AstroGK Manual

2007

Contents

Part I

Gyrokinetics and `AstroGK`

Chapter 1

Derivation of the Gyrokinetic Equation

This chapter is intended as a brief derivation of the gyrokinetic equations for the astrophysical community; the plasma physics literature contains detailed papers relating the development of linear [Antonsen and Lane, 1980] and nonlinear [Frieman and Chen, 1982] gyrokinetic theory [NOTE: more references to be added].

The general approach to deriving the gyrokinetic equations follows. We begin with the Fokker-Planck equation and Maxwell's equations and define the limits of the gyrokinetic approximation. First, the Fokker-Planck equation is transformed to guiding center coordinates and the ordering specified by the gyrokinetic approximations is applied. At lowest order, we discover that the equilibrium distribution function is independent of gyrophase angle. At the next order, it is proven that the equilibrium distribution function must take the form of a Maxwellian; this solution is used to solve the equation at this order which consists of of a particular solution related to the equilibrium and a homogeneous part that is indepedent of the gyrophase angle. At the next order, a ring average at fixed guiding center yields a closed equation for the unknown homogeneous solution—the Gyrokinetic equation.

The remaining equations necessary to define a complete system of gyrokinetic equations are derived similarly from Maxwell's equations: integration of the distribution function (to the same order as the gyrokinetic equation) over the velocity and performing a ring average ar constant position gives the charge and current necessary in Coulomb's Law and Ampere's Law. Performing this procedure on Coulomb's Law yields the Quasineutrality condition; the parallel and perpendicular components of Ampere's Law provide the remaining two equations.

NOTE: This chapter uses the variable g for the gyrokinetic distribution function; in our standard convention, it should actually be h. I will change this when I get around to it.

1.1 The Gyrokinetic Approximations and Ordering

Two approximations are necessary to derive the gyrokinetic equations: weak coupling and strong magnetization.

The weak coupling approximation, $g = n_e \lambda_{De} \gg 1$, where n_e is the electron number density and λ_{De} is the electron Debye length, means that a given charge interacts with many electrons that fall within its Debye sphere. This is the standard approximation in plasma physics that allows the use of Fokker-Planck equation to describe plasma species.

For the discussion to follow, we specify standard definitions of thermal velocity, cyclotron frequency, and Larmour radius of a plasma species s (in cgs units): $v_{th_s}^2 = 2T_s/m_s$, $\Omega_s = q_s B_0/(m_s c)$, and $\rho_s = v_{th_s}/\Omega_s$ where we have mass m_s, charge q_s, temperature T_s (absorbing Boltzmann's constant to give temperature in units of energy), mean magnetic field strength B_0, and speed of light c.

The condition of strong magnetization for gyrokinetics restricts the scales of validity in space and time

$$\tfrac{\rho_i}{L} \ll 1 \qquad \omega \ll \Omega_i. \tag{1.1}$$

where L is the macroscopic length scale of the equilibrium plasma and ω is the frequency of fluctuations described by the theory.

Gyrokinetics employs a formal ordering to manage the disparate length and time scales typical of magnetized plasmas. Two length and three time scales are specified: a large macroscopic length L, the small ion larmour radius ρ_i, the fast time scale associated with the ion cyclotron frequency Ω_i, the intermediate time scale of the fluctuation freqency $\omega \sim v_{th_i}/L$, and the slow time scale of the transport rate $1/\tau$. A small dimensionless parameter is defined by $\epsilon = \rho_i/L \ll 1$. Splitting the distribution function into an equilibrium and fluctuating component $f = F_0 + \delta f$, we define the formal gyrokinetic ordering by

$$\frac{\rho_i}{L} \sim \frac{\omega}{\Omega_i} \sim \frac{\delta f}{F_0} \sim \frac{|\delta \mathbf{B}|}{B_0} \sim \frac{|\delta \mathbf{E}|c}{B_0 v_{th_i}} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon \ll 1. \tag{1.2}$$

Here $\delta\mathbf{B}$ and $\delta\mathbf{E}$ are the fluctuating components of the magnetic and electric field and $k_\parallel$ and $k_\perp$ are typical wavenumbers parallel and perpendicular to the mean magnetic field $\mathbf{B}_0$. In this ordering, the equlibrium quantities F_0 and $\mathbf{B}_0$ vary at the slow transport rate $1/\tau \sim \epsilon^3\Omega_i$. Note that fluctuations perpendicular to the magnetic field are of the same order as the ion Larmour radius

$$k_\perp \rho_i \sim 1. \tag{1.3}$$

1.2 The Gyrokinetic Ordering

1.3 Definitions

1.3.1 General

1. Cyclotron frequency: The cyclotron frequency (in cgs units) for a species s in a equilibrium field of strength B_0 is given by

$$\Omega_s = \frac{q_s B_0}{m_s c} \tag{1.4}$$

2. Thermal velocity: The thermal velocity of a particle species s is defined by

$$v_{th_s}^2 = \frac{2T_s}{m_s} \tag{1.5}$$

1.3.2 Spatial Coordinates and Transformations

1. Guiding Center Coordinates: The position $\mathbf{r}$, guiding center $\mathbf{R}$, and gyroradius ρ are related by

$$\mathbf{r} = \mathbf{R} + \rho \tag{1.6}$$

2. Gyroradius: The gyroradius is related to the perpendicular velocity by

$$\rho = \frac{\hat{\mathbf{b}} \times \mathbf{v}(\theta)}{\Omega} \tag{1.7}$$

NOTE: The gyroradius is given by (1.7) for a positive charge q. If the charge is negative, there is a negative sign.

3. Particle velocity:

$$\mathbf{v} = v_{\parallel}\hat{\mathbf{z}} + v_{\perp}(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \tag{1.8}$$

1.3.3 Velocity Coordinates and Transformations

1. Velocity coordinates $\mathcal{E} = \frac{v^2}{2}$ $\quad \mu = \frac{v_{\perp}^2}{2B_0}$ $\quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

2. The Jacobian for this transformation is

$$d^3\mathbf{v} = \frac{d\mathcal{E}d\mu d\theta}{|\frac{\partial\mathcal{E}}{\partial\mathbf{v}} \times \frac{\partial\mu}{\partial\mathbf{v}} \cdot \frac{\partial\theta}{\partial\mathbf{v}}|} = \frac{B_0 d\mathcal{E}d\mu d\theta}{|v_{\parallel}|} \tag{1.9}$$

3. This can be seen using

$$\frac{\partial\mathcal{E}}{\partial\mathbf{v}} = \mathbf{v} \tag{1.10}$$

$$\frac{\partial\mu}{\partial\mathbf{v}} = \frac{\mathbf{v}_{\perp}}{B_0} \tag{1.11}$$

$$\frac{\partial\theta}{\partial\mathbf{v}} = \frac{\hat{\mathbf{z}} \times \mathbf{v}_{\perp}}{\mathbf{v}_{\perp}^2} \tag{1.12}$$

1.3.4 Ring Averages

1. Ring Average at constant guiding center $\mathbf{R}$

$$\langle a(\mathbf{r}, \mathbf{v}, t)\rangle_{\mathbf{R}} = \frac{1}{2\pi}\oint d\theta a(\mathbf{R} - \frac{\mathbf{v} \times \hat{\mathbf{z}}}{\Omega}, \mathbf{v}, t) \tag{1.13}$$

2. Ring Average at fixed position $\mathbf{r}$

$$\langle a(\mathbf{R}, \mathcal{E}, \mu, \theta, t)\rangle_{\mathbf{r}} = \frac{1}{2\pi}\oint d\theta a(\mathbf{r} + \frac{\mathbf{v} \times \hat{\mathbf{z}}}{\Omega}, \mathcal{E}, \mu, \theta, t) \tag{1.14}$$

Note here that

$$\frac{\mathbf{v}(\theta) \times \hat{\mathbf{z}}}{\Omega} = -\rho(\theta) \tag{1.15}$$

1.4 Maxwell's Equations

1. The set of Maxwell's Equations (in cgs units) are

$$\nabla \cdot \delta\mathbf{E} = 4\pi\rho \tag{1.16}$$

$$\frac{1}{c}\frac{\partial \delta\mathbf{E}}{\partial t} = \nabla \times \delta\mathbf{B} - \frac{4\pi}{c}\delta\mathbf{J} \tag{1.17}$$

$$\nabla \cdot \delta\mathbf{B} = 0 \tag{1.18}$$

$$\frac{1}{c}\frac{\partial \delta\mathbf{B}}{\partial t} = -\nabla \times \delta\mathbf{E} \tag{1.19}$$

We'll look at each of these equations in the Gyrokinetic Approximation.

2. Faraday's Law:

$$\frac{1}{c}\frac{\partial \delta\mathbf{B}}{\partial t} = -\nabla \times \delta\mathbf{E} \tag{1.20}$$

We will use the standard potentials

$$\delta\mathbf{B} = \nabla \times \mathbf{A} \tag{1.21}$$

and

$$\delta\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{1.22}$$

3. Ampere/Maxwell Law:

$$\nabla \times \delta\mathbf{B} = \frac{4\pi}{c}\delta\mathbf{J} + \frac{1}{c}\frac{\partial \delta\mathbf{E}}{\partial t} \tag{1.23}$$

Here we drop the displacement current under the approximations for gyrokinetics. This leaves us Ampere's Law:

$$\nabla \times \delta\mathbf{B} = \frac{4\pi}{c}\delta\mathbf{J} \tag{1.24}$$

4. The Poisson equation from Coulomb's Law:

$$\nabla \cdot \delta\mathbf{E} = 4\pi\sum_s q_s n_s \tag{1.25}$$

Under our approximations, the divergence is small, leaving us the Quasineutrality Condition:

$$\sum_s q_s n_s = 0 \tag{1.26}$$

1.5 Fokker-Planck Equation

1. The Fokker-Planck Equation: For a species α with the distribution function $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, we have

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} \{\mathbf{E} + \mathbf{v} \times \mathbf{B}\} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \sum_\beta \mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta) \tag{1.27}$$

2. The Landau form of the Collision operator is

$$\mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta) = -\frac{\partial \mathbf{J}_{\alpha\beta}}{\partial \mathbf{v}} \tag{1.28}$$

where

$$\mathbf{J}_{\alpha\beta} = 2\pi \ln \Lambda_{\alpha\beta} \frac{q_\alpha^2 q_\beta^2}{m_\alpha} \int \frac{d^3\mathbf{v}'}{u^3} \left(u^2 \mathcal{I} - \mathbf{uu}\right) \cdot \left\{ \frac{1}{m_\alpha} f_\alpha(\mathbf{v}) \frac{\partial f_\beta(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{1}{m_\beta} f_\beta(\mathbf{v}') \frac{\partial f_\alpha(\mathbf{v})}{\partial \mathbf{v}} \right\} \tag{1.29}$$

3. We begin with the Fokker-Planck equation and split the distribution function f into a slowly changing equilibrium solution F_0 and a quickly changing behavior δf

$$f = F_0(\mathbf{v}, \tau) + \delta f(\mathbf{r}, \mathbf{v}, t, \tau) \tag{1.30}$$

where the long time behavior $\tau = t/\epsilon^2$.

1.6 Moments of the Distribution Function

1. Number Density n_s

$$n_s = \int d^3\mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \tag{1.31}$$

2. Current Density $\mathbf{J}$

$$\mathbf{J} = \sum_s \int d^3\mathbf{v} q_s \mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \tag{1.32}$$

1.7 Ordering the Equation

1.7.1 The Ordered Fokker-Planck Equation

$$\epsilon^2 \frac{\partial F_0}{\partial \tau} + \frac{\partial \delta f}{\partial t} + \epsilon^2 \frac{\partial \delta f}{\partial \tau} + \mathbf{v}_\perp \cdot \nabla \delta f + v_\parallel \hat{\mathbf{z}} \cdot \nabla \delta f + \frac{q}{m} \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \delta\mathbf{B} + \mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial F_0}{\partial \mathbf{v}} \tag{1.33}$$

$$+ \frac{q}{m} \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \delta\mathbf{B} + \mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial \delta f}{\partial \mathbf{v}} = C(F_0, F_0) + C(\delta f, F_0) + C(F_0, \delta f) + C(\delta f, \delta f) \tag{1.34}$$

$$
\begin{array}{ccccc}
\epsilon^2\partial F_0/\partial\tau & +\partial\delta f/\partial t & +\epsilon^2\partial\delta f/\partial\tau & +\mathbf{v}_\perp\cdot\nabla\delta f & +v_\parallel\hat{\mathbf{z}}\cdot\nabla\delta f \\
\epsilon^2 & \epsilon & \epsilon^3 & 1 & \epsilon \\
+\frac{q}{m} & (-\nabla\phi & -\partial\mathbf{A}/\partial t & +\mathbf{v}\times\delta\mathbf{B} & +\,\mathbf{v}\times\mathbf{B}_0)\cdot\partial F_0/\partial\mathbf{v} \\
 & 1 & \epsilon & 1 & 1/\epsilon \\
+\frac{q}{m} & (-\nabla\phi & -\partial\mathbf{A}/\partial t & +\mathbf{v}\times\delta\mathbf{B} & +\,\mathbf{v}\times\mathbf{B}_0)\cdot\partial\delta f/\partial\mathbf{v} \\
 & \epsilon & \epsilon^2 & \epsilon & 1 \\
= & C(F_0,F_0) & +C(\delta f,F_0) & +C(F_0,\delta f) & +C(\delta f,\delta f) \\
 & 1 & \epsilon & \epsilon & \epsilon^2
\end{array}
\tag{1.35}
$$

1.7.2 Order $\mathcal{O}(1/\epsilon)$

At lowest order we find

$$\frac{\partial F_0}{\partial\theta} = 0, \tag{1.36}$$

so the equilibirum distribution function does not depend on gyrophase,

$$F_0 = F_0(\mathcal{E}, \mu, \tau) \tag{1.37}$$

1.7.3 Order $\mathcal{O}(1)$

1. At first order we have the equation

$$\mathbf{v}_\perp\cdot\nabla\delta f_1 + \frac{q}{m}\{-\nabla\phi + \mathbf{v}\times\delta\mathbf{B}\}\cdot\frac{\partial F_0}{\partial\mathbf{v}} - \Omega_c\frac{\partial\delta f_1}{\partial\theta} = \mathcal{C}(F_0, F_0) \tag{1.38}$$

2. From Boltzmann's H theorem, we can show that the equilibrium distribution function takes the form

$$F_0 = \frac{n_0}{\pi^{3/2}}\frac{1}{v_{th}^3}\exp\left(-\frac{v^2}{v_{th}^2}\right) \tag{1.39}$$

   where the mean density $n_0 = n_0(\tau)$ evolves only on the long timescale.

