2.2

$$\langle S_x;-|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(\langle+| - e^{-i\delta_1}\langle-|\right)\frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$= \frac{1}{2}\left[\langle+|+\rangle - e^{i\delta_1 - i\delta_1}\langle-|-\rangle\right] = 0 \quad \checkmark$$

$$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle \pm e^{i\delta_1}|-\rangle\right]e^{i\theta_\pm}$$

$$|S_y;\pm\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle \pm e^{i\delta_2}|-\rangle\right]e^{i\gamma_\pm}$$

Construct operator $\underset{\sim}{S}_x$ by $\sum$(eigenvalue)×(projection op)

$$\underset{\sim}{S}_x = \frac{\hbar}{2}|S_x+\rangle\langle S_x+| - \frac{\hbar}{2}|S_x-\rangle\langle S_x-|$$

$$= \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left[|+\rangle + e^{i\delta_1}|-\rangle\right]e^{i\theta_+}\cdot\frac{1}{\sqrt{2}}\left[\langle+| + e^{-i\delta_1}\langle-|\right]e^{-i\theta_+}$$

$$-\frac{\hbar}{2}\frac{1}{\sqrt{2}}\left[|+\rangle - e^{i\delta_1}|-\rangle\right]e^{i\theta_-}\frac{1}{\sqrt{2}}\left[\langle+| - e^{-i\delta_1}\langle-|\right]e^{-i\theta_-}$$

$$= \frac{\hbar}{2}\left[e^{-i\delta_1}|+\rangle\langle-| + e^{i\delta_1}|-\rangle\langle+|\right]$$

matrix form :
$$\begin{pmatrix}|+\rangle\\|-\rangle\end{pmatrix}^{+}\frac{\hbar}{2}\begin{pmatrix}0 & e^{-i\delta_1}\\ e^{i\delta_1} & 0\end{pmatrix}\begin{pmatrix}|+\rangle\\|-\rangle\end{pmatrix}$$

2.3

Similarly

$$\underset{\sim}{S}_y = \frac{\hbar}{2}\left[e^{-i\delta_2} |+\rangle\langle -| + e^{i\delta_2} |-\rangle\langle +| \right]$$

matrix form: $\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}^{\dagger} \frac{\hbar}{2} \begin{pmatrix} 0 & e^{-i\delta_2} \\ e^{i\delta_2} & 0 \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$

what are δ_1 & δ_2 ?

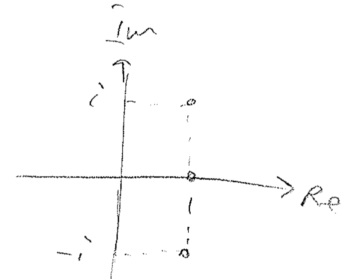
just as $|\langle + | S_x + \rangle| = |\langle - | S_x + \rangle| = \frac{1}{\sqrt{2}}$

also $\quad \left| \langle S_x \pm | \, S_y + \rangle \right| = \frac{1}{\sqrt{2}} \quad$ by rotating axes

$$\left| e^{-i\theta_+} \frac{1}{\sqrt{2}} \left(\langle + | \pm e^{-i\delta_1} \langle - | \right) \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\delta_2} |-\rangle \right) e^{i\theta_+} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{1}{2} \left| \langle + | + \rangle \pm e^{i(\delta_2 - \delta_1)} \langle - | - \rangle \right| = \frac{1}{\sqrt{2}}$$

$$\frac{1}{2} \left| 1 \pm \underbrace{e^{i(\delta_2 - \delta_1)}}_{\pm i} \right| = \frac{1}{\sqrt{2}}$$



$$\rightarrow \delta_2 - \delta_1 = \pm \pi/2$$

Only the difference is determined, so we're free to choose one of δ's. Convention: $\delta_1 = 0 \rightarrow \delta_2 = \pm \pi/2$

2.4

$\delta_2 = \pi/2$ (not $-\pi/2$ because... reasons.)

