Grading Scheme Problem Set 1

In general, $-\frac{1}{2}$ point for a minor algebraic or copying error.

1.2 10 points for the whole thing; It's pretty much right or wrong.

1.4 (a) 2 points for $\operatorname{tr} \underset{\sim}{X}$
4 points for $\operatorname{tr} \underset{\sim}{\sigma}_K \underset{\sim}{X}$

(b) 1 point each for a_0 through a_3

1.6 (a) 3 points
(b) 3 points
(c) 3 points
(d) 1 point – (we haven't covered this yet!)

1.8 10 points. This is another "binary" problem that is pretty much right or wrong.

1.9 (a) 3 points
(b) 3 points
(c) 2 points for (a) operator
2 points for (b) operator

Sakurai 1.2 work backwards:

$$-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$$

cancel

$$= -ACDB - ACBD + ACBD + ABCD$$

cancel

$$-CDAB - CADB + CADB + ACDB$$

cancel

$$= ABCD - CDAB = [AB, CD] \checkmark$$

Sakurai 1.8

$$\underset{\sim}{A}\left(|i\rangle + |j\rangle\right) = a_i |i\rangle + a_j |j\rangle$$

this is $\neq |i\rangle + |j\rangle$ unless $a_i = a_j$. So $|i\rangle + |j\rangle$ isn't an eigenvector of $\underset{\sim}{A}$ unless $|i\rangle$ & $|j\rangle$ are degenerate

Sakurai 1.4

$$\underset{\sim}{X} = a_0 \underset{\sim}{1} + \vec{a}\cdot\vec{\underset{\sim}{\sigma}} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) $\quad \mathrm{tr}\, \underset{\sim}{\sigma}_i = 0$

$$\mathrm{tr}\, \underset{\sim}{X} = \mathrm{tr}\left(a_0 \underset{\sim}{1} + \sum_i a_i \underset{\sim}{\sigma}_i\right) = a_0\, \mathrm{tr}\, \underset{\sim}{1} + \sum_i a_i\, \mathrm{tr}\, \underset{\sim}{\sigma}_i = 2a_0$$

$$\mathrm{tr}\left(\underset{\sim}{\sigma}_K \underset{\sim}{X}\right) = \mathrm{tr}\left(a_0 \underset{\sim}{\sigma}_K + \underset{\sim}{\sigma}_K \sum_i a_i \underset{\sim}{\sigma}_i\right)$$

$$= a_0\, \mathrm{tr}\, \underset{\sim}{\sigma}_K + \sum_i a_i\, \mathrm{tr}\, \underset{\sim}{\sigma}_K \underset{\sim}{\sigma}_i$$

$$= 0 + \sum_i a_i \tfrac{1}{2}\, \mathrm{tr}\left(\underset{\sim}{\sigma}_K \underset{\sim}{\sigma}_i + \underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_K\right)$$

$$= \sum_i \frac{a_i}{2}\, \mathrm{tr}\, \underset{\sim}{1}\, \delta_{Ki} = 2a_K$$

b) $\quad a_0 = \frac{1}{2}\, \mathrm{tr}\, \underset{\sim}{X} \qquad a_K = \frac{1}{2}\, \mathrm{tr}\left(\underset{\sim}{\sigma}_K \underset{\sim}{X}\right) \qquad$ from (a)

so

$$a_0 = \tfrac{1}{2}\, \mathrm{tr}\, \underset{\sim}{X} = \tfrac{1}{2}\left(X_{11} + X_{22}\right)$$

$$a_1 = \tfrac{1}{2}\, \mathrm{tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \tfrac{1}{2}\left(X_{21} + X_{12}\right)$$

$$a_2 = \tfrac{1}{2}\, \mathrm{tr} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \tfrac{i}{2}\left(-X_{21} + X_{12}\right)$$

$$a_3 = \tfrac{1}{2}\, \mathrm{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \tfrac{1}{2}\left(X_{11} - X_{22}\right)$$