3. Using this solution for F_0 and the fact that for an equilibirum state $\mathcal{C}(F_0, F_0) = 0$, we can simplify (1.38) to yield

$$\mathbf{v}_\perp\cdot\nabla\delta f_1 - \Omega_c\frac{\partial\delta f_1}{\partial\theta} = -\mathbf{v}\cdot\nabla\left(\frac{q\phi}{T}\right)F_0 \tag{1.40}$$

   This inhomogeneous equation supports a particular solution and a homogeneous solution.

4. The particular solution to this equation can be picked from this equation to be

$$\delta f_p = -\frac{q\phi}{T}F_0 \tag{1.41}$$

5. We add a homogeneous solution g to the solution for δf_1 to get a complete solution

$$\delta f_1 = -\frac{q\phi}{T}F_0 + g \tag{1.42}$$

where the homogeneous part of the solution must satisfy the equation

$$\mathbf{v}_\perp \cdot \frac{\partial g}{\partial \mathbf{r}} - \Omega_c \frac{\partial g}{\partial \theta} = 0 \tag{1.43}$$

If we express g in terms of the guiding center variables $g = g(\mathbf{R}, \mathcal{E}, \mu, \theta, t, \tau)$, changing variables simplifies this equation to

$$\left(\frac{\partial g}{\partial \theta}\right)_{\mathbf{R}} = 0. \tag{1.44}$$

This part of the solution is independent of gyrophase at constant guiding center $\mathbf{R}$ (but not at constant position $\mathbf{r}$).

6. We must go to the next order to learn how g evolves on the medium timescale.

7. NOTE: In our solution thus far, we can take the term $1 - q\phi/T = \exp(q\phi/T) + \mathcal{O}(\epsilon^2)$ and combine the ϵ^2 bit with the second order part of the solution δf_2.

8. Summary: We have found a solution of the form

$$f = F_0(\mathcal{E}, \tau)\exp(-\frac{q\phi(\mathbf{r}, t)}{T}) + g(\mathbf{R}, \mathcal{E}, \mu, t, \tau) + \delta f_2 + \cdots \tag{1.45}$$

The first term in the solution is the Boltzmann, or adiabatic, term; the second term is the gyroaveraged distribution function that we want to solve.

1.7.4 Order $\mathcal{O}(\epsilon)$

1. Here we go to second order to find an equation for g. We will also need to find equations for ϕ, $A_\parallel$, and $\delta B_\parallel$.

2. At second order we have the equation

$$\begin{aligned}\frac{\partial g}{\partial t} + \frac{\partial \mathbf{R}}{\partial t} \cdot \frac{\partial g}{\partial \mathbf{R}} + \frac{q}{m}\{-\nabla\phi + \mathbf{v} \times \delta\mathbf{B}\} \cdot \left\{\mathbf{v}\frac{\partial g}{\partial \mathcal{E}} + \frac{\mathbf{v}_\perp}{B_0}\frac{\partial g}{\partial \mu}\right\} - \mathcal{C}_L(g, F_0) = \\ \Omega_c \left(\frac{\partial \delta f_2}{\partial \theta}\right)_{\mathbf{R}} + \frac{q}{T}\left(\frac{\partial \phi}{\partial t} - \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial t}\right) F_0 + \mathcal{O}\epsilon^2 + \cdots\end{aligned} \tag{1.46}$$

3. We need to eliminate δf_2 from this equation; we accomplish this by gyroaveraging over θ at fixed guiding center position $\mathbf{R}$. We eliminate it using the fact that δf_2 must be periodic in θ.

$$\int_0^{2\pi} d\theta \frac{\partial \delta f_2}{\partial \theta} = 0 \tag{1.47}$$

4. After this ring average, we are left with

$$\frac{\partial g}{\partial t} + \left\langle \frac{\partial \mathbf{R}}{\partial t} \right\rangle_{\mathbf{R}} \cdot \frac{\partial g}{\partial \mathbf{R}} - \langle \mathcal{C}_L(g, F_0) \rangle_{\mathbf{R}} = \frac{q}{T} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} F_0 \tag{1.48}$$

where

$$\chi = \phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \tag{1.49}$$

5. A little more simplification yields the Gyrokinetic Equation

$$\frac{\partial g}{\partial t} + v_{\parallel} \cdot \hat{\mathbf{z}} \frac{\partial g}{\partial \mathbf{R}} + \frac{1}{B_0} \left[\langle \chi \rangle, g \right] - \langle \mathcal{C}_L(g, F_0) \rangle = \frac{q}{T} \frac{\partial \langle \chi \rangle}{\partial t} F_0 \tag{1.50}$$

Here the Poisson bracket, which is the nonlinear term, is defined by

$$[\langle \chi \rangle_{\mathbf{R}}, g] = \left(\frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial \mathbf{R}} \times \frac{\partial g}{\partial \mathbf{R}} \right) \cdot \hat{\mathbf{z}} = \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial y} \frac{\partial g}{\partial x} \tag{1.51}$$

1.8 Maxwell's Equations in Gyrokinetics

1. Ampere's Law:

$$\nabla \times \delta \mathbf{B} = \frac{4\pi}{c} \delta \mathbf{J} \tag{1.52}$$

2. Quasineutrality

$$\rho = 0 \tag{1.53}$$

3. Potentials

$$\delta \mathbf{B} = \nabla \times \mathbf{A} \tag{1.54}$$

and

$$\delta \mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \tag{1.55}$$

4. We choose the Coulomb Gauge

$$\nabla \cdot \mathbf{A} = 0 \tag{1.56}$$

Thus, we can decompose the current into

$$\nabla \times \delta \mathbf{B} = -\nabla^2 A_{\parallel} \hat{\mathbf{z}} + \nabla \delta B_{\parallel} \times \hat{\mathbf{z}} \tag{1.57}$$

This is true in the gyrokinetic limit, since, for the case Fourier components $\mathbf{k} = k_{\parallel} \hat{\mathbf{z}} + k_{\perp} \hat{\mathbf{x}}$,

$$\mathbf{J} = \nabla \times \mathbf{B} = (k_{\perp}^2 A_{\parallel} - k_{\parallel} k_{\perp} A_{\perp_x}) \hat{\mathbf{z}} + (k_{\parallel}^2 A_{\perp_x} - k_{\parallel} k_{\perp} A_{\parallel}) \hat{\mathbf{x}} + (k_{\perp}^2 A_{\perp_y} + k_{\parallel}^2 A_{\perp_y}) \hat{\mathbf{y}} \tag{1.58}$$

In the gyrokinetic limit, $k_{\parallel} \ll k_{\perp}$, so we can drop the terms with $k_{\parallel}$ to yield

$$\mathbf{J} = \nabla \times \mathbf{B} \simeq k_{\perp}^2 A_{\parallel} \hat{\mathbf{z}} + k_{\perp}^2 A_{\perp_y} \hat{\mathbf{y}} \tag{1.59}$$

We know that

$$B_{\parallel} = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{A} = i(\mathbf{k}_{\perp} \times \mathbf{A}_{\perp}) \cdot \hat{\mathbf{z}} = ik_{\perp} A_{\perp_y}, \tag{1.60}$$

so the relation for current can be expressed by (1.57)

5. Discussion of potentials in gyrokinetics, or how we reduce to just three free independent variables for the fields (from six).

   We specify the potentials

$$\delta\mathbf{B} = \nabla \times \mathbf{A} \tag{1.61}$$

and

$$\delta\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{1.62}$$

and choose the Coulomb Gauge

$$\nabla \cdot \mathbf{A} = 0. \tag{1.63}$$

The gyrokinetic ordering means that, to lowest order, $\nabla_{\perp} \cdot \mathbf{A}_{\perp} = 0$, so we can write the perpendicular component of the vector potential as the curl of a scalar $\mathbf{A}_{\perp} = \nabla \times \xi\hat{\mathbf{z}} = \nabla\xi \times \hat{\mathbf{z}}$. Thus, the vector potential depends on only two independent components $A_{\parallel}$ and ξ,

$$\mathbf{A} = A_{\parallel}\hat{\mathbf{z}} + \nabla\xi \times \hat{\mathbf{z}} \tag{1.64}$$

Solving for the magnetic field gives

$$\delta\mathbf{B} = \nabla A_{\parallel} \times \hat{\mathbf{z}} + \nabla(\nabla \cdot \xi\hat{\mathbf{z}}) - \nabla^2\xi\hat{\mathbf{z}} \tag{1.65}$$

For the case that $\mathbf{k} = k_{\perp}\hat{\mathbf{x}} + k_{\parallel}\hat{\mathbf{z}}$, we can Fourier transform the equation and solve for the magnetic field to find

$$\delta\hat{\mathbf{B}} = -k_{\perp}k_{\parallel}\xi\hat{\mathbf{x}} - ik_{\perp}\hat{A}_{\parallel}\hat{\mathbf{y}} + k_{\perp}^2\xi\hat{\mathbf{z}} \tag{1.66}$$

Choosing to express our potentials in terms of $\delta\hat{B}_{\parallel}$ instead of ξ, we substitute

$$\xi = \frac{\delta\hat{\mathbf{B}}_{\parallel}}{k_{\perp}^2} \tag{1.67}$$

Hence, the vector potential is expressed as

$$\hat{\mathbf{A}} = \hat{A}_{\parallel}\hat{\mathbf{z}} - i\frac{\delta\hat{\mathbf{B}}_{\parallel}}{k_{\perp}}\hat{\mathbf{y}}. \tag{1.68}$$

Hence, in the gyrokinetic ordering, the electric and magnetic fields are then given by

$$\delta\hat{\mathbf{E}} = -ik_{\perp}\hat{\phi}\hat{\mathbf{x}} + \frac{\omega\delta\hat{\mathbf{B}}_{\parallel}}{k_{\perp}c}\hat{\mathbf{y}} - ik_{\parallel}\left(\hat{\phi} - \frac{\omega\hat{A}_{\parallel}}{k_{\parallel}c}\right)\hat{\mathbf{z}} \tag{1.69}$$

and

$$\delta\hat{\mathbf{B}} = -\frac{k_{\parallel}}{k_{\perp}}\delta\hat{\mathbf{B}}_{\parallel}\hat{\mathbf{x}} - ik_{\perp}\hat{A}_{\parallel}\hat{\mathbf{y}} + \delta\hat{\mathbf{B}}_{\parallel}\hat{\mathbf{z}} \tag{1.70}$$

1.8.1 Quasineutrality

1. Quasineutrality

$$n_{0i}q = n_{0e}e \tag{1.71}$$

or

$$\sum_s n_s q_s = 0 \tag{1.72}$$

2. Since the number density is found by integrating the distribution function over velocity, we can find this by (1.31) using the distribution function from (1.45). The integration over velocity of the Boltzmann term becomes

$$q_s \int d^3\mathbf{v} F_{0s}(\mathcal{E},\tau) e^{-q_s\phi(\mathbf{r},t)/T_s} = q_s n_s e^{-q_s\phi(\mathbf{r},t)/T_s} \tag{1.73}$$

Expanding the exponential and dropping terms higher then order $\mathcal{O}(\epsilon)$, this terms yields

$$\sum_s q_s n_s \left(1 - \frac{q_s\phi(\mathbf{r},t)}{T_s}\right) = \sum_s \frac{q_s^2 n_s}{T_s}\phi(\mathbf{r},t) \tag{1.74}$$

since the equilibrium must satisfy neutrality $\sum_s q_s n_s = 0$

3. At order $\mathcal{O}(\epsilon)$ we have

$$\sum_s -\frac{q_s^2 n_s}{T_s}\phi + q_s \int d^3\mathbf{v} g_s(\mathbf{r} + \frac{\mathbf{v}\times\hat{\mathbf{z}}}{\Omega_s}, \mathcal{E}, \mu, t) = 0 \tag{1.75}$$

4. Performing a ring average at fixed $\mathbf{r}$, this becomes

$$\sum_s \left(-\frac{q_s^2 n_s}{T_s}\phi + q_s \int d^3\mathbf{v} \langle g_s\rangle_{\mathbf{r}}\right) = 0 \tag{1.76}$$

where

$$\int d^3\mathbf{v} = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} d\theta \tag{1.77}$$

1.8.2 Parallel Ampere's Law

1. Here we take the parallel component of Ampere's Law

$$\hat{\mathbf{z}} \cdot (\nabla \times \delta\mathbf{B}) = \frac{4\pi}{c}\delta\mathbf{J}_\parallel \tag{1.78}$$

2. The current can be found by taking a moment of the distribution function (1.45) according to (1.32) and integrating the over velocity. For the parallel component of the current, the contribution from the Boltzmann part of the distribution function vanishes upon integrating over $v_\parallel$.

3. After performing a ring average at fixed $\mathbf{r}$ and using (1.57), the parallel component of Ampere's Law yields the equation

$$-\nabla_\perp^2 A_\| = \frac{4\pi}{c}\delta J_\| = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} v_\| \langle g_s \rangle_\mathbf{r} \tag{1.79}$$

1.8.3 Perpendicular Ampere's Law

1. Here we take the perpendicular component of Ampere's Law

$$\hat{\mathbf{z}} \times (\nabla \times \delta\mathbf{B}) = \frac{4\pi}{c}\hat{\mathbf{z}} \times \delta\mathbf{J} \tag{1.80}$$

2. Again,the current is found by taking a moment of the distribution function (1.45) according to (1.32) and integrating the over velocity. For the perpendicular component of the current, the contribution from the Boltzmann part of the distribution function vanishes upon integrating over θ.