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right) \qquad \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$S_y = \frac{\hbar}{2}\left(-i|+\rangle\langle -| + i|-\rangle\langle +|\right) \qquad \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right) \qquad \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

Pauli matrices

S_i obey

commutation relation $\quad S_i S_j - S_j S_i = \underbrace{[S_i, S_j]}_{\text{commutator}} = i\epsilon_{ijk}\hbar S_k$

$\epsilon_{xyz} = 1 = -\epsilon_{zyx}$

$\epsilon_{xxy} = 0$ etc.

anticommutation relation $\quad S_i S_j + S_j S_i = \underbrace{\{S_i, S_j\}}_{\text{anti-commutator}} = \frac{1}{2}\hbar^2\delta_{ij}$

$\vec{S} \equiv (S_x, S_y, S_z)$ form a <u>vector of operators</u>

define $S^2 = \vec{S}\cdot\vec{S} = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{2}\delta_{xx} + \frac{\hbar^2}{2}\delta_{yy} + \frac{\hbar^2}{2}\delta_{zz} = \frac{3}{4}\hbar^2$

2.5

1) $\underset{\sim}{S}^2 = (\underset{\sim}{S}^2)^\dagger$

2) $\underset{\sim}{S}^2 \propto \underset{\sim}{1}$ so $[\underset{\sim}{S}_i, \underset{\sim}{S}^2] = 0$

Operators $[\underset{\sim}{A}, \underset{\sim}{B}] = 0$ compatible

$[\underset{\sim}{A}, \underset{\sim}{B}] \neq 0$ incompatible

Degenerate states: more than one w/ same eigenvalue

If $[\underset{\sim}{A}, \underset{\sim}{B}] = 0$ & we have eigenstates $\underset{\sim}{A} |a'\rangle = a' |a'\rangle$

then $\langle a'' | \underset{\sim}{B} | a' \rangle$ is diagonal

2.6

(for $\hat{A}$ non degenerate)

Proof ↑ $\langle a''|[\hat{A},\hat{B}]|a'\rangle = (a''-a')\underbrace{\langle a''|B|a'\rangle}_{\substack{=0 \\ \text{unless } a''=a'}} = 0$

$$\langle a''|\underset{\sim}{B}|a'\rangle = \delta_{a''a'}\langle a'|\underset{\sim}{B}|a'\rangle$$

Also if $[\hat{A},\hat{B}]=0$ then $|a'\rangle$'s are e-kets of $\hat{B}$ too

proof $\underset{\sim}{B} = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$

$$\underset{\sim}{B}|a'\rangle = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle$$

gives $\delta_{a'a''}$

Above assuming $\hat{A}$ has nondegenerate states, ALSO TRUE IF DEGENERATE c.f. Sakurai

Among the degenerate e-kets of $\hat{A}$ will be e-kets of $\hat{B}$ w/ various e-vals b' which will distinguish states in the degenerate (wrt $\hat{A}$) set.

maximal set of commuting observables $\hat{A}, \hat{B}, \hat{C}, \ldots$

States: $|a', b', c', \ldots\rangle$

$$\sum_{a'}\sum_{b'}\sum_{c'} \ldots |a', b', c', \ldots\rangle\langle a', b', c', \ldots| = \mathbb{1}$$

2.7 <u>Uncertainty</u>

define $\Delta A \equiv A - \langle A \rangle$

$$\langle \Delta A^2 \rangle = \langle (A^2 - 2A\langle A\rangle + \langle A\rangle^2) \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

dispersion / mean square deviation / variance

any state $\Rightarrow$ statement about OPERATORS

$$* \quad \langle \Delta A^2 \rangle \langle \Delta B^2 \rangle \geq \frac{1}{4} | \langle [A, B] \rangle |^2$$

generalized uncertainty relation

Mon $\longrightarrow$

(1) Schwarz inequality

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq | \langle \alpha | \beta \rangle |^2 \qquad \left[|\vec{a}|^2 |\vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|^2 \right.$$

<u>Proof</u> $\quad (\langle \alpha | + \lambda^* \langle \beta |)(|\alpha\rangle + \lambda |\beta\rangle) \geq 0$

set $\quad \lambda = -\langle \beta | \alpha \rangle / \langle \beta | \beta \rangle$

