So far, discrete eigenvalues for observables

Now continuous ones (position, momentum, etc.)

positions x_i : $\begin{pmatrix} C_{x_1} \\ \vdots \\ C_{x_N} \end{pmatrix} \rightarrow \lim_{N\to\infty} \begin{pmatrix} C_{x_1} \\ \vdots \\ C_{x_N} \end{pmatrix}$

$$|\psi\rangle = \sum_n C_{x_n} |x_n\rangle \longrightarrow \int dx\, f(x)\, |x\rangle$$

eigenstate labeled by its eigenvalue

$$\underset{\sim}{X} |x\rangle = x\, |x\rangle$$

continuous operator — eigenvalue

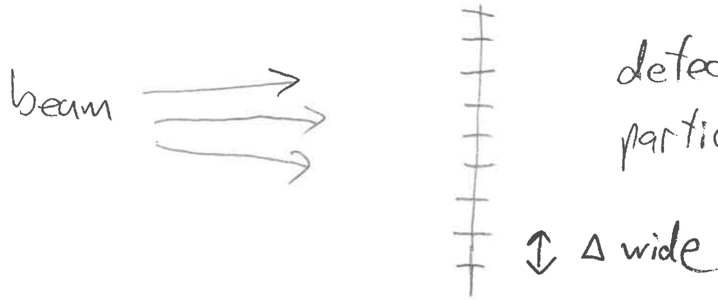
Discrete	Continuous
$\sum_i \lvert a_i\rangle\langle a_i\rvert = \underset{\sim}{1}$	$\int d\xi\, \lvert\xi\rangle\langle\xi\rvert = \underset{\sim}{1}$
$\lvert\psi\rangle = \sum_i \lvert a_i\rangle\langle a_i\lvert\psi\rangle$	$\lvert\psi\rangle = \int d\xi\, \lvert\xi\rangle\langle\xi\lvert\psi\rangle$
$\sum_i \lvert\langle a_i\lvert\psi\rangle\rvert^2 = 1$	$\int d\xi\, \lvert\langle\xi\lvert\psi\rangle\rvert^2 = 1$
$\langle a'\lvert \underset{\sim}{A} \rvert a''\rangle = a'\,\delta_{a''a'}$	$\langle\xi''\lvert \underset{\sim}{\xi} \rvert\xi'\rangle = \xi'\,\delta(\xi'-\xi'')$

$$\int dx\, \delta(x-x_0)\, f(x) = f(x_0)$$

3.2

Position Operator

$$\underset{\sim}{X}|x'\rangle = x'|x'\rangle \qquad \{|x'\rangle\} \text{ is complete}$$



detectors produce signal if particle lands in $x \pm \Delta/2$

$$|\psi_{beam}\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi_{beam}\rangle \xrightarrow{\text{measure}} \int_{x'-\Delta/2}^{x'+\Delta/2} dx\, |x\rangle\langle x|\psi_{beam}\rangle$$

$$\begin{pmatrix}\text{probability of particle} \\ \text{measured within} \\ x \pm dx/2\end{pmatrix} = |\langle x|\psi\rangle|^2\, dx$$

<u>3D</u> 3 vector components

$$\vec{\underset{\sim}{X}}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

or $$(\underset{\sim}{X}\hat{x} + \underset{\sim}{Y}\hat{y} + \underset{\sim}{Z}\hat{z})|x,y,z\rangle = (x\hat{x} + y\hat{y} + z\hat{z})|x,y,z\rangle$$