3. After performing a ring average at fixed $\mathbf{r}$ and using (1.57), the perpendicular component of Ampere's Law yields the equation

$$\nabla_\perp \delta B_\| = \frac{4\pi}{c}\hat{\mathbf{z}} \times \delta\mathbf{J} = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} \langle \hat{\mathbf{z}} \times \mathbf{v}_\perp g_s \rangle_\mathbf{r} \tag{1.81}$$

4. This can be written in a more physical way as a pressure balance in the direction perpendicular to the field

$$\nabla_\perp \cdot \left[\delta B_\| B_0 \mathcal{I} + \delta\mathcal{P}_\perp\right] = 0 \tag{1.82}$$

where $\mathcal{I}$ is the identity matrix and $\delta\mathcal{P}_\perp$ is the ring averaged perturbed perpendicular pressure tensor. Rewriting this, we find

$$\nabla \delta B_\| B_0 = -\frac{4\pi}{c}\nabla \cdot \delta\mathcal{P}_\perp = \sum_s \frac{4\pi}{c} \int d^3\mathbf{v} \langle m_s (\mathbf{v}_\perp \mathbf{v}_\perp) g_s(\mathbf{R}, \mathcal{E}, mu, t) \rangle_\mathbf{r} \tag{1.83}$$

1.9 The Driven Gyrokinetic Equation

To see the changes in the gyrokinetic equation when it is driven with an external antenna, we first must look at the driven Fokker-Planck Equation.

1.9.1 The Driven Fokker-Planck Equation

We begin with the full Fokker-Planck equation including a source on the right hand side due to an external driving antenna that creates a driving electric field $-\mathbf{E}_a$ (the negative sign is just so that the total electric field, plasma plus antenna, ends up as $\mathbf{E}+\mathbf{E}_a$)

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \sum_r C_{sr}(f_s,f_r) + C_{ss}(f_s,f_s) - \frac{q_s}{m_s}\mathbf{E}_a\cdot\frac{\partial f_s}{\partial \mathbf{v}} \tag{1.84}$$

Note that if we take the energy moment of this equation by multiplying by $\frac{1}{2}m_s v^2$, integrating over all space and velocity, and simplifying according to Section (2.1.1), we obtain

$$\frac{3}{2}n_{0s}\frac{dT_{0s}}{dt} + \frac{\partial}{\partial t}\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2\delta f_{2s} = \int\frac{d^3\mathbf{r}}{V}\mathbf{J}\cdot(\mathbf{E}+\mathbf{E}_a) + \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 C_{sr}(f_s,f_r). \tag{1.85}$$

1.9.2 Ordering the Driven Gyrokinetic Equation

Folowing the method of Section (1.7), we will note here the additions to the derivation of the gyrokinetic equation created by including the driving term. Splitting the driving antenna electric field into antenna potentials, $\mathbf{E}_a = -\nabla\phi_a - \frac{1}{c}\frac{\partial \mathbf{A}_a}{\partial t}$, and splitting the distribution function into its equilibrium and perturbed quantities, the right hand side of the gains the additional terms

$$\frac{q_s}{m_s c}\Big(\underset{1}{c\nabla\phi_a} \quad \underset{\epsilon}{+\partial\mathbf{A}_a/\partial t}\Big) \quad \cdot\partial F_{0s}/\partial\mathbf{v} \quad +\frac{q_s}{m_s c}\Big(\underset{\epsilon}{c\nabla\phi_a} \quad \underset{\epsilon^2}{+\partial\mathbf{A}_a/\partial t}\Big) \quad \cdot\partial\delta f_s/\partial\mathbf{v} \tag{1.86}$$

Now we will look at each of the orders of the equation and find the differences.

Order $\mathcal{O}(1/\epsilon)$

At lowest order there is no change from the usual derivation.

Order $\mathcal{O}(1)$

At first order, the equation for the first order perturbation can be found to be

$$\mathbf{v}_\perp\cdot\nabla\delta f_1 - \Omega_c\frac{\partial\delta f_1}{\partial\theta} = -\mathbf{v}\cdot\nabla\left(\frac{q\phi}{T}\right)F_0 - \mathbf{v}\cdot\nabla\left(\frac{q\phi_a}{T}\right)F_0 \tag{1.87}$$

which has the particular solution

$$\delta f_p = -\frac{q(\phi + \phi_a)}{T} F_0. \tag{1.88}$$

Hence, we eventually end up with a solution of the form

$$f_s = F_{0s}(\mathcal{E}, \tau) \exp(-\frac{q_s(\phi + \phi_a)}{T_s}) + h_s(\mathbf{R}_s, \mathcal{E}, \mu, t, \tau) + \delta f_{2s} + \cdots \tag{1.89}$$

Order $\mathcal{O}(\epsilon)$

At second order, we find we must replace all instances of ϕ by $\phi + \phi_a$; we also must change only the instances of $\mathbf{A}$ that are multiplied by F_{0s} to $\mathbf{A} + \mathbf{A}_a$. The resulting driven gyrokinetic equation for h_s becomes

$$\frac{\partial h_s}{\partial t} + v_\| \frac{\partial h_s}{\partial z} + \frac{c}{B}\left[\langle \chi \rangle_{\mathbf{R}_s}, h_s\right] + \frac{c}{B}\left[\langle \phi_a \rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}} \right\rangle_{\mathbf{R}_s} = \frac{q_s F_{0s}}{T_s} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} + \frac{q_s F_{0s}}{T_s} \frac{\partial \langle \phi_a - \mathbf{v}\cdot \mathbf{A}_a/c \rangle_{\mathbf{R}_s}}{\partial t} \tag{1.90}$$

1.9.3 Driven Alfvén Wave Antenna

In our case, we drive with an antenna whose only nonzero component is $A_{\|a}$, so the effective gyrokinetic equation that we need is

$$\frac{\partial h_s}{\partial t} + v_\| \frac{\partial h_s}{\partial z} + \frac{c}{B}\left[\langle \chi \rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}} \right\rangle_{\mathbf{R}_s} = \frac{q_s F_{0s}}{T_s} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} - \frac{q_s F_{0s}}{T_s} \frac{\partial \langle v_\| A_{\|a}/c \rangle_{\mathbf{R}_s}}{\partial t} \tag{1.91}$$

1.9.4 The Entropy Equation from the Driven Gyrokinetic Equation

We multiply the driven gyrokinetic equation (1.91) by $T_{0s}h_s/F_{0s}$ and integrate over all space and velocity at constant guiding center $\mathbf{R}$ to obtain the **driven entropy equation**

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q \frac{\partial \langle \chi \rangle}{\partial t} h_s - \frac{d}{dt} \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s C(h_s) \rangle_{\mathbf{R}_s} - \int \frac{d^3\mathbf{r}}{V} J_{\|s} E_{\|a} = 0. \tag{1.92}$$

where we have manipulated the driving antenna term by

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial \langle v_\| A_{\|a}/c \rangle_{\mathbf{R}_s}}{\partial t} = \int \frac{d^3\mathbf{r}}{V} \frac{1}{c} \frac{\partial A_{\|a}}{\partial t} \int d^3\mathbf{v} q_s v_\| \langle h_s \rangle_{\mathbf{r}} = \int \frac{d^3\mathbf{r}}{V} J_{\|s} E_{\|a} \tag{1.93}$$

and used the definition of the species parallel current $J_{\|s} = \int d^3\mathbf{v} q_s v_\| \langle h_s \rangle_{\mathbf{r}}$.

Chapter 2

Derivation of Heating in Gyrokinetics

2.1 Derivation of Entropy-Balance and Power-Balance Equations

2.1.1 Energy Equation: Second Moment of the Fokker-Planck Equation

We begin with the full Fokker-Planck equation

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s). \tag{2.1}$$

We begin by multiplying this equation by $\frac{1}{2} m_s v^2$, integrating over all space and velocity, and dividing by the box volume V.

Expanding the distribution function using

$$f_s = F_{0s} + \delta f_{1s} + \delta f_{2s} + \cdots \tag{2.2}$$

the first term becomes

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \frac{\partial f_s}{\partial t} = \frac{3}{2} n_{0s} \frac{\partial T_{0s}}{\partial t} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 f_{2s} \tag{2.3}$$

where we have used the definition

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 F_{0s} = p_{0s} = \frac{3}{2} n_{0s} T_{0s} \tag{2.4}$$

and the fact that

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \delta f_{1s} = 0 \tag{2.5}$$

because first-order perturbations spatially average to zero.

The second term can be integrated by parts to give

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \mathbf{v} \cdot \nabla f_s = \int d^3\mathbf{v} \int \frac{d^3\mathbf{r}}{V} \nabla \cdot \left(\frac{1}{2} m_s v^2 \mathbf{v} f_s \right) = 0 \tag{2.6}$$

where the boundary terms disappear by periodicity and the divergence theorem shows this term is zero also by periodicity.

The third term can be written using a similar integration by parts as

$$\begin{aligned}\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \frac{q_s}{m_s} \left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} &= -\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{q_s v^2}{2} \left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) f_s\right] \\ = \quad -\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \mathbf{v} \cdot \left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) f_s &= -\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \mathbf{v} \cdot \mathbf{E} f_s \end{aligned} \tag{2.7}$$

The collisional terms can be written

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m v^2 \sum_r C_{sr}(f_s, f_r) = \sum_r n_{0s} \nu_\epsilon^{sr} (T_r - T_s) \tag{2.8}$$

and

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m v^2 C_{ss}(f_s, f_s) = 0 \tag{2.9}$$

since like-particle collsions do not produce a loss of energy.

Thus, the second moment of the Fokker-Planck equation yields the **Energy Equation**

$$\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \delta f_{2s} = \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s (\mathbf{E}\cdot\mathbf{v}) f_s + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 C_{sr}(f_s, f_r). \tag{2.10}$$

An alternative way of writing this expression involves the plasma current

$$\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \delta f_{2s} = \int \frac{d^3\mathbf{r}}{V} \mathbf{J}_s \cdot \mathbf{E} + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 C_{sr}(f_s, f_r). \tag{2.11}$$

2.1.2 The Gyrokinetic Poynting's Theorem

Poynting's Theorem tells us that

$$\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) + \frac{1}{V} \frac{c}{4\pi} \oint (\mathbf{E}\times\mathbf{B}) \cdot d\mathbf{S} = -\int \frac{d^3\mathbf{r}}{V} (\mathbf{J} + \mathbf{J}_a) \cdot \mathbf{E}, \tag{2.12}$$

where $\mathbf{J}_a$ is the current in the antenna driving the system and $\mathbf{J}$ is the plasma current. For periodic boundary conditions, the surface term from the Poynting flux is zero. The gyrokinetic ordering specifies that the order of the electric field energy is $|\delta\mathbf{E}|^2 \sim \mathcal{O}(\epsilon^2 B_0^2 v_{th_i}^2/c^2)$ and the magnetic field energy is $|\delta\mathbf{B}|^2 \sim \mathcal{O}(\epsilon^2 B_0^2)$. Thus, in the non-relativistic limit, the magnetic energy dominates and we may drop the electric field energy; this is expected since the displacement current is dropped in the non-relativistic ordering. We are left with

$$\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \frac{|\delta\mathbf{B}|^2}{8\pi} + \int \frac{d^3\mathbf{r}}{V} \mathbf{J}_a \cdot \mathbf{E} = -\int \frac{d^3\mathbf{r}}{V} \mathbf{J} \cdot \mathbf{E}. \tag{2.13}$$

Derivation of the Plasma Current in Gyrokinetic Variables

To determine $\int \frac{d^3\mathbf{r}}{V}\mathbf{J}\cdot\mathbf{E}$, we note that

$$\mathbf{J} = \sum_s \int d^3\mathbf{v} q_s \mathbf{v} f_s \tag{2.14}$$

where f_s is the full distribution function from the Fokker-Planck equation (without any ring averaging). We also choose to express the electric field in terms of potentials,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{2.15}$$

Hence, for each species we have

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} = -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s}{c}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} f_s \tag{2.16}$$

First we want to manipulate the first term to extract a piece in the form of the gyrokinetic potential $\chi = \phi - \mathbf{v}\cdot\mathbf{A}/c$. Adding and subtracting $q_s f_s \frac{\partial\phi}{\partial t}$ produces

$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s \frac{\partial\phi}{\partial t} - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s\left(\frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi\right) \tag{2.17}$$

Using

$$f_s\frac{\partial\phi}{\partial t} = \frac{\partial\phi f_s}{\partial t} - \phi\frac{\partial f_s}{\partial t} \tag{2.18}$$

and integrating by parts in space on the last term we obtain

$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s = \tag{2.19}$$

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s \frac{\partial\phi}{\partial t} + \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \phi\left(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial\phi f_s}{\partial t}$$

We can use the Fokker-Planck equation to substitute for the second term

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s = -\frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} + \sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s) \tag{2.20}$$

and we find that this term becomes

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \phi\left(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s\right) = -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} \tag{2.21}$$

$$+ \quad \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \phi\left(\sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s)\right) \tag{2.22}$$

The integrals over the collision operators give zero because all collisions conserve particles. Integration by parts in velocity on the first term gives

$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right) f_s = 0 \tag{2.23}$$

Putting all of this together, we find

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s \frac{\partial}{\partial t}\left(\phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \phi f_s}{\partial t} \tag{2.24}$$

Next, we decompose the distribution function into a Maxwellian equilibrium and a fluctuating component $f_s = F_{0s} + \delta f_s$. The terms

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s F_{0s} \frac{\partial}{\partial t}\left(\phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \phi F_{0s}}{\partial t} = 0 \tag{2.25}$$

because the Maxwellian is spatially uniform and first order perturbations spatially average to zero. Now we can connect to the gyrokinetic system by identifying the gyrokinetic potential and splitting the fluctuating component of the distribution function into its gyrokinetic and Boltzmann pieces

$$\delta f_{1s} = \langle h_s \rangle_{\mathbf{r}} - \frac{q_s\phi}{T_s}F_{0s}. \tag{2.26}$$