QED

(2) $\langle$ (Hermitian op) $\rangle$ is $\mathbb{R}$

(3) $\langle$ (anti-Hermitian) $\rangle$ is pure Im

2.8 uncertainty proof

1 with $|\alpha\rangle = \Delta A |\overset{\text{any}}{\ }\rangle \qquad |\beta\rangle = \Delta B |\ \rangle$

$$\langle \Delta A^2 \rangle \langle \Delta B^2 \rangle \geq |\langle \Delta A\, \Delta B \rangle|^2 \quad (\Delta A \,\&\, \Delta B \text{ Hermitian})$$

$$\Delta A\, \Delta B = \frac{1}{2}\underbrace{[\Delta A, \Delta B]}_{\text{anti-Herm}} + \frac{1}{2}\underbrace{\{\Delta A, \Delta B\}}_{\text{Herm}}$$

$$\langle \Delta A\, \Delta B \rangle = \underset{\text{Im}}{\frac{1}{2}\langle [A,B] \rangle} + \underset{\mathbb{R}}{\frac{1}{2}\langle \{\Delta A, \Delta B\} \rangle}$$

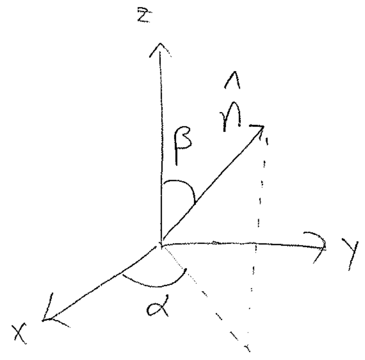
$$|\langle \Delta A\, \Delta B \rangle|^2 = \frac{1}{4}|\langle [A,B] \rangle|^2 + \underbrace{\frac{1}{4}|\langle \{A,B\} \rangle|^2}_{\text{only makes RHS bigger}}$$

$$|\langle \Delta A\, \Delta B \rangle|^2 \leq \frac{1}{4}|\langle [A,B] \rangle|^2$$

W-7

1.9
p 1/2

Find $\hat{n}\cdot\vec{\underset{\sim}{S}}$ such that $\hat{n}\cdot\underset{\sim}{\vec{S}}\,|\hat{n}\cdot\underset{\sim}{\vec{S}};+\rangle = \frac{\hbar}{2}\,|\hat{n}\cdot\underset{\sim}{\vec{S}};+\rangle$



$$\hat{n}\cdot\underset{\sim}{\vec{S}} = \cos\alpha\,\sin\beta\,\underset{\sim}{S}_x + \sin\alpha\,\sin\beta\,\underset{\sim}{S}_y + \cos\beta\,\underset{\sim}{S}_z$$

$$\underset{\sim}{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \underset{\sim}{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \underset{\sim}{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

so $$\hat{n}\cdot\underset{\sim}{\vec{S}} = \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta\\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta\end{pmatrix}$$

We want the eigenvector of this with eigenvalue $+\hbar/2$

i.e. $$\begin{pmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$$

and $|a|^2 + |b|^2 = 1$

choose phase so $a > 0$ [whole $\begin{pmatrix}a\\ b\end{pmatrix}$ can be multiplied by $e^{i\theta}$]

$$a\cos\beta + b\,e^{-i\alpha}\sin\beta = a$$

$$a^2(1-\cos\beta)^2 = |b|^2\sin^2\beta = (1-a^2)\sin^2\beta$$

$$a^2\cdot 4\sin^2\beta/2 = (1-a^2)\,4\sin^2\tfrac{\beta}{2}\cos^2\tfrac{\beta}{2}$$

example problem

1.9
p 2/2

$$4a^2 \sin^2\frac{\beta}{2} = (1-a^2)\, 4 \sin^2\frac{\beta}{2} \cos^2\frac{\beta}{2}$$

$$a = \cos\frac{\beta}{2}$$

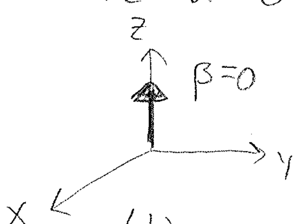
plug back into $a\cos\beta + b e^{-i\alpha}\sin\beta = a$

$$b = a e^{i\alpha}\frac{1-\cos\beta}{\sin\beta} = \cos\frac{\beta}{2}\, e^{i\alpha}\frac{2\sin^2\frac{\beta}{2}}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}$$

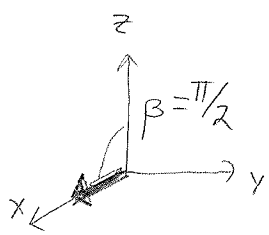
$$= e^{i\alpha}\sin\frac{\beta}{2}$$

$$|\hat{n}\cdot\vec{S};+\rangle = \begin{pmatrix}\cos\beta/2 \\ e^{i\alpha}\sin\beta/2\end{pmatrix}$$