Sakurai 1.6

a) $\operatorname{Tr}(XY) = \sum_a \langle a | X Y | a \rangle$

$$= \sum_{a,b} \langle a | X | b \rangle \langle b | Y | a \rangle$$

matrix elements are numbers

$$= \sum_{a,b} \langle b | Y | a \rangle \langle a | X | b \rangle$$

$$= \sum_b \langle b | Y X | b \rangle = \operatorname{tr}(Y X)$$

b) The dual of $X|\alpha\rangle$ is $\langle \alpha | X^\dagger$ (Sakurai 1.2.24)

let $|\beta\rangle = Y|\alpha\rangle$

the dual of $|\beta\rangle$ is $\langle \beta | = \langle \alpha | Y^\dagger$

the dual of $X|\beta\rangle$ is $\langle \beta | X^\dagger$

$$= \langle \alpha | Y^\dagger X^\dagger$$

But the dual of $XY|\alpha\rangle$ is $\langle \alpha | (X Y)^\dagger$ $\qquad \Rightarrow (XY)^\dagger = Y^\dagger X^\dagger$

c) $e^{if(A)} = e^{if(A)} \sum_a |a\rangle\langle a| = \sum_a e^{if(a)} |a\rangle\langle a|$

exponential of an operator $\qquad$ exponential of a number

d) $\sum_a \psi_a^*(x') \psi_a(x) = \sum_a \langle x' | a \rangle^* \langle x'' | a \rangle = \sum_a \langle x'' | a \rangle \langle a | x' \rangle = \langle x'' | x' \rangle$

$$= \delta(x'' - x')$$

Sakurai P1/2 1.9

a) Since $\{|a'\rangle\}$ is complete, a general state is

$$|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle$$

So let's look at the action of $\prod_{a'}(\underset{\sim}{A} - a')$ on an $|a_i\rangle$

$$\prod_{a'}(\underset{\sim}{A} - a')|a_i\rangle = \prod_{a'}(a_i - a')|a_i\rangle$$

Since this product is over <u>all</u> a', one will be $a' = a_i$, giving zero. Therefore $\prod_{a'}(\underset{\sim}{A} - a')|a_i\rangle = 0 \quad \forall\ |a_i\rangle$

and $\prod_{a'}(\underset{\sim}{A} - a')|\psi\rangle = 0$ since $|\psi\rangle$ is a sum of $|a_i\rangle$'s

b) $$\prod_{a''\neq a'} \frac{\underset{\sim}{A} - a''}{a' - a''}|a_i\rangle = \prod_{a''\neq a'} \frac{a_i - a''}{a' - a''}|a_i\rangle$$

As in (a) there will be one factor zero in the product, EXCEPT if $a_i = a'$, in which case we get

$$\prod_{a''\neq a'} \frac{a' - a''}{a' - a''}|a'\rangle = |a'\rangle$$

So $$\prod_{a''\neq a'} \frac{\underset{\sim}{A} - a''}{a' - a''}|a_i\rangle = \begin{cases} 0 & \text{if } a_i \neq a' \\ |a'\rangle & \text{if } a_i = a' \end{cases}$$

i.e. it's a projection operator onto $|a'\rangle$

c) if $\underset{\sim}{A} = \underset{\sim}{S}_z$ with $\{|a'\rangle\} = \{|+\rangle, |-\rangle\}$

$$\underset{\sim}{Q}_0 = \prod_{a'} (\underset{\sim}{A} - a') = (\underset{\sim}{S}_z - \tfrac{\hbar}{2})(\underset{\sim}{S}_z + \tfrac{\hbar}{2})$$

This is clearly the null operator since

$\underset{\sim}{Q}_0|+\rangle = \underset{\sim}{Q}_0|-\rangle = 0$

For the operator of part (b),

$$\prod_{a'' \neq a'} \frac{\underset{\sim}{A} - a''}{a' - a''} = \begin{cases} \dfrac{\underset{\sim}{S}_z + \hbar/2}{\hbar} & \text{if } a' = +\hbar/2 \\[2ex] \dfrac{\underset{\sim}{S}_z - \hbar/2}{-\hbar} & \text{if } a' = -\hbar/2 \end{cases}$$

using $\underset{\sim}{S}_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$ & $\underset{\sim}{1} = |+\rangle\langle+| + |-\rangle\langle-|$

<u>$a' = \hbar/2$ case</u>

$$\frac{1}{\hbar}\underset{\sim}{S}_z + \frac{1}{2}\underset{\sim}{1} = \frac{1}{2}(|+\rangle\langle+| - |-\rangle\langle-|) + \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$$

$= |+\rangle\langle+|$ the "up" projection op.

<u>$a' = -\hbar/2$ case</u>

$$-\frac{1}{\hbar}S_z + \frac{1}{2}\underset{\sim}{1} = -\frac{1}{2}(|+\rangle\langle+| - |-\rangle\langle-|) + \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$$

$= |-\rangle\langle-|$ the "down" projection op.