Here we have defined the non-Boltzmann part, a function of position $\mathbf{r}$ and not guiding center $\mathbf{R}$, by the symbol $\langle h_s \rangle_{\mathbf{r}}$. We will specifically define the meaning of this symbol in terms of the standard gyrophase independent gyrokinetic distribution function later—for now we take it simply as some function of position $\mathbf{r}$. Doing this we obtain

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t}\langle h_s \rangle_{\mathbf{r}} - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2\phi}{T_s}F_{0s}\frac{\partial \chi}{\partial t} - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s\phi\langle h_s \rangle_{\mathbf{r}} - \frac{q_s^2\phi^2}{T_s}F_{0s}\right) \tag{2.27}$$

which can be rearranged to form

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t}\langle h_s \rangle_{\mathbf{r}} + \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2\phi}{T_s c}F_{0s}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s\phi\langle h_s \rangle_{\mathbf{r}} - \frac{q_s^2\phi^2}{2T_s}F_{0s}\right) \tag{2.28}$$

We can show (HOW?—$v_\parallel$ part is odd and integrates to zero, what about $v_\perp$ part?) that

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2\phi}{T_s c}F_{0s}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} = 0, \tag{2.29}$$

leaving us with the final result

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t}\langle h_s \rangle_{\mathbf{r}} - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s\phi\langle h_s \rangle_{\mathbf{r}} - \frac{q_s^2\phi^2}{2T_s}F_{0s}\right) \tag{2.30}$$

Completing the Gyrokinetic Poynting's Theorem

By substituting eq (2.30) into eq (2.13) we arrive at the **Gyrokinetic Poynting's Theorem**

$$\frac{\partial}{\partial t}\int\frac{d^3\mathbf{r}}{V}\left[\frac{|\delta\mathbf{B}|^2}{8\pi}+\int d^3\mathbf{v}\sum_s\left(-q_s\phi\langle h_s\rangle_{\mathbf{r}}+\frac{q_s^2\phi^2}{2T_s}F_{0s}\right)\right]+\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$
$$+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s q_s\frac{\partial\chi}{\partial t}\langle h_s\rangle_{\mathbf{r}}=0 \tag{2.31}$$

2.1.3 Derivation of the Entropy Equation

Next, we must look at the gyrokinetic equation to investigate the entropy balance

$$\frac{\partial h_s}{\partial t}+v_\parallel\hat{\mathbf{z}}\cdot\frac{\partial h_s}{\partial\mathbf{R}_s}+\frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s},h_s\right]-\left\langle\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}=\frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s}. \tag{2.32}$$

We multiply the gyrokinetic equation by $T_{0s}h_s/F_{0s}$ and integrate over space and velocity at constant guiding center $\mathbf{R}$ to obtain the **entropy equation**

$$\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q\frac{\partial\langle\chi\rangle}{\partial t}h_s-\frac{d}{dt}\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{2F_{0s}}h_s^2+\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\langle h_sC(h_s)\rangle_{\mathbf{R}_s}=0. \tag{2.33}$$

2.1.4 Derivation of the Power Balance Equation

By interchanging $\mathbf{r}$ and $\mathbf{v}$ integrations, we can show that

$$\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\frac{\partial\chi}{\partial t}\langle h_s\rangle_{\mathbf{r}}=\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}h_s. \tag{2.34}$$

Summing eq (2.33) over species, we can substitute into eq (2.103) to obtain

$$\frac{d}{dt}\left[\int\sum_s\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}h_s^2}{2F_{0s}}-\int\frac{d^3\mathbf{r}}{V}\sum_s\left(-q_s\phi\langle h_s\rangle_{\mathbf{r}}+\frac{q_s^2\phi^2}{2T_s}F_{0s}\right)+\int\frac{d^3\mathbf{r}}{V}\frac{|\delta\mathbf{B}|^2}{8\pi}\right]+\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$
$$-\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}=0 \tag{2.35}$$

Note that the first term can be alternatively expressed as

$$\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}h_s^2}{2F_{0s}}=\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}\langle h_s\rangle_{\mathbf{r}}^2}{2F_{0s}} \tag{2.36}$$

which enables us to combine the first three terms and factor to produce the **Power Balance Equation**

$$\frac{d}{dt}\left[\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{2F_{0s}}\left(\langle h_s\rangle_{\mathbf{r}}-\frac{q_s\phi}{T_s}F_{0s}\right)^2+\int\frac{d^3\mathbf{r}}{V}\frac{|\delta\mathbf{B}|^2}{8\pi}\right]+\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$
$$-\int\frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}=0 \tag{2.37}$$

Using the definition of the first-order perturbed distribution function δf_{1s} given by eq (2.26), this is seen to be

$$\frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\left(\int d^3\mathbf{v}\sum_s \frac{T_{0s}}{F_{0s}}\frac{|\delta f_{1s}|^2}{2}+\frac{|\delta\mathbf{B}|^2}{8\pi}\right)+\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$
$$-\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\sum_s \frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}=0 \tag{2.38}$$

2.2 Heating Using g

We begin with equation (15) from GKI.

$$\frac{\partial h_s}{\partial t}+v_\parallel\hat{\mathbf{z}}\cdot\frac{\partial h_s}{\partial \mathbf{R}_s}+\frac{c}{B}\left[\langle\chi\rangle_{\mathbf{R}_s},h_s\right]-\left\langle\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}=\frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s}. \tag{2.39}$$

NOTE: Is this c/B or c/B_0? First we define two quantities

$$\tilde{\phi}\equiv J_0(\frac{k_\perp v_\perp}{\Omega})\phi+\frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta B_\parallel}{B_0} \tag{2.40}$$

and

$$A\equiv J_0(\frac{k_\perp v_\perp}{\Omega})\frac{v_\parallel A_\parallel}{c} \tag{2.41}$$

so that we have

$$\langle\chi\rangle=\tilde{\phi}-A \tag{2.42}$$

Dropping the species subscript s and equilibrium subscripts 0, using $\mathcal{C}(h)$ to denote the collision operator, and substituting for $\langle\chi\rangle$, we get

$$\frac{\partial h}{\partial t}+v_\parallel\frac{\partial h}{\partial z}+\frac{c}{B}\left[\tilde{\phi}-A,h\right]-\langle\mathcal{C}(h)\rangle=\frac{q}{T}\frac{\partial(\tilde{\phi}-A)}{\partial t}F \tag{2.43}$$

Next, we define an alternative version of the gyrokinetic collision operator,

$$g\equiv h-\frac{q\tilde{\phi}}{T}F \tag{2.44}$$

Thus, the relation between g and h is

$$g\equiv h-\frac{q\langle\phi\rangle}{T}F-\frac{mv_\perp^2}{T}\frac{\langle\delta B_\parallel\rangle}{B_0}F \tag{2.45}$$

Writing the gyrokinetic equation in terms of g gives

$$\frac{\partial g}{\partial t}+v_\parallel\frac{\partial g}{\partial z}+\frac{q}{T}v_\parallel F\frac{\partial\tilde{\phi}}{\partial z}+\frac{c}{B}\left[\tilde{\phi}-A,h\right]-\langle\mathcal{C}(h)\rangle=-\frac{q}{T}F\frac{\partial A}{\partial t} \tag{2.46}$$

2.2.1 Heating in terms of g

Multiplying (12.24) by g and integrating over all space and velocity yields

$$\int g\frac{\partial g}{\partial t} + \int v_\parallel g\frac{\partial g}{\partial z} + \frac{q}{T}\int v_\parallel Fg\frac{\partial\tilde{\phi}}{\partial z} + \frac{c}{B}\int g\left[\tilde{\phi} - A, h\right] - \int g\langle \mathcal{C}(h)\rangle = -\frac{q}{T}\int Fg\frac{\partial A}{\partial t} \tag{2.47}$$

The second term on the LHS is zero for periodic BCs and the nonlinear (fourth term on LHS) can be written

$$\frac{c}{B}\int g\left[\tilde{\phi} - A, h\right] = \frac{c}{B}\int (h - \frac{q\tilde{\phi}}{T}F)\left[\tilde{\phi} - A, h\right] = \frac{q}{T}\frac{c}{B}\int \tilde{\phi}F\left[A, h\right] \tag{2.48}$$

since any terms multiplying the Poisson Bracket that are the same as terms within the Poisson bracket will vanish for periodic BCs. We are left with

$$\frac{1}{2}\frac{\partial}{\partial t}\int g^2 + \frac{q}{T}\int v_\parallel Fg\frac{\partial\tilde{\phi}}{\partial z} + \frac{q}{T}\frac{c}{B}\int \tilde{\phi}F\left[A, h\right] - \int g\langle \mathcal{C}(h)\rangle = -\frac{q}{T}\int Fg\frac{\partial A}{\partial t} \tag{2.49}$$

The second and third terms in (2.49) are problematic. We can eliminate these terms by multiplying (12.24) by $q\tilde{\phi}F/T$ and integrating over all space and velocity to find

$$\frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} + \frac{q}{T}\int v_\parallel\tilde{\phi}F\frac{\partial g}{\partial z} + \frac{q^2}{T^2}\int v_\parallel F^2\tilde{\phi}\frac{\partial\tilde{\phi}}{\partial z} + \frac{q}{T}\frac{c}{B}\int \tilde{\phi}F\left[\tilde{\phi} - A, h\right] - \frac{q}{T}\int \tilde{\phi}F\langle \mathcal{C}(h)\rangle = -\frac{q^2}{T^2}\int F^2\tilde{\phi}\frac{\partial A}{\partial t} \tag{2.50}$$

Integrating the second term by parts, noting that the third term will vanish for periodic BCs, and eliminating terms in the Poisson bracket in the fourth term yields

$$\frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} - \frac{q}{T}\int v_\parallel Fg\frac{\partial\tilde{\phi}}{\partial z} - \frac{q}{T}\frac{c}{B}\int \tilde{\phi}F\left[A, h\right] - \frac{q}{T}\int \tilde{\phi}F\langle \mathcal{C}(h)\rangle = -\frac{q^2}{T^2}\int F^2\tilde{\phi}\frac{\partial A}{\partial t} \tag{2.51}$$

Rearranging this equation, we see that we have found an alternative form of the two problematic terms in (2.49):

$$\frac{q}{T}\int v_\parallel Fg\frac{\partial\tilde{\phi}}{\partial z} + \frac{q}{T}\frac{c}{B}\int \tilde{\phi}F\left[A, h\right] = \frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} - \frac{q}{T}\int \tilde{\phi}F\langle \mathcal{C}(h)\rangle + \frac{q^2}{T^2}\int F^2\tilde{\phi}\frac{\partial A}{\partial t} \tag{2.52}$$

Therefore substituting into (2.49) gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int g^2 + \frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} - \frac{q}{T}\int \tilde{\phi}F\langle \mathcal{C}(h)\rangle + \frac{q^2}{T^2}\int F^2\tilde{\phi}\frac{\partial A}{\partial t} - \int g\langle \mathcal{C}(h)\rangle = -\frac{q}{T}\int Fg\frac{\partial A}{\partial t} \tag{2.53}$$

Collecting like terms gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int g^2 + \frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} + \frac{q}{T}\int F(g + \frac{q}{T}\tilde{\phi}F)\frac{\partial A}{\partial t} = \int (g + \frac{q}{T}\tilde{\phi}F)\langle \mathcal{C}(h)\rangle \tag{2.54}$$

Using the definition of g (12.22), this becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\int g^2 + \frac{q}{T}\int \tilde{\phi}F\frac{\partial g}{\partial t} + \frac{q}{T}\int Fh\frac{\partial A}{\partial t} = \int h\langle \mathcal{C}(h)\rangle \tag{2.55}$$

2.2.2 Connection to h form of heating

We can manipulate the form of (2.55) to demonstrate the result agrees with the heating in terms of h from GKI. First, we combine the first two terms

$$\int (g + \frac{q\tilde{\phi}}{T}F)\frac{\partial g}{\partial t} + \frac{q}{T}\int Fh\frac{\partial A}{\partial t} = \int h\langle \mathcal{C}(h)\rangle \tag{2.56}$$

Then, using again the definition of g (12.22), it can be written,

$$\int h\frac{\partial (h - \frac{q\tilde{\phi}}{T}F)}{\partial t} + \frac{q}{T}\int Fh\frac{\partial A}{\partial t} = \int h\langle \mathcal{C}(h)\rangle \tag{2.57}$$

Expanding the first term gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int h^2 - \frac{q}{T}\int Fh\frac{\partial(\tilde{\phi} - A)}{\partial t} = \int h\langle \mathcal{C}(h)\rangle \tag{2.58}$$

and identifying $\langle \chi\rangle = \tilde{\phi} - A$ gives the final form

$$\frac{1}{2}\frac{\partial}{\partial t}\int h^2 - \frac{q}{T}\int Fh\frac{\partial\langle\chi\rangle}{\partial t} = \int h\langle \mathcal{C}(h)\rangle \tag{2.59}$$

which agrees with equation (B10) from GKI.

2.2.3 Explicit Derivation of Equation (B19)

Neglecting interspecies collisions, we have equation (B5)

$$\frac{3}{2}n_{0s}\frac{dT_{0s}}{dt} + \frac{\partial}{\partial t}\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2\delta f_{2s} = \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\mathbf{v}\cdot\mathbf{E}f_s \tag{2.60}$$

where we can write

$$\mathbf{J}_s\cdot\mathbf{E} = \int d^3\mathbf{v}q_s\mathbf{v}\cdot\mathbf{E}f_s \tag{2.61}$$

Thus,

$$\frac{3}{2}n_{0s}\frac{dT_{0s}}{dt} + \frac{\partial}{\partial t}\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2\delta f_{2s} = \int\frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} \tag{2.62}$$

Also, we have equation (B9)

$$\frac{3}{2}n_{0s}\frac{dT_{0s}}{dt} + \frac{d}{dt}\left[\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(\frac{1}{2}m_s v^2\delta f_{2s} + q_s\phi h_s\right) - \int\frac{d^3\mathbf{r}}{V}\frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\right] = \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\frac{\partial\chi}{\partial t}h_s \tag{2.63}$$

Combining these two equations gives

$$\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_s\cdot\mathbf{E} + \frac{d}{dt}\left[\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\phi h_s - \int\frac{d^3\mathbf{r}}{V}\frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\right] = \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\frac{\partial\chi}{\partial t}h_s \tag{2.64}$$

NOTE: This combination is unnecessary. All we need is the gyrokinetic version of $\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E}$ which is just what this equation is (although the energy moment of the Fokker-Planck equation is used to get this result).