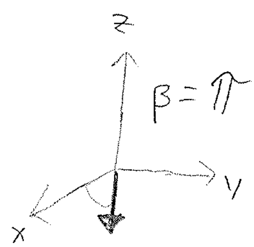
take $\alpha = 0$



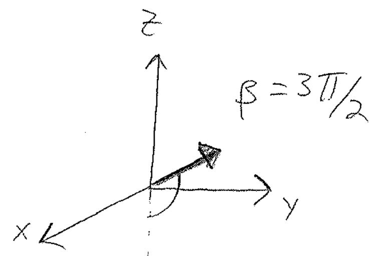
$\begin{pmatrix}1\\0\end{pmatrix}$



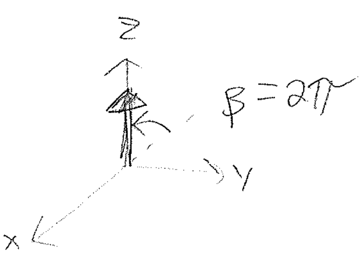
$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$



$\begin{pmatrix}0\\1\end{pmatrix}$



$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$



$$\begin{pmatrix}\cos 2\pi/2 \\ \sin 2\pi/2\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

example problem

1.12
P 1

spin-½ in eigenstate $|\hat{n}\cdot\vec{S};+\rangle$ with $\hat{n}$ in

x-z plane making angle β with $\hat{z}$

S_x is measured. What is the probability of $+\hbar/2$?

$$|\hat{n}\cdot\vec{S};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$$

$$\text{prob} = |\langle S_x;+|\hat{n}\cdot\vec{S};+\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(\langle+| + \langle-|)\left(\cos\frac{\beta}{2}|+\rangle + \overset{1}{\cancel{e^{i\alpha}}}\sin\frac{\beta}{2}|-\rangle\right)\right|^2$$

$$= \frac{1}{2}\left|\sqrt{\frac{1+\cos\beta}{2}} + \sqrt{\frac{1-\cos\beta}{2}}\right|^2 = \frac{1+\sin\beta}{2}$$

example problem

1.10

P1

two-state system: $\underset{\sim}{H} = H_{11}\,|1\rangle\langle 1| + H_{22}\,|2\rangle\langle 2| + H_{12}\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$

let $H_{11} = H_{22} = \varepsilon \qquad H_{12} = H_{21} = \Delta$

$$\rightarrow \quad \underset{\sim}{H} = \begin{pmatrix} \varepsilon_0 & \Delta \\ \Delta & \varepsilon_0 \end{pmatrix} \qquad \underset{\sim}{H}\, u = \varepsilon\, u$$

$$\det\left[\begin{pmatrix} \varepsilon_0 & \Delta \\ \Delta & \varepsilon_0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} \varepsilon_0 - \lambda & \Delta \\ \Delta & \varepsilon_0 - \lambda \end{pmatrix} = 0$$

$$(\varepsilon_0 - \lambda)^2 - \Delta^2 = 0$$

$$\lambda = \varepsilon_0 \pm \Delta$$

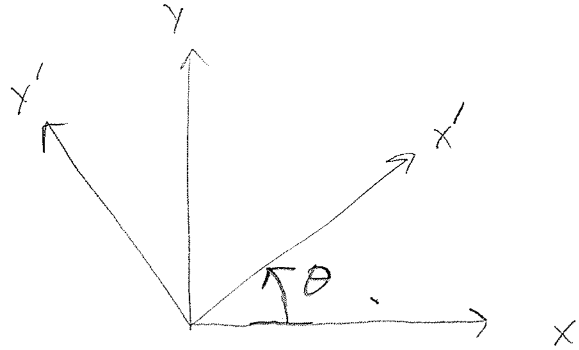
$$\underset{\sim}{H} = \begin{pmatrix} \varepsilon_0 & \Delta \\ \Delta & \varepsilon_1 \end{pmatrix} : \quad \det\left[\begin{pmatrix} \varepsilon_0 & \Delta \\ \Delta & \varepsilon_1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0$$

$$(\varepsilon_0 - \lambda)(\varepsilon_1 - \lambda) - \Delta^2 = 0$$

$$\lambda = \frac{1}{2}\left(\varepsilon_0 + \varepsilon_1 \pm \sqrt{(\varepsilon_0 + \varepsilon_1)^2 + 4\Delta^2}\right)$$