3.3 Translation

$$\underset{\sim}{T}(d\vec{x}')\,|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$$

$$|\psi\rangle \rightarrow \underset{\sim}{T}(d\vec{x}') \int d^3x\, |\vec{x}'\rangle\langle\vec{x}'|\psi\rangle$$

$$= \int d^3x\, |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\psi\rangle$$

$$= \int d^3x\, |\vec{x} + d\vec{x}' - d\vec{x}'\rangle\langle\vec{x} - d\vec{x}'|\psi\rangle$$

$$= \int d^3x\, |\vec{x}\rangle\langle\vec{x} - d\vec{x}'|\psi\rangle$$

Normalization Preserved

$$\text{if } \langle\psi|\psi\rangle = 1 \Rightarrow \left[\underset{\sim}{T}(d\vec{x})\,|\psi\rangle\right]^{\dagger}\left[\underset{\sim}{T}(d\vec{x})\,|\psi\rangle\right]$$

$$= \langle\psi|\underbrace{\underset{\sim}{T}^{\dagger}(d\vec{x})\,\underset{\sim}{T}(d\vec{x})}_{=\underset{\sim}{1}}|\psi\rangle = 1$$

Some Reasonable conditions:

1) Addition of translations

$$\underset{\sim}{T}(d\vec{x}')\,\underset{\sim}{T}(d\vec{x}'') = \underset{\sim}{T}(d\vec{x}' + d\vec{x}'')$$

2) Inverse

$$\underset{\sim}{T}^{-1}(d\vec{x}) = \underset{\sim}{T}(-d\vec{x})$$

3) Limit:

$$\lim_{d\vec{x}\to 0} \underset{\sim}{T}(d\vec{x}) = \underset{\sim}{1}$$

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4) infinitesimal translation: $\underset{\sim}{T}(d\vec{x}') = \underset{\sim}{1} - i\, d\vec{x}' \cdot \underset{\sim}{\vec{K}}$

$\uparrow$

unknown operator

(3) has right limit: $\underset{\sim}{1}$ as $d\vec{x}' \to 0$

$\underset{\sim}{T}$ satisfies $\quad \underset{\sim}{T}^{+}(d\vec{x})\, \underset{\sim}{T}(d\vec{x}) =$

$$\left(\underset{\sim}{1} + i\, d\vec{x}' \cdot \underset{\sim}{\vec{K}}^{+}\right)\left(\underset{\sim}{1} - i\, d\vec{x}' \cdot \underset{\sim}{\vec{K}}'\right)$$

$$\underset{\sim}{1} + i\, d\vec{x}' \cdot \left(\underset{\sim}{\vec{K}}^{+} - \underset{\sim}{\vec{K}}\right) + O(d\vec{x}'^{2})$$

$$= \underset{\sim}{1} + O(d\vec{x}'^{2}) \quad \text{if} \quad \underset{\sim}{\vec{K}}^{+} = \underset{\sim}{\vec{K}}$$

right inverse $\quad \underset{\sim}{T}(-d\vec{x})\, \underset{\sim}{T}(d\vec{x}) \overset{?}{=} \underset{\sim}{1}$

$$\left(\underset{\sim}{1} - -d\vec{x} \cdot \underset{\sim}{\vec{K}}\right)\left(\underset{\sim}{1} - d\vec{x} \cdot \underset{\sim}{\vec{K}}\right)$$

$$\underset{\sim}{1} + d\vec{x} \cdot \left(\underset{\sim}{\vec{K}} - \underset{\sim}{\vec{K}}\right) + O(dx^{2}) \overset{\checkmark}{=} \underset{\sim}{1} + O(dx^{2})$$

3.5

Commutator $[\underset{\sim}{\vec{X}}, \underset{\sim}{T}(d\vec{x})] = ?$

$$\underset{\sim}{\vec{X}}\ \underset{\sim}{T}(d\vec{x}')\,|\vec{x}\rangle = \underset{\sim}{\vec{X}}\,|\vec{x}+d\vec{x}'\rangle = (\vec{x}+d\vec{x}')\,|\vec{x}+d\vec{x}'\rangle$$

different same

$$\underset{\sim}{T}(d\vec{x}')\,\underset{\sim}{\vec{X}}\,|\vec{x}\rangle = \underset{\sim}{T}(d\vec{x}')\,\vec{x}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}+d\vec{x}'\rangle$$