But we also have equation (B10),

$$\frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{2F_{0s}}h_s^2 - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s} = \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v} q_s \frac{\partial \langle \chi\rangle_{\mathbf{R}_s}}{\partial t}h_s. \tag{2.65}$$

Thus we can subsitute to get

$$\begin{aligned}\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} \quad &+ \quad \frac{d}{dt}\left[\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s\phi h_s - \int \frac{d^3\mathbf{r}}{V}\frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\right] \\ &= \quad \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{2F_{0s}}h_s^2 - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}\end{aligned} \tag{2.66}$$

We can combine time derivative terms, use the definition of density

$$n_{0s} = \int d^3\mathbf{v} F_{0s}(\mathbf{r}, \mathbf{v}, t), \tag{2.67}$$

sum over species and note that $\sum_s \mathbf{J}_s = \mathbf{J}$ to get

$$\begin{aligned}\int \frac{d^3\mathbf{r}}{V}\mathbf{J} \cdot \mathbf{E} \quad &+ \quad \frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\left(-\frac{T_{0s}}{2F_{0s}}h_s^2 + q_s\phi h_s - \frac{F_{0s}q_s^2\phi^2}{2T_{0s}}\right) \\ &= \quad -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}\end{aligned} \tag{2.68}$$

But Poynting's Theorem in the gyrokinetic approximation gives

$$\frac{d}{dt}\int d^3\mathbf{r}\frac{|\delta\mathbf{B}|^2}{8\pi} = -\int d^3\mathbf{r}(\mathbf{J} + \mathbf{J}_a) \cdot \mathbf{E}, \tag{2.69}$$

so we can subsitute to get

$$\begin{aligned}-\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_a \cdot \mathbf{E} \quad &+ \quad \frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\left(-\frac{T_{0s}}{2F_{0s}}h_s^2 + q_s\phi h_s - \frac{F_{0s}q_s^2\phi^2}{2T_{0s}}\right) - \frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\frac{|\delta\mathbf{B}|^2}{8\pi} \\ &= \quad -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}\end{aligned} \tag{2.70}$$

Finally, we get equation (B19),

$$\begin{aligned}\frac{d}{dt}\int \frac{d^3\mathbf{r}}{V} \quad &\left[\int d^3\mathbf{v}\sum_s\left(\frac{T_{0s}}{2F_{0s}}h_s^2 - q_s\phi h_s + \frac{F_{0s}q_s^2\phi^2}{2T_{0s}}\right) + \frac{|\delta\mathbf{B}|^2}{8\pi}\right] \\ = \quad &\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s} - \int \frac{d^3\mathbf{r}}{V}\mathbf{J}_a \cdot \mathbf{E}\end{aligned} \tag{2.71}$$

Another way of writing this is

$$
\begin{aligned}
\frac{d}{dt}\int\frac{d^3\mathbf{r}}{V} & \left[\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{2F_{0s}}\left(h_s-\frac{q_s\phi}{T_{0s}}F_{0s}\right)^2+\frac{|\delta\mathbf{B}|^2}{8\pi}\right] \\
&= \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}-\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}
\end{aligned}
\tag{2.72}
$$

2.3 Hyperviscous Heating

2.3.1 Energy Equation: Second Moment of the Fokker-Planck Equation

We begin with the full Fokker-Planck equation

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s) - \nu_H \nabla^4 f_s \tag{2.73}$$

where we have added to the right-side a hyperviscous term of the form $\nu_H \nabla^4 f_s$, although in general any even power of the ∇ operator will do. We begin by multiplying this equation by $\frac{1}{2}m_s v^2$, integrating over all space and velocity, and dividing by the box volume V.

Expanding the distribution function using

$$f_s = F_{0s} + \delta f_{1s} + \delta f_{2s} + \cdots \tag{2.74}$$

the first term becomes

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 \frac{\partial f_s}{\partial t} = \frac{3}{2}n_{0s}\frac{\partial T_{0s}}{\partial t} + \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 f_{2s} \tag{2.75}$$

where we have used the definition

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 F_{0s} = p_{0s} = \frac{3}{2}n_{0s}T_{0s} \tag{2.76}$$

and the fact that

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 \delta f_{1s} = 0 \tag{2.77}$$

because first-order perturbations spatially average to zero.

The second term can be integrated by parts to give

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 \mathbf{v}\cdot\nabla f_s = \int d^3\mathbf{v}\int \frac{d^3\mathbf{r}}{V}\nabla\cdot\left(\frac{1}{2}m_s v^2 \mathbf{v} f_s\right) = 0 \tag{2.78}$$

where the boundary terms disappear by periodicity and the divergence theorem shows this term is zero also by periodicity.

The third term can be written using a similar integration by parts as

$$\begin{aligned}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}m_s v^2 \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} &= -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\frac{q_s v^2}{2}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)f_s\right] \\ = \quad -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)f_s &= -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\mathbf{E}\end{aligned} \tag{2.79}$$

The collisional terms can be written

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{1}{2}mv^2 \sum_r C_{sr}(f_s, f_r) = \sum_r n_{0s}\nu_\epsilon^{sr}(T_r - T_s) \tag{2.80}$$

and

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m v^2 C_{ss}(f_s, f_s) = 0 \tag{2.81}$$

since like-particle collsions do not produce a loss of energy.

Finally, the hyperviscous term becomes

$$-\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m v^2 \nu_H \nabla^4 f_s = -\int d^3\mathbf{v} \int \frac{d^3\mathbf{r}}{V} \nabla^4 \left(\frac{1}{2} m v^2 \nu_H f_s \right) = 0 \tag{2.82}$$

due to the periodic boundary conditions.

Thus, the second moment of the Fokker-Planck equation yields the **Energy Equation**

$$\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \delta f_{2s} = \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s (\mathbf{E} \cdot \mathbf{v}) f_s + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 C_{sr}(f_s, f_r). \tag{2.83}$$

We see that hyperviscosity does not change the form of this expression. An alternative way of writing this expression involves the plasma current

$$\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 \delta f_{2s} = \int \frac{d^3\mathbf{r}}{V} \mathbf{J} \cdot \mathbf{E} + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{1}{2} m_s v^2 C_{sr}(f_s, f_r). \tag{2.84}$$

2.3.2 The Gyrokinetic Poynting's Theorem

Poynting's Theorem tells us that

$$\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right) + \frac{1}{V} \frac{c}{4\pi} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S} = -\int \frac{d^3\mathbf{r}}{V} (\mathbf{J} + \mathbf{J}_a) \cdot \mathbf{E}, \tag{2.85}$$

where $\mathbf{J}_a$ is the current in the antenna driving the system and $\mathbf{J}$ is the plasma current. For periodic boundary conditions, the surface term from the Poynting flux is zero. The gyrokinetic ordering specifies that the order of the electric field energy is $|\delta\mathbf{E}|^2 \sim \mathcal{O}(\epsilon^2 B_0^2 v_{th_i}^2 / c^2)$ and the magnetic field energy is $|\delta\mathbf{B}|^2 \sim \mathcal{O}(\epsilon^2 B_0^2)$. Thus, in the non-relativistic limit, the magnetic energy dominates and we may drop the electric field energy; this is expected since the displacement current is dropped in the non-relativistic ordering. We are left with

$$\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \frac{|\delta\mathbf{B}|^2}{8\pi} + \int \frac{d^3\mathbf{r}}{V} \mathbf{J}_a \cdot \mathbf{E} = -\int \frac{d^3\mathbf{r}}{V} \mathbf{J} \cdot \mathbf{E}. \tag{2.86}$$

Derivation of the Plasma Current in Gyrokinetic Variables

To determine $\int \frac{d^3\mathbf{r}}{V} \mathbf{J} \cdot \mathbf{E}$, we note that

$$\mathbf{J} = \sum_s \int d^3\mathbf{v} q_s \mathbf{v} f_s \tag{2.87}$$

where f_s is the full distribution function from the Fokker-Planck equation (without any ring averaging). We also choose to express the electric field in terms of potentials,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{2.88}$$

Hence, for each species we have

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} = -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s}{c}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} f_s \tag{2.89}$$

First we want to manipulate the first term to extract a piece in the form of the gyrokinetic potential $\chi = \phi - \mathbf{v}\cdot\mathbf{A}/c$. Adding and subtracting $q_s f_s \frac{\partial\phi}{\partial t}$ produces

$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s \frac{\partial\phi}{\partial t} - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s\left(\frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi\right) \tag{2.90}$$

Using

$$f_s\frac{\partial\phi}{\partial t} = \frac{\partial\phi f_s}{\partial t} - \phi\frac{\partial f_s}{\partial t} \tag{2.91}$$

and integrating by parts in space on the last term we obtain

$$\begin{aligned} -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \mathbf{v}\cdot\nabla\phi f_s = & \\ \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s f_s\frac{\partial\phi}{\partial t} + \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s\phi\left(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s\frac{\partial\phi f_s}{\partial t} & \end{aligned} \tag{2.92}$$

We can use the Fokker-Planck equation to substitute for the second term

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s = -\frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} + \sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s) - \nu_H\nabla^4 f_s \tag{2.93}$$

and we find that this term becomes

$$\begin{aligned} \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s\phi\left(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s\right) &= -\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} \\ &+ \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s\phi\left(\sum_r C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s)\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\nu_H\nabla^4 f_s \end{aligned} \tag{2.94}$$

The integrals over the collision operators give zero because all collisions conserve particles and the integral over the hyperviscous term gives zero after application of the divergence theorem with periodic boundary conditions. Integration by parts in velocity on the first term gives

$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2}{m_s}\phi\frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) f_s = 0 \tag{2.95}$$

Putting all of this together, we find

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} = \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s f_s \frac{\partial}{\partial t}\left(\phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \phi f_s}{\partial t} \tag{2.96}$$

Next, we decompose the distribution function into a Maxwellian equilibrium and a fluctuating component $f_s = F_{0s} + \delta f_s$. The terms

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s F_{0s} \frac{\partial}{\partial t}\left(\phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \phi F_{0s}}{\partial t} = 0 \tag{2.97}$$

because the Maxwellian is spatially uniform and first order perturbations spatially average to zero. Now we can connect to the gyrokinetic system by identifying the gyrokinetic potential and splitting the fluctuating component of the distribution function into its gyrokinetic and Boltzmann pieces

$$\delta f_s = h_s - \frac{q_s \phi}{T_s} F_{0s} \tag{2.98}$$

to obtain

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t} h_s - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} \frac{q_s^2 \phi}{T_s} F_{0s}\frac{\partial \chi}{\partial t} - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s \phi h_s - \frac{q_s^2\phi^2}{T_s}F_{0s}\right) \tag{2.99}$$

which can be rearranged to form

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t} h_s + \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} \frac{q_s^2 \phi}{T_s c} F_{0s}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s \phi h_s - \frac{q_s^2\phi^2}{2T_s}F_{0s}\right) \tag{2.100}$$

We can show (HOW?—$v_\parallel$ part is odd and integrates to zero, what about $v_\perp$ part?) that

$$\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} \frac{q_s^2 \phi}{T_s c} F_{0s}\frac{\partial \mathbf{v}\cdot\mathbf{A}}{\partial t} = 0, \tag{2.101}$$

leaving us with the final result

$$\int \frac{d^3\mathbf{r}}{V}\mathbf{J}_s \cdot \mathbf{E} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t} h_s - \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(q_s \phi h_s - \frac{q_s^2\phi^2}{2T_s}F_{0s}\right) \tag{2.102}$$

Completing the Gyrokinetic Poynting's Theorem

By substituting eq (2.30) into eq (2.13) we arrive at the **Gyrokinetic Poynting's Theorem**

$$\begin{aligned}
&\frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\left[\frac{|\delta \mathbf{B}|^2}{8\pi} + \int d^3\mathbf{v}\sum_s\left(-q_s\phi h_s + \frac{q_s^2\phi^2}{2T_s}F_{0s}\right)\right] + \int \frac{d^3\mathbf{r}}{V}\mathbf{J}_a \cdot \mathbf{E} \\
&+ \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s q_s \frac{\partial \chi}{\partial t} h_s = 0
\end{aligned} \tag{2.103}$$

2.3.3 Derivation of the Entropy Equation: Hyperviscosity

Next, we must look at the gyrokinetic equation, including the hyperviscous term, to investigate the entropy balance

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{\mathbf{z}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} + \frac{c}{B_0}\left[\langle \chi \rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}} \right\rangle_{\mathbf{R}_s} + \nu_H \nabla^4 h_s = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{2.104}$$

[QUESTION: Is the form of the hyperviscous term here consistent with that from the Fokker-Planck Equation?] We multiply the gyrokinetic equation by $T_{0s} h_s / F_{0s}$ and integrate over space and velocity to obtain the **entropy equation**

$$\begin{aligned} &\int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} q \frac{\partial \langle \chi \rangle}{\partial t} h_s - \frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s C(h_s) \rangle_{\mathbf{R}_s} \\ &- \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \nu_H \left[\left(\frac{\partial^2 h_s}{\partial^2 x}\right)^2 + 2\left(\frac{\partial^2 h_s}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y}\right)^2 \right] = 0 \end{aligned} \tag{2.105}$$

where we have applied the gyrokinetic approximation $k_\parallel \ll k_\perp$ to the hyperviscous operator $\nabla^4 \to \nabla_\perp^4$ and note here that

$$\nabla_\perp^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{2.106}$$

2.3.4 Derivation of the Power Balance Equation: Hyperviscosity

By interchanging $\mathbf{r}$ and $\mathbf{v}$ integrations, we can show that

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \frac{\partial \chi}{\partial t} h_s = \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} h_s. \tag{2.107}$$

Summing eq (2.105) over species, we can substitute into eq (2.103) to obtain the **Power Balance Equation**

$$\begin{aligned}
&\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \left[\int d^3\mathbf{v} \sum_s \frac{T_{0s}}{2F_{0s}} \left(h_s - \frac{q_s \phi}{T_{0s}} F_{0s} \right)^2 + \frac{|\delta \mathbf{B}|^2}{8\pi} \right] + \int \frac{d^3\mathbf{r}}{V} \mathbf{J}_a \cdot \mathbf{E} \\
&- \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \sum_s \frac{T_{0s}}{F_{0s}} \left\langle h_s \left(\frac{\partial h_s}{\partial t} \right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} \\
&+ \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \sum_s \frac{T_{0s}}{F_{0s}} \nu_H \left[\left(\frac{\partial^2 h_s}{\partial^2 x} \right)^2 + 2 \left(\frac{\partial^2 h_s}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y} \right)^2 \right] = 0
\end{aligned} \tag{2.108}$$

2.3.5 GS2 Normalization of Hyperviscous Heating Term

Note that in a Fourier decomposition we have

$$\left(\frac{\partial^2 h_s}{\partial^2 x} \right)^2 + 2 \left(\frac{\partial^2 h_s}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y} \right)^2 = (k_x^4 + 2k_x^2 k_y^2 + k_y^4) h_s^2 = k_\perp^4 h_s^2 \tag{2.109}$$

so we will use this as a shorthand for the term in the normalization below.