example problem

2.9 Rotate axes



$$|\psi\rangle = \begin{pmatrix} \psi_{x'} \\ \psi_{y'} \end{pmatrix} = \begin{pmatrix} \langle x'|\psi\rangle \\ \langle y'|\psi\rangle \end{pmatrix}$$

$$\langle x'|\psi\rangle = \langle x'| \Big(\overbrace{|x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle}^{|\psi\rangle \text{ in original } x,y \text{ basis}} \Big)$$

$$\langle y'|\psi\rangle = \langle y'| \Big(|x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle \Big)$$

WRITE AS ...

$$\begin{pmatrix} \langle x'|\psi\rangle \\ \langle y'|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix}$$

example: rotation by θ

$$|x'\rangle = \cos\theta\, |x\rangle + \sin\theta\, |y\rangle$$
$$|y'\rangle = -\sin\theta\, |x\rangle + \cos\theta\, |y\rangle$$

so $\underset{\sim}{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ transformation matrix

2.10 We can view this transformation as

1) rotating axes by θ (counter clockwise) to new x', y'

OR

2) rotating state by θ (clockwise)

new state rotated by θ clockwise: $|\psi'\rangle = \underset{\sim}{R}(\theta)|\psi\rangle$

In general, components get all scrambled

We can ask if there are $|\psi\rangle$'s such that operating w/ $\underset{\sim}{R}(\theta)$ gives a simple multiple

$$\underset{\sim}{R}(\theta)|\psi\rangle = c|\psi\rangle$$ eigenvalue, eigenvector

Classical Noether's Theorem: symmetry $\Rightarrow$ conservation law

If $|\psi\rangle$ is an eigenvector of $\underset{\sim}{R}(\theta)$, it's unchanged (only norm changed, but same polarization state)

$$\underset{\sim}{R}(\theta) = \cos\theta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin\theta\underbrace{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}_{\equiv \underset{\sim}{S}}$$

implicit $\mathbb{1}$

$$= \cos\theta + i\underset{\sim}{S}\sin\theta$$

Find evals of $\underset{\sim}{S}$ ... $\underset{\sim}{S}^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \mathbb{1}$

denote eval of $\underset{\sim}{S}$, λ: $\underset{\sim}{S}^2|\psi\rangle = \lambda^2|\psi\rangle = \mathbb{1}|\psi\rangle = |\psi\rangle$

$$\rightarrow \lambda = \pm 1$$

$\lambda = +1$: $|\psi_{+1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \underbrace{|R\rangle}_{\text{from before!}}$ $\lambda = -1$: $|\psi_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = |L\rangle$

W $\rightarrow$

2.11 So, $|R\rangle$ & $|L\rangle$ are eigenvectors of $\underset{\sim}{R}(\theta)$

$$\underset{\sim}{R}(\theta)\,|R\rangle = [\cos\theta + i\underset{\sim}{S}\sin\theta]\,|R\rangle = [\cos\theta + i\cdot 1\cdot \sin\theta]\,|R\rangle$$
$$= e^{i\theta}\,|R\rangle$$

$$\underset{\sim}{R}(\theta)\,|L\rangle = [\cos\theta + i\underset{\sim}{S}\sin\theta]\,|L\rangle = [\cos\theta + i\cdot(-1)\sin\theta]\,|L\rangle$$
$$= e^{-i\theta}\,|L\rangle$$

Since $|R\rangle, |L\rangle$ form an orthonormal basis

$$|\psi\rangle = |R\rangle\langle R|\psi\rangle + |L\rangle\langle L|\psi\rangle$$

& under rotation

$$|\psi\rangle \rightarrow \underset{\sim}{R}(\theta)|\psi\rangle = \underset{\sim}{R}(\theta)|R\rangle\langle R|\psi\rangle + \underset{\sim}{R}(\theta)|R\rangle\langle R|\psi\rangle$$
$$= e^{i\theta}|R\rangle\langle R|\psi\rangle + e^{-i\theta}|L\rangle\langle L|\psi\rangle$$

we call $\underset{\sim}{S}$ the spin matrix or operator for the photon, with eigenvalues ± 1

photon = spin 1

to see why, let's look at angular momentum