any $|\vec{x}\rangle$ so... $[\underset{\sim}{\vec{X}}, \underset{\sim}{T}(d\vec{x}')] = [(\vec{x}+d\vec{x}') - (\vec{x})]\underset{\sim}{1} = d\vec{x}'\,\underset{\sim}{1}$

infinitesimal : $[\underset{\sim}{\vec{X}}, \underset{\sim}{1} - i\,d\vec{x}\cdot\underset{\sim}{\vec{K}}] = d\vec{x}\ \underset{\sim}{1}$

$$[\underset{\sim}{\vec{X}}, \underset{\sim}{1}] - i[\underset{\sim}{\vec{X}}, d\vec{x}\cdot\underset{\sim}{\vec{K}}] = d\vec{x}\cdot\underset{\sim}{1}$$

$$[\underset{\sim}{\vec{X}}, d\vec{x}\circ\underset{\sim}{\vec{K}}] = i\,d\vec{x}\ \underset{\sim}{1}$$

let $d\vec{x}$ be in the direction $\hat{x}_j$

$$[\underset{\sim}{\vec{X}}, dx_j\,\underset{\sim}{K}_j] = i\,dx_j\ \hat{x}_j$$

$$\hat{x}_i \circ \{\qquad\qquad\qquad\}$$

$$[\underset{\sim}{X}_i, dx_j\,\underset{\sim}{K}_j] = i\,dx_j\ \delta_{ij}$$

dx_j = number : divide by it

$$[\underset{\sim}{X}_i, \underset{\sim}{K}_j] = i\,\delta_{ij}\ \underset{\sim}{1}$$

3.6

Classical mechanics:

symmetry under translation $\Longleftrightarrow$ momentum conservation

by analogy, $\underset{\sim}{K} \propto$ proportional to momentum

but $[\underset{\sim}{x} \quad \underset{\sim}{K}] = 1 \qquad [\underset{\sim}{K}] = \frac{1}{\text{Length}}$

dimensions of

$\hbar$ has units (Length) $\times$ (momentum) $=$ (Energy) $\cdot$ (time)

$$\Longrightarrow \quad \underset{\sim}{K} = \text{wavenumber} = \frac{2\pi}{\lambda} = \frac{p}{\hbar}$$

$$\underset{\sim}{T}(d\vec{x}) = 1 - \frac{i\, d\vec{x}' \cdot \underset{\sim}{\vec{p}}}{\hbar}$$

$$[\underset{\sim}{x}_i, \underset{\sim}{p}_j] = i\hbar\, \delta_{ij}$$

plus into uncertainty relation

$$\langle (\Delta \underset{\sim}{x}_i)^2 \rangle \langle (\Delta \underset{\sim}{p}_j)^2 \rangle \geq \frac{\hbar^2}{4}$$

Finite translation by x' $\qquad \Delta x' = \frac{x'}{N}$

$$\underset{\sim}{T}(x') = \lim_{N\to\infty} \left(\underset{\sim}{1} - \frac{i\, \Delta x'\, \underset{\sim}{p}_x}{\hbar} \right)^N$$

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$$\underset{\sim}{T}(x') = \exp\left(-\frac{i \underset{\sim}{p}_x \Delta x'}{\hbar}\right)$$

$$= \underset{\sim}{I} + \left(-\frac{i\Delta x'}{\hbar}\underset{\sim}{p}_x\right) + \frac{1}{2!}\left(-\frac{i\Delta x'}{\hbar}\underset{\sim}{p}_x\right)^2 + \ldots$$