Now we need to normalize the hyperviscous term in the power balance equation; we'll focus on only this term here, but we'll do it in exactly the same way as the rest of the equation was done in section (26.2.3). We multiply the equation by $\frac{a_0}{v_{t0}} \left(\frac{a_0}{\rho_0} \right)^2 \frac{1}{v_{t0}^3 F_{00} T_0}$ to get

$$\begin{aligned}
&\left[\frac{a_0}{v_{t0}} \left(\frac{a_0}{\rho_0} \right)^2 \frac{1}{v_{t0}^3 F_{00} T_0} \right] \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \sum_s \frac{T_{0s}}{F_{0s}} \nu_H k_\perp^4 h_s^2 \\
&= \int \frac{d^3\mathbf{r}/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \sum_s \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}} \right)^3 \frac{T_s}{T_0} \frac{F_{0s}}{F_{00}} \left(\frac{\nu_H a_0}{\rho_0^4 v_{t0}} \right) (k_\perp^4 \rho_0^4) \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \right) \\
&= \int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \sum_s \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{T}_s \hat{\nu}_H \hat{k}_\perp^4 \hat{h}_s^2
\end{aligned} \tag{2.110}$$

Note that in GS2, the hyperviscosity is scaled by

```
D_v = D_hypervisc / akperp4_max
```
(2.111)

when the option `gridnorm = T`.

2.4 Hyperresistive Heating

Hyperresistivity is implemented by replacing $\partial \mathbf{A}/\partial t$ by $\partial \mathbf{A}/\partial t + \eta_H \nabla^4 \mathbf{A}$ in the gyrokinetic equation (which arises in the gyrokinetic potential χ on the right hand side).

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{\mathbf{z}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} + \frac{c}{B_0} \left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\rm coll} \right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial \langle\chi\rangle_{\mathbf{R}_s}}{\partial t} F_{0s} - \frac{q_s F_{0s} \eta_H}{T_{0s} c} \frac{\partial \langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \rangle_{\mathbf{R}_s}}{\partial t} \tag{2.112}$$

NOTE: One question is whether this will affect the χ in nonlinear term; but I will neglect that for the time being.

2.4.1 Derivation of the Entropy Equation: Hyperresistivity

We multiply the gyrokinetic equation by $T_{0s} h_s / F_{0s}$ and integrate over space and velocity to obtain the **Entropy Equation**

$$\begin{aligned} \frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 - \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \left\langle h_s C(h_s) \right\rangle_{\mathbf{R}_s} & \\ = \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} q_s \frac{\partial \left\langle \chi \right\rangle_{\mathbf{R}_s}}{\partial t} h_s - \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \frac{\eta_H}{c} q_s \left\langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \right\rangle_{\mathbf{R}_s} h_s & \end{aligned} \tag{2.113}$$

where we have applied the gyrokinetic approximation $k_\parallel \ll k_\perp$ to the hyperviscous operator $\nabla^4 \to \nabla_\perp^4$ and note here that

$$\nabla_\perp^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{2.114}$$

2.4.2 Positive Definite Form of Hyperresistive Heating Term

We can write the hyperresistive term alternatively by

$$\int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \frac{\eta_H}{c} q_s \left\langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \right\rangle_{\mathbf{R}_s} h_s = \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{\eta_H}{c} q_s \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \left\langle h_s \right\rangle_{\mathbf{r}} \tag{2.115}$$

and using the definition for the current due to a given species s (NOTE: I have left out the Boltzmann term here!!! ERROR!!!)

$$\mathbf{J}_s = \int d^3\mathbf{v} q_s \mathbf{v} \left\langle h_s \right\rangle_{\mathbf{r}} \tag{2.116}$$

the equation can be written

$$\begin{aligned} \frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 - \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \left\langle h_s C(h_s) \right\rangle_{\mathbf{R}_s} & \\ = \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} q \frac{\partial \left\langle \chi \right\rangle}{\partial t} h_s - \int \frac{d^3\mathbf{r}}{V} \frac{\eta_H}{c} \mathbf{J}_s \cdot \nabla_\perp^4 \mathbf{A} & \end{aligned} \tag{2.117}$$

We can derive a positive definite form of the heating due to hyperresistivity by noting that (in the Coulomb Gauge $\nabla \cdot \mathbf{A} = 0$), the gyrokinetic form of Ampere's Law is

$$-\nabla^2 \mathbf{A} = \sum_s \frac{4\pi}{c} \mathbf{J}_s \tag{2.118}$$

Hence, if we sum over species we find

$$\begin{aligned} &\frac{d}{dt} \sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 - \sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \left\langle h_s C(h_s) \right\rangle_{\mathbf{R}_s} \\ = \; &\sum_s \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} q \frac{\partial \langle \chi \rangle}{\partial t} h_s + \frac{\eta_H}{4\pi} \int \frac{d^3\mathbf{r}}{V} \nabla_\perp^2 \mathbf{A} \cdot \nabla_\perp^4 \mathbf{A} \end{aligned} \tag{2.119}$$

where have again taken the gyrokinetic limit to give $\nabla^2 \to \nabla_\perp^2$. Focusing on the hyperresistive term, we can integrate by parts an odd number of times on each term to yield a positive definite form of the heating

$$-\frac{\eta_H}{4\pi} \int \frac{d^3\mathbf{r}}{V} \left(\left| \frac{\partial^3 \mathbf{A}}{\partial x^3} \right|^2 + 2 \left| \frac{\partial^3 \mathbf{A}}{\partial x^2 \partial y} \right|^2 + 2 \left| \frac{\partial^3 \mathbf{A}}{\partial x \partial y^2} \right|^2 + \left| \frac{\partial^3 \mathbf{A}}{\partial y^3} \right|^2 \right). \tag{2.120}$$

Note that it may be possible to create an alternative positive definite form of the hyperresistive term involving only the distribution function by substituting for $-\nabla_\perp^4 \mathbf{A} = \sum_s \frac{4\pi}{c} \nabla^2 \mathbf{J}_s$, but this will involve cross terms between the electron and ion distribution functions that present computational difficulties with parallelization in `GS2` and so may not be practically useful.

2.4.3 Derivation of the Power Balance Equation: Hyperresistivity

In order to apply the Poynting's Theorem eq (2.13) to find the Power Balance Equation, we must find the form of $\int \frac{d^3\mathbf{r}}{V} \mathbf{J} \cdot \mathbf{E}$. [QUESTION: What is the form of hyperresistivity inthe Fokker-Planck Equation?] Assuming that the hyperresistive term does not contribute to any changes (as expected since the hyperviscous term changed nothing), we will use eq (2.30) as before, which gives the Gyrokinetic Poynting's Theorem eq (2.103).

Noting eq (2.107), we can summing eq (2.113) over species, we can substitute into eq (2.103) to obtain the **Power Balance Equation**

$$\begin{aligned} &\frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \left[\int d^3\mathbf{v} \sum_s \frac{T_{0s}}{2F_{0s}} \left(h_s - \frac{q_s \phi}{T_{0s}} F_{0s} \right)^2 + \frac{|\delta \mathbf{B}|^2}{8\pi} \right] + \int \frac{d^3\mathbf{r}}{V} \mathbf{J}_a \cdot \mathbf{E} \\ &- \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \sum_s \frac{T_{0s}}{F_{0s}} \left\langle h_s \left(\frac{\partial h_s}{\partial t} \right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} + \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \sum_s \frac{\eta_H}{c} q_s \left\langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \right\rangle_{\mathbf{R}_s} h_s = 0 \end{aligned} \tag{2.121}$$

2.4.4 Normalized Hyperresistive Power Balance

Now we need to determine the normalization of the hyperresistive term in the Power Balance equation. First, we must determine the form of this term in terms of the fields $A_\parallel$ and $\delta B_\parallel$. Taking the Fourier transform we we find the ring averaged part of the hyperresistive term becomes

$$\left\langle \frac{\mathbf{v} \cdot \nabla_\perp^4 \mathbf{A}}{c} \right\rangle_{\mathbf{R}_s} \to k_\perp^4 \left\langle \frac{\mathbf{v} \cdot \hat{\mathbf{A}}}{c} \right\rangle_{\mathbf{R}_s} = k_\perp^4 \left\langle \frac{v_\parallel \hat{A}_\parallel}{c} \right\rangle_{\mathbf{R}_s} + k_\perp^4 \left\langle \frac{\mathbf{v}_\perp \cdot \hat{\mathbf{A}}_\perp}{c} \right\rangle_{\mathbf{R}_s} \tag{2.122}$$

and we note that

$$\left\langle \frac{v_\parallel \hat{A}_\parallel}{c} \right\rangle_{\mathbf{R}_s} = J_0(\frac{k_\perp v_\perp}{\Omega_s}) \frac{v_\parallel}{c} \hat{A}_\parallel \tag{2.123}$$

and

$$\left\langle \frac{\mathbf{v}_\perp \cdot \hat{\mathbf{A}}_\perp}{c} \right\rangle_{\mathbf{R}} = -\frac{J_1(\frac{k_\perp v_\perp}{\Omega_s})}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m_s v_\perp^2}{q_s} \frac{\delta \hat{B}_\parallel}{B_0}. \tag{2.124}$$

Therefore the hyperresistive heating term for each species becomes

$$\begin{aligned} \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \quad \frac{\eta_H}{c} q_s \left\langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \right\rangle_{\mathbf{R}_s} h_s &= \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \eta_H k_\perp^4 q_s J_0(\gamma_s) \frac{v_\parallel}{c} \hat{A}_\parallel h_s \\ &- \int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \eta_H k_\perp^4 \frac{J_1(\gamma_s)}{\gamma_s} m_s v_\perp^2 \frac{\delta \hat{B}_\parallel}{B_0} h_s \end{aligned} \tag{2.125}$$

where $\gamma_s = \frac{k_\perp v_\perp}{\Omega_s}$.

As with the rest of the Power Balance Equation, we multiply by $\frac{a_0}{v_{t0}} \left(\frac{a_0}{\rho_0}\right)^2 \frac{1}{v_{t0}^3 F_{00} T_0}$ to obtain

$$\begin{aligned} &\int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} \frac{\eta_H}{c} q_s \left\langle \mathbf{v} \cdot \nabla_\perp^4 \mathbf{A} \right\rangle_{\mathbf{R}_s} h_s \\ &= \int \frac{d^3\mathbf{R}/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}} \left(\frac{\eta_H a_0}{\rho_0^4 v_t 0}\right) (k_\perp \rho_0)^4 J_0 \frac{q_s}{q_0} \frac{v_\parallel}{v_{ts}} \frac{v_{ts}}{v_{t0}} \left(\frac{v_{t0}}{c} \frac{q_0 A_\parallel}{T_0} \frac{a_0}{\rho_0}\right) \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0}\right) \\ &- \int \frac{d^3\mathbf{R}/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}} \left(\frac{\eta_H a_0}{\rho_0^4 v_t 0}\right) (k_\perp \rho_0)^4 \left(\frac{J_1}{\gamma_s}\right) \frac{m_s}{m_0} \left(\frac{v_\perp}{v_{ts}} \frac{v_{ts}}{v_{t0}}\right)^2 \left(\frac{\delta B_\parallel}{B_0} \frac{a_0}{\rho_0}\right) \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0}\right) \frac{v_{t0}^2 m_0}{T_0}. \end{aligned} \tag{2.126}$$

Noting that

$$\frac{v_{t0}^2 m_0}{T_0} = \left(\frac{2T_0}{m_0}\right) \frac{m_0}{T_0} = 2 \tag{2.127}$$

we can write this in normalized quantities as

$$\begin{aligned} &= \int \frac{d^3\hat{\mathbf{R}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{\eta_H} \hat{k}_\perp^4 \hat{J}_0 \hat{v}_{\parallel s} \hat{n}_s \hat{q}_s \sqrt{\frac{\hat{T}_s}{\hat{m}_s}} \hat{A}_\parallel \hat{h}_s \\ &- \int \frac{d^3\hat{\mathbf{R}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{\eta_H} \hat{k}_\perp^4 \hat{J}_1 2 \hat{v}_{\perp s}^2 \hat{n}_s \hat{T}_s \delta \hat{B}_\parallel \hat{h}_s \end{aligned} \tag{2.128}$$

2.4.5 Summary of Hyperdiffusivity Results

The equations here are written with the sign with which they are computed in GS2. NOTE: In GS2, Bill's sign convention for the collision term is the opposite of what it is here.

The general hyperdiffusive **Entropy Equation** is

$$\int \frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v} q_s \frac{\partial \langle\chi\rangle}{\partial t} h_s - \frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\langle h_s C(h_s)\rangle_{\mathbf{R}_s}$$
$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{\nu_H T_{0s}}{F_0}\left[\left(\frac{\partial^2 h_s}{\partial^2 x}\right)^2 + 2\left(\frac{\partial^2 h_s}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y}\right)^2\right]$$
$$-\int \frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v}\frac{\eta_H}{c} q_s \left\langle \mathbf{v}\cdot\nabla_\perp^4\mathbf{A}\right\rangle_{\mathbf{R}_s} h_s = 0 \tag{2.129}$$

The general hyperdiffusive **Power Balance Equation** is

$$\frac{d}{dt}\int \frac{d^3\mathbf{r}}{V}\left[\int d^3\mathbf{v}\sum_s \frac{T_{0s}}{2F_{0s}}\left(h_s - \frac{q_s\phi}{T_{0s}}F_{0s}\right)^2 + \frac{|\delta\mathbf{B}|^2}{8\pi}\right] + \int \frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$
$$-\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s \frac{T_{0s}}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\text{coll}}\right\rangle_{\mathbf{R}_s} + \int \frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v}\sum_s \frac{\eta_H}{c} q_s \left\langle \mathbf{v}\cdot\nabla_\perp^4\mathbf{A}\right\rangle_{\mathbf{R}_s} h_s$$
$$+\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s \frac{T_{0s}}{F_{0s}}\nu_H\left[\left(\frac{\partial^2 h_s}{\partial^2 x}\right)^2 + 2\left(\frac{\partial^2 h_s}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y}\right)^2\right] = 0 \tag{2.130}$$

2.4.6 Full Hyperresistive Term in Gyrokinetic Equation

We begin with the gyrokinetic equation including the hyperresistive term

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{\mathbf{z}}\cdot\frac{\partial h_s}{\partial \mathbf{R}_s} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \left\langle\left(\frac{\partial h_s}{\partial t}\right)_{\text{coll}}\right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s} - \frac{q_s F_{0s}\eta_H}{T_{0s}c}\frac{\partial\langle\mathbf{v}\cdot\nabla_\perp^4\mathbf{A}\rangle_{\mathbf{R}_s}}{\partial t} \tag{2.131}$$

We note that in the Fourier decomposition,

$$\left\langle\frac{\mathbf{v}\cdot\nabla_\perp^4\mathbf{A}}{c}\right\rangle_{\mathbf{R}_s} \to k_\perp^4 J_0(\frac{k_\perp v_\perp}{\Omega_s})\frac{v_\parallel}{c}\hat{A}_\parallel - k_\perp^4\frac{J_1(\frac{k_\perp v_\perp}{\Omega_s})}{\frac{k_\perp v_\perp}{\Omega_s}}\frac{m_s v_\perp^2}{q_s}\frac{\delta\hat{B}_\parallel}{B_0}. \tag{2.132}$$

Writing $\gamma_s = \frac{k_\perp v_\perp}{\Omega_s}$ and concentrating on only the normalization of this term, we follow the normalization of the gyrokinetic equation as before. Multiplying by $a_0^2/(F_{0s}\rho_0 v_{t0})$ we get

$$-\quad\left(\frac{a_0}{\rho_0}\frac{a_0}{v_{t0}}\frac{1}{F_{0s}}\right)\left(\frac{q_s F_{0s}}{T_{0s}}\eta_H k_\perp^4 J_0\frac{v_\parallel}{c}\hat{A}_\parallel + \frac{F_{0s}}{T_{0s}}\eta_H k_\perp^4\frac{J_1}{\gamma_s}m_s v_\perp^2\frac{\delta\hat{B}_\parallel}{B_0}\right) = \tag{2.133}$$
$$\frac{T_0}{T_s}\left(\frac{\eta_H a_0}{\rho_0^4 v_t 0}\right)(k_\perp\rho_0)^4\left[-\frac{q_s}{q_0}J_0\frac{q_s}{q_0}\frac{v_\parallel}{v_{ts}}\frac{v_{ts}}{v_{t0}}\left(\frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0}\right) + \left(\frac{J_1}{\gamma_s}\right)\frac{m_s}{m_0}\left(\frac{v_\perp}{v_{ts}}\frac{v_{ts}}{v_{t0}}\right)^2\left(\frac{\delta B_\parallel}{B_0}\frac{a_0}{\rho_0}\right)\frac{v_{t0}^2 m_0}{T_0}\right]$$
$$= -\hat{\eta_H}\hat{k}_\perp^4\frac{\hat{q}_s}{\sqrt{\hat{T}_s\hat{m}_s}}\hat{J}_0\hat{v_\parallel}_s\hat{A}_\parallel + \hat{\eta_H}\hat{k}_\perp^4\hat{J}_1 2\hat{v}_{\perp s}^2\delta\hat{B}_\parallel$$

Chapter 3

Derivation of Driven Gyrokinetics

In this chapter are assembled the equations for generally driven gyrokinetics and the heating equations in this case.

3.1 The Equations for Driven Gyrokinetics

We begin with the nonlinear, collisional gyrokinetic equation for h_s

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left[\langle \chi \rangle_{\mathbf{R}_s}, h_s \right] - \left\langle \left(\frac{\partial h_s}{\partial t} \right)_{\rm coll} \right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s} \tag{3.1}$$

and the gyrokinetic Maxwell's equations

$$-\frac{1}{4\pi} \nabla_\perp^2 \phi_a + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi = q_s \int d^3\mathbf{v} \langle h_s \rangle_{\mathbf{r}}, \tag{3.2}$$

$$-\frac{c}{4\pi} \nabla_\perp^2 (A_{\parallel a} + A_\parallel) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle h_s \rangle_{\mathbf{r}}, \tag{3.3}$$

$$\frac{c}{4\pi} \nabla_\perp (\delta B_{\parallel a} + \delta B_\parallel) = \sum_s q_s \int d^3\mathbf{v} \langle (\hat{\mathbf{z}} \times \mathbf{v}_\perp) h_s \rangle_{\mathbf{r}}. \tag{3.4}$$

The gyrokinetic-Maxwell equations can be driven generally by the terms ϕ_a, $A_{\parallel a}$, and $\delta B_{\parallel a}$.

3.2 The Equations for Driven Gyrokinetics

Chapter 4

Gyrokinetic Linear Dispersion Relation

4.1 Linear Collisionless Gyrokinetics

4.1.1 Summary of Gyrokinetic Equations

The starting equations for our linear analysis are

1. Linearized Collisionless Gyrokinetic Equation: Dropping the nonlinear term $[\langle\chi\rangle, g]$ and the collisional term $\langle C(g)\rangle$, we obtain the linearized gyrokinetic equation

$$\frac{\partial g}{\partial t} + v_\parallel \frac{\partial g}{\partial z} = \frac{q}{T} F_0 \frac{\partial \langle\chi\rangle_{\mathbf{R}}}{\partial t} \tag{4.1}$$

2. Quasineutrality

$$\sum_s \left(-\frac{q_s^2 n_s}{T_s}\phi + q_s \int d^3\mathbf{v} \langle g_s\rangle_{\mathbf{r}} \right) = 0 \tag{4.2}$$

3. Parallel Ampere's Law

$$-\nabla_\perp^2 A_\parallel = \frac{4\pi}{c}\delta J_\parallel = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} v_\parallel \langle g_s\rangle_{\mathbf{r}} \tag{4.3}$$

4. Perpendicular Ampere's Law

$$\nabla_\perp \delta B_\parallel = \frac{4\pi}{c}\hat{\mathbf{z}} \times \delta\mathbf{J} = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} \langle \hat{\mathbf{z}} \times \mathbf{v}_\perp g_s\rangle_{\mathbf{r}} \tag{4.4}$$

5. The perpendicular Ampere's Law can also be written as a Perpendicular Pressure Balance

$$\nabla \delta B_\parallel B_0 = -\frac{4\pi}{c}\nabla \cdot \delta\mathcal{P}_\perp = \sum_s \int d^3\mathbf{v} q_s \langle m_s(\mathbf{v}_\perp \mathbf{v}_\perp) g_s(\mathbf{R}, \mathcal{E}, \mu, t)\rangle_{\mathbf{r}} \tag{4.5}$$

4.1.2 Bessel Functions Definitions and Relations

1. One definition of the Bessel Function J_0 (9.1.18) is

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{iz\cos\theta} \tag{4.6}$$

2. Derivatives: (9.1.28)

$$\frac{dJ_0(z)}{dz} = -J_1(z) \tag{4.7}$$

3. Higher Order Forms: (9.1.21)

$$J_n(z) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} d\theta \cos(n\theta) e^{iz\cos\theta} \tag{4.8}$$

4. The form of an Ascending Series expansion for the Bessel functions (9.1.10) is given by

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)} \tag{4.9}$$

   Thus for the zeroth Bessel Function

$$J_0(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \cdots \tag{4.10}$$

   This can be used to find the small argument expansion for the Bessel Function.

5. The large argument expansion for the Bessel Function (9.2.1) is

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{\cos(z - \nu\pi/2 - \pi/4) + e^{|\mathcal{J}_z|}\mathcal{O}(|z|^{-1})\right\} \tag{4.11}$$

   This tells us that for large z

$$J_0(z) \sim \mathcal{O}(|z|^{-1}) \tag{4.12}$$

6. The modified Bessel function $I_n(z)$ obeys the derivative relation (9.6.27)

$$\frac{dI_0(z)}{dz} = I_1(z) \tag{4.13}$$

7. The recurrence relation for modified Bessel functions (9.6.26) is

$$\frac{dI_\nu(z)}{dz} = I_{\nu-1}(z) - \frac{\nu}{z}I_\nu(z) \tag{4.14}$$

8. Ascending series form for modified Bessel functions (9.6.10) is

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)} \tag{4.15}$$

9. From Watson, Basic Integrals of Bessel Functions,

$$\int_0^\infty e^{-a^2x^2} J_n(px)J_n(qx)xdx = \frac{1}{2a^2}e^{-(p^2+q^2)/(4a^2)}I_n\left(\frac{pq}{2a^2}\right) \tag{4.16}$$

4.1.3 Decompose into Bessel Functions

We decompose the quantities into plane waves. In this case, the ring averages at fixed guiding center $\mathbf{R}$ and at fixed position $\mathbf{r}$ simply becomes multiplications by Bessel functions.

1. We will assume fluctuating quantities of the form

$$q(\mathbf{r}, t) = \hat{q}(t)\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{4.17}$$

2. Consider first the effect of the spatial plane waves on the ring average at fixed guiding center $\mathbf{R}$ using $\mathbf{r} = \mathbf{R} + \rho$

$$\langle\phi(\mathbf{r}, t)\rangle_{\mathbf{R}} = \frac{1}{2\pi}\oint d\theta\hat{\phi}(t)e^{i\mathbf{k}\cdot(\mathbf{R}+\rho(\theta))} = \frac{1}{2\pi}\hat{\phi}(t)e^{i\mathbf{k}\cdot\mathbf{R}}\oint d\theta e^{i\mathbf{k}\cdot\rho(\theta)} \tag{4.18}$$

We can write

$$\mathbf{k}\cdot\rho = \mathbf{k}_\perp\cdot\rho = \frac{k_\perp v_\perp}{\Omega}\cos\alpha \tag{4.19}$$

where α is the angle between $\mathbf{k}_\perp$ and ρ. Changing integration variables from θ to α does not change the ring integral.

3. Thus, using the definition of the Bessel function (4.6), we can write the the ring average at fixed guiding center $\mathbf{R}$ as a multiplication by a Bessel function

$$\langle\phi(\mathbf{r}, t)\rangle_{\mathbf{R}} = J_0(\frac{k_\perp v_\perp}{\Omega})\hat{\phi}(t)e^{i\mathbf{k}\cdot\mathbf{R}} \tag{4.20}$$

4. Similarly, we can write the ring averages at constant $\mathbf{r}$ for variables in guiding center coordinates in terms of Bessel Functions

$$\langle g_\alpha(\mathbf{R}, \mathcal{E}, \mu_0, t)\rangle_{\mathbf{r}} = J_0(\frac{k_\perp v_\perp}{\Omega})\hat{g_\alpha}(\mathcal{E}, \mu_0, t)e^{i\mathbf{k}\cdot\mathbf{r}} \tag{4.21}$$

5. Taking plane waves for all of the potentials in $\langle\chi\rangle_{\mathbf{R}} = \langle\phi - v_\| A_\|/c - \mathbf{v}_\perp\cdot\mathbf{A}_\perp/c\rangle_{\mathbf{R}}$, we can derive an expression for the gyroaveraged potential in terms of Bessel Functions.