$\rightarrow$

Translate by dx then dy = translate by dy then dx

$\Downarrow$

$$[\underset{\sim}{p}_x, \underset{\sim}{p}_y] = 0$$

$$[\underset{\sim}{p}_i, \underset{\sim}{p}_j] = 0$$

since $\underset{\sim}{p}_x$, $\underset{\sim}{p}_y$, $\underset{\sim}{p}_z$ all commute with each other they are compatible $|\vec{p}\rangle = |p_x, p_y, p_z\rangle$

$$\vec{\underset{\sim}{p}}_i |\vec{p}\rangle = \vec{p}_i |\vec{p}\rangle$$

3.8

Δx infinitesimal:

$$\underset{\sim}{T}(\Delta x)|\psi\rangle = \left(\underset{\sim}{1} - \frac{i\Delta x}{\hbar}\underset{\sim}{p}\right)|\psi\rangle = \int dx' |x'+\Delta x\rangle\langle x'|\psi\rangle$$

$$= \int dx' |x'\rangle \underbrace{\langle x'-\Delta x|\psi\rangle}_{}$$

f(x) is just a function
so expand in small Δx

$$= \int dx' |x'\rangle \left[\langle x'|\psi\rangle - \Delta x\, \partial_{x'}\langle x'|\psi\rangle\right]$$

$$|\psi\rangle - \frac{i\Delta x}{\hbar}\underset{\sim}{p}|\psi\rangle = |\psi\rangle - \Delta x \int dx' |x'\rangle \partial_{x'}\langle x'|\psi\rangle$$

$$\underset{\sim}{p}|\psi\rangle = \int dx' |x'\rangle \underbrace{\left(\frac{\hbar}{i}\partial_{x'}\right)}_{}\langle x'|\psi\rangle$$

momentum operator in
coordinate basis!

multiply both sides by $\langle x|$

$$\langle x|\underset{\sim}{p}|\psi\rangle = \int dx' \langle x|x'\rangle \frac{\hbar}{i}\partial_{x'}\langle x'|\psi\rangle$$

$$\langle x|\underset{\sim}{p}|\psi\rangle = \frac{\hbar}{i}\partial_x\langle x|\psi\rangle$$

3.9

Eigenstates of momentum operator

$$\underset{\sim}{p}|p\rangle = p|p\rangle$$

we just saw $\langle x|\underset{\sim}{p}|\psi\rangle = \frac{\hbar}{i}\partial_x\langle x|\psi\rangle$

so $\langle x|\underset{\sim}{p}|p\rangle = \frac{\hbar}{i}\partial_x\langle x|p\rangle$

$$p\underbrace{\langle x|p\rangle}_{f(x,p)} = \frac{\hbar}{i}\partial_x\underbrace{\langle x|p\rangle}_{f(x,p)}$$

$$p\,f(x,p) = \frac{\hbar}{i}\partial_x f(x,p)$$

$\rightarrow$ $f(x,p) = \langle x|p\rangle = N e^{ipx/\hbar}$

N — normalization

Generalization to continuous : $\langle x'|x''\rangle = \delta(x'-x'')$

$$\langle x'|x''\rangle = \int dp\,\langle x'|p\rangle\langle p|x''\rangle$$

$$\delta(x'-x'') = |N|^2\int dp\; e^{\frac{i}{\hbar}p\cdot(x'-x'')}$$

$N = \frac{1}{\sqrt{2\pi\hbar}}$ normalization to make δ-function

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

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$$\langle x|\alpha\rangle = \int dp \, \langle x|p\rangle \langle p|\alpha\rangle \qquad \langle p|\alpha\rangle = \int dx \, \langle p|x\rangle \langle x|\alpha\rangle$$

$$\psi_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \, e^{ipx/\hbar} \, \phi_\alpha(p) \qquad \phi_\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-ipx/\hbar} \, \psi_\alpha(x)$$

Localize with a Gaussian

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \, e^{ikx} \, e^{-x^2/2d^2}$$ Gaussian wave packet

Some matrix elements

$$\langle\alpha|\underset{\sim}{x}|\alpha\rangle = \int_{-\infty}^{\infty} dx \, \langle\alpha|x\rangle x \langle x|\alpha\rangle = \frac{1}{\pi^{1/2}d} \int_{-\infty}^{\infty} dx \, e^{-x^2/d^2} x = 0$$

$$\langle\alpha|\underset{\sim}{x}^2|\alpha\rangle = \int_{-\infty}^{\infty} dx \, x^2 |\langle x|\alpha\rangle|^2 = \frac{1}{\sqrt{\pi}d} \int_{-\infty}^{\infty} dx \, x^2 e^{-x^2/d^2} = d^2/2$$

$$\langle (\Delta \underset{\sim}{x})^2 \rangle = \langle \underset{\sim}{x}^2\rangle - \langle \underset{\sim}{x}\rangle^2 = d^2/2$$

$$\langle\alpha|p|\alpha\rangle = \int_{-\infty}^{\infty} dx \, \langle\alpha|x\rangle \frac{\hbar}{i}\partial_x \langle x|\alpha\rangle$$

$$= \frac{\hbar}{i} \frac{1}{\sqrt{\pi}d} \int dx \, e^{-x^2/d^2} \left(ik - \frac{x}{d^2}\right) = \hbar k$$

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$$\langle \alpha | \underset{\sim}{p}^2 | \alpha \rangle = \int_{-\infty}^{\infty} dx \, \langle \alpha | x \rangle \frac{\hbar}{i}\partial_x \frac{\hbar}{i}\partial_x \langle \alpha | x \rangle$$

$$= \hbar^2 \frac{1}{\sqrt{\pi}\, d} \int dx \left[-\frac{1}{d^2} - k^2 - i\frac{2kx}{d^2} + \frac{x^2}{d^4} \right] e^{-x^2/d^2}$$

$$= \hbar^2 k^2 + \frac{\hbar^2}{2d^2}$$

$$\langle \alpha | (\Delta \underset{\sim}{p})^2 | \alpha \rangle = \langle \underset{\sim}{p}^2 \rangle - \langle \underset{\sim}{p} \rangle^2$$

$$= \left(\hbar^2 k^2 + \frac{\hbar^2}{2d^2} \right) - (\hbar k)^2 = \frac{\hbar^2}{2d^2}$$

$$\langle \alpha | (\Delta \underset{\sim}{p})^2 | \alpha \rangle \langle \alpha | (\Delta \underset{\sim}{x})^2 | \alpha \rangle \geq |\langle \alpha | [\underset{\sim}{x}, \underset{\sim}{p}] | \alpha \rangle|^2 / 4$$

$$\frac{d^2}{2} \qquad \frac{\hbar^2}{2d^2} \qquad\qquad \hbar^2/4$$

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3D

$$\underset{\sim}{\vec{X}}\,|\vec{X}\rangle = \vec{X}\,|\vec{X}\rangle \qquad \underset{\sim}{\vec{P}}\,|\vec{P}\rangle = \vec{P}\,|\vec{P}\rangle$$

$$\langle\vec{X}|\vec{X}'\rangle = \delta^{(3)}(\vec{X}-\vec{X}') \qquad \langle\vec{P}|\vec{P}'\rangle = \delta^{(3)}(\vec{P}-\vec{P}')$$

$$\delta^{(3)}(\vec{X}-\vec{X}') \equiv \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int d^3x\,|\vec{x}\rangle\langle\vec{x}| = \mathbb{1} \qquad \int d^3p\,|\vec{p}\rangle\langle\vec{p}|$$

<u>dimensions</u> $[\psi(x)] = \frac{1}{L^{1/2}}$ $\qquad [\phi(p)] = L^{1/2}/[\hbar]$

$[\psi(\vec{x})] = \frac{1}{L^{3/2}}$ $\qquad [\phi(\vec{p})] = L^{3/2}/[\hbar^3]$

$$1 = \int dx\,|\psi(x)|^2$$

$$1 = [x]\cdot[\psi(x)]^2$$

$$[\psi] = \frac{1}{L^{1/2}}$$

$$1 = \int dp\,|\phi(p)|^2$$

$$1 = [p]\cdot[\phi]^2$$

$$1 = \frac{[\hbar]}{L}\cdot[\phi]^2$$

$$[\phi] = \frac{L^{1/2}}{[\hbar]}$$