6. Both $\langle\phi\rangle_{\mathbf{R}}$ and $\langle v_\| A_\|/c\rangle_{\mathbf{R}}$ are derived as in (4.20). For the perpendicular component, we find

$$\langle\frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\rangle_{\mathbf{R}} = e^{i\mathbf{k}\cdot\mathbf{R}}\frac{1}{2\pi c}\oint d\theta\mathbf{v}_\perp(\theta)\cdot\mathbf{A}_\perp(t)e^{i\mathbf{k}\cdot\rho(\theta)} \tag{4.22}$$

Here we define an orthogonal coordiante system $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$ such that $\mathbf{k}_\perp = k_\perp\hat{\mathbf{e}}_1$ and, for θ as the angle between $\mathbf{k}_\perp$ and ρ, we have

$$\rho(\theta) = \rho(\cos\theta\hat{\mathbf{e}}_1 + \sin\theta\hat{\mathbf{e}}_2) \tag{4.23}$$

By (1.7) we get

$$\mathbf{v}_\perp(\theta) = v_\perp(\sin\theta\hat{\mathbf{e}}_1 - \cos\theta\hat{\mathbf{e}}_2) \tag{4.24}$$

and substituting above, we get

$$\langle\frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\rangle_\mathbf{R} = e^{i\mathbf{k}\cdot\mathbf{R}}\frac{1}{2\pi c}\oint d\theta v_\perp(\hat{A}_{\perp_1}\sin\theta - \hat{A}_{\perp_2}\cos\theta)e^{i(k_\perp v_\perp/\Omega)\cos\theta} \tag{4.25}$$

The first term on the right hand side is zero, while the second can be written using the $n = 1$ Bessel function (A.17). Using the definition of the cyclotron frequency (1.4), the result is

$$\langle\frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\rangle_\mathbf{R} = -\frac{J_1(\frac{k_\perp v_\perp}{\Omega})}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{ik_\perp\hat{A}_{\perp_2}}{B_0}e^{i\mathbf{k}\cdot\mathbf{R}} \tag{4.26}$$

Using

$$\delta\hat{B}_\parallel = i(\mathbf{k}_\perp \times \hat{A}_\perp)\cdot\hat{\mathbf{z}} = ik_\perp\hat{A}_{\perp_2}, \tag{4.27}$$

we can thus write the result as

$$\langle\frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\rangle_\mathbf{R} = -\frac{J_1(\frac{k_\perp v_\perp}{\Omega})}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta B_\parallel}{B_0}e^{i\mathbf{k}\cdot\mathbf{R}} \tag{4.28}$$

Therefore, we can express the potential in terms of $n = 0$ and $n = 1$ Bessel functions

$$\langle\chi\rangle_\mathbf{R} = \sum_\mathbf{k}\left[J_0(\frac{k_\perp v_\perp}{\Omega})(\hat{\phi} - \frac{v_\parallel\hat{A}_\parallel}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_\parallel}{B_0}\right]e^{i\mathbf{k}\cdot\mathbf{R}} \tag{4.29}$$

which can also be written as

$$\langle\chi\rangle_\mathbf{R} = \sum_\mathbf{k}\left[J_0(\frac{k_\perp v_\perp}{\Omega})(\hat{\phi} - \frac{v_\parallel\hat{A}_\parallel}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{2v_\perp^2}{v_{ts}^2}\frac{T_s}{q_s}\frac{\delta\hat{B}_\parallel}{B_0}\right]e^{i\mathbf{k}\cdot\mathbf{R}} \tag{4.30}$$

7. Assuming the time dependence for all quantities to be $e^{-i\omega t}$, we can solve for the distribution function $\hat{g}_\alpha$ from the linearized collisionless gyrokinetic equation (4.1) using the potential (6.36) and assuming a form of the distribution function of

$$g_s(\mathbf{R}, \mathcal{E}, \mu_0, t) = \hat{g}_s(\mathcal{E}, \mu_0)e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)}; \tag{4.31}$$

the resulting solution is

$$\hat{g}_s = \frac{q_s F_{0s}}{T_s}\left\{J_0(\gamma_s)\frac{\omega\hat{A}_\parallel}{k_\parallel c} + \frac{\omega}{\omega - k_\parallel v_\parallel}\left[J_0(\gamma_s)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \frac{J_1(\gamma_s)}{\gamma_s}\frac{m_s v_\perp^2}{q_s}\frac{\delta\hat{B}_\parallel}{B_0}\right]\right\} \tag{4.32}$$

where

$$\gamma_s = \frac{k_\perp v_\perp}{\Omega_s} \tag{4.33}$$

Here we have used the expression

$$\frac{\omega}{\omega - v_\parallel k_\parallel}v_\parallel = \frac{\omega}{\omega - v_\parallel k_\parallel}\frac{\omega}{k_\parallel} - \frac{\omega}{k_\parallel} \tag{4.34}$$

to simplify the result.

4.1.4 Velocity Integrals of Maxwell Equation Relations

The solution to the distribution function $\hat{g}_s$, (6.39), is a product of functions of $v_\parallel$ and $v_\perp$. This allows us to perform the integrals over velocity space and express the results in terms of modified Bessel function and plasma dispersion functions. In this way we can derive algebraic relations for the fields $\hat{\phi}$, $\hat{A}_\parallel$, and $\hat{B}_\parallel$ from the quaisneutrality condition (4.2), the parallel Ampere's Law (4.3), and the perpendicular Ampere's Law (4.4).

The velocity integrals are performed over $\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} v_\perp dv_\parallel dv_\perp d\theta$. The velocity integrals are the same for each species, so we need only calcualte these integrals for a general species s; accordingly, we drop the species subscript in this subsection.

Parallel Velocity Integrals: Plasma Dispersion Function

1. Integrating over $v_\parallel$ from $-\infty$ to $+\infty$, many of the integrals can be reduced to the form

$$Z(\xi_s) = \frac{1}{\sqrt{\pi}} \int_L dx \frac{e^{-x^2}}{x - \xi_s} \tag{4.35}$$

where the argument

$$\xi_s = \frac{\omega}{k_\parallel v_{th_s}} \tag{4.36}$$

and the integral is performed over the Landau contour from $-\infty$ to $+\infty$ below the pole at ξ_s in the complex plane.

Perpendicular Velocity Integrals: Modified Bessel Functions

In performing the integrations over $v_\perp$ from 0 to $+\infty$, we make use of several integrals derived here.

1. The first integral that arises is

$$\Gamma_0 = \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/v_{th}^2} \tag{4.37}$$

Using (4.16) this becomes

$$\Gamma_0(\beta) = I_0(\beta) e^{-\beta} \tag{4.38}$$

where

$$\beta = \frac{k_\perp^2 \rho^2}{2} \tag{4.39}$$

and the gyroradius is defined by $\rho = v_{th}/\Omega$.

2. The second integral that arises is given by

$$\Gamma_1 = \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} \frac{2v_\perp^2}{v_{th}^2} \frac{J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} e^{-v_\perp^2/v_{th}^2} \tag{4.40}$$

Denoting $p = k_\perp \rho = k_\perp v_{th}/\Omega$ and $x = v_\perp / v_{th}$, we note that

$$\frac{d\Gamma_0(px)}{dp} = -\int_0^\infty 2xdx2xJ_0(px)J_1(px)e^{-x^2} \tag{4.41}$$

using (A.16). Thus, we find

$$\Gamma_1 = -\frac{1}{p}\frac{\Gamma_0}{dp} = -\frac{\Gamma_0}{d\beta} = [I_0(\beta) - I_1(\beta)]e^{-\beta} \tag{4.42}$$

where we have used the relation for derivatives of the modified Bessel function (4.13).

3. To solve the third important derivative

$$\Gamma_2 = \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2}\frac{4v_\perp^4}{v_{th}^4}\left[\frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\right]^2 e^{-v_\perp^2/v_{th}^2} \tag{4.43}$$

we note that

$$\int_0^\infty x^3 e^{-a^2x^2}\left(\frac{J_1(px)}{p}\right)^2 dx = -\frac{d}{da^2}\left[\int_0^\infty xe^{-a^2x^2}\left(\frac{J_1(px)}{p}\right)^2 dx\right]. \tag{4.44}$$

Thus, performing the integration within the brackets on the right hand side of (4.44) using (4.16), we find

$$\Gamma_2 = -\frac{d}{da^2}\left[\frac{8}{2a^2}\frac{e^{2p^2/(4a^2)}}{p^2}I_1\left(\frac{p^2}{2a^2}\right)\right] \tag{4.45}$$

Doing the derivative and evaluating the result at $a = 1$ produces the result

$$\Gamma_2 = 2[I_0(\beta) - I_1(\beta)]e^{-\beta} = 2\Gamma_1 \tag{4.46}$$

4.1.5 Gyrokinetic and Maxwell's Equations

Quasineutrality

1. We begin with the gyrokinetic quasineutrality condition (4.2); in the velocity integral, we write the ring average at constant position $\mathbf{r}$ as a multiplication by a Bessel function as in (4.21). Thus, the integral becomes

$$2\pi\int_{-\infty}^\infty dv_\parallel \int_0^\infty v_\perp dv_\perp J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{q^2F_0}{T}\left\{J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{\omega\hat{A}_\parallel}{k_\parallel c} + \right.$$
$$\left.\frac{\omega}{\omega - v_\parallel k_\parallel}\left[J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_\parallel}{B_0}\right]\right\}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{4.47}$$

2. Using (1.39) for the equilibrium distribution function F_0 and performing the integrations over $v_\parallel$ and $v_\perp$, and using (A.1) the first term becomes

$$\frac{q^2n}{T}\Gamma_0(\beta)\frac{\omega\hat{A}_\parallel}{k_\parallel c}. \tag{4.48}$$

Writing the parallel velocity integral in terms of the plasma dispersion function by (4.35) and performing the perpendicular velocity integration in the same way as above, the second term becomes

$$-\frac{q^2 n}{T}\Gamma_0(\beta)\xi Z(\xi)\left(\hat{\phi}-\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right). \tag{4.49}$$

The third term, after employing (4.40) for the perpendicular integral and the plasma dispersion function (4.35) for the parallel integral, becomes

$$-\frac{q^2 n}{T}\Gamma_1(\beta)\xi Z(\xi)\left(\frac{T}{q}\frac{\delta \hat{B}_\parallel}{B_0}\right). \tag{4.50}$$

3. Using (4.20) for the potential in the Boltzmann term (the first term in the sum in (4.2)), dropping the factor $\exp(i\mathbf{k}\cdot\mathbf{r})$, and summing over species, the quasineutrality condition becomes

$$\sum_s \frac{q_s^2 n_s}{T_s}\left\{[1+\Gamma_{0s}\xi_s Z(\xi_s)]\left(\hat{\phi}-\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right)+(1-\Gamma_{0s})\left(\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right)+\Gamma_{1s}\xi_s Z(\xi_s)\left(\frac{T_s}{q_s}\frac{\delta \hat{B}_\parallel}{B}\right)\right\}=0 \tag{4.51}$$

Parallel Ampere's Law

1. To perform the velocity integrals for the parallel component of Ampere's Law, we use the linearized gyrokinetic equation (4.1) to find an expression for $J_\parallel$ by multiplying the equation for both species by the charge, integrating over velocity, summing over species, and taking the ring average at constant position $\mathbf{r}$.

$$\left\langle \sum_s \int d^3\mathbf{v}\left[\frac{\partial g}{\partial t}+v_\parallel \frac{\partial g}{\partial z}-\frac{q}{T}F_0\frac{\partial \langle\chi\rangle_{\mathbf{R}}}{\partial t}\right]\right\rangle_{\mathbf{r}}=0 \tag{4.52}$$

2. The first term of (4.52) simply becomes, using the condition $\sum_s n_s q_s = 0$,

$$\langle \sum_s \int d^3\mathbf{v} q_s \frac{\partial g_s}{\partial t}\rangle_{\mathbf{r}} = -\frac{\partial}{\partial t}\sum_s \frac{q_s^2 n_s}{T_s}\phi \tag{4.53}$$

3. The second term of (4.52), using (1.32), becomes

$$\langle \sum_s \int d^3\mathbf{v} q_s v_\parallel \frac{\partial g_s}{\partial z}\rangle_{\mathbf{r}} = \frac{\partial}{\partial z}\sum_s \int d^3\mathbf{v} q_s v_\parallel \langle g_s\rangle_{\mathbf{r}} = \frac{\partial}{\partial z}J_\parallel \tag{4.54}$$

4. The third term of (4.52) becomes

$$-\langle \sum_s \int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_s}\frac{\partial \langle\chi\rangle_{\mathbf{R}_s}}{\partial t}\rangle_{\mathbf{r}} = -\frac{\partial}{\partial t}\sum_s \int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_s}\langle\langle\chi\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} \tag{4.55}$$

The parallel term in the definition of χ (1.49) integrates to zero, leaving only the ϕ and perpendicular term.

5. Combining terms yields the relation

$$\frac{\partial}{\partial z}J_{\|} = -\frac{\partial}{\partial t}\left\{\sum_s \frac{q_s^2 n_s}{T_s}\left(\phi - \langle\langle\phi\rangle\rangle_s + \langle\langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp \rangle\rangle_s\right)\right\} \tag{4.56}$$

where the double brackets denote the integration

$$\langle\langle a(\mathbf{r},\mathbf{v},t)\rangle\rangle_s = \int_0^\infty v_\perp dv_\perp \frac{m_s}{T_s} e^{-m_s v_\perp^2/(2T_s)} \langle\langle a(\mathbf{r},\mathbf{v},t)\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} \tag{4.57}$$

6. The parallel component of Ampere's Law demands that

$$k_\perp^2 \hat{A}_{\|} = \frac{4\pi}{c}\hat{J}_{\|}, \tag{4.58}$$

so first we use (4.56) to find $\hat{J}_{\|}$ assuming it to vary as $\exp -i(\mathbf{k}\cdot\mathbf{r} - \omega t)$.

7. First we show that

$$\langle\langle \phi(\mathbf{r},t)\rangle\rangle_s = \langle J_0(\frac{k_\perp v_\perp}{\Omega})\hat{\phi}(t)e^{i\mathbf{k}\cdot\mathbf{R}}\rangle_{\mathbf{r}} = \hat{\phi}J_0^2(\frac{k_\perp v_\perp}{\Omega})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{4.59}$$

Then, using (A.1), the first and second terms on right hand side of (4.56) combine to give

$$i\omega \sum_s \frac{q_s^2 n_s}{T_s}[1 - \Gamma_0(\beta)]\hat{\phi}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4.60}$$

8. Evaluating the double bracket in the third term on the right hand side of (4.56) with the help of (A.17) gives

$$\langle\langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp\rangle\rangle_s = \langle -\frac{J_1(\frac{k_\perp v_\perp}{\Omega})}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_{\|}}{B_0}e^{i(\mathbf{k}\cdot\mathbf{R}-\omega t)}\rangle_{\mathbf{r}} = -\frac{J_0(\frac{k_\perp v_\perp}{\Omega})J_1(\frac{k_\perp v_\perp}{\Omega})}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_{\|}}{B_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4.61}$$

With this result and (4.40), this term becomes

$$-i\omega \sum_s \frac{q_s^2 n_s}{T_s}\Gamma_1(\beta_s)\frac{T_s}{q_s}\frac{\delta\hat{B}_{\|}}{B_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4.62}$$

9. Thus, putting these results together into (4.58) and cancelling the exponentials, we obtain the result

$$k_\perp^2 A_{\|} = \frac{4\pi\omega}{ck_{\|}}\sum_s \frac{q_s^2 n_s}{T_s}\left\{[1-\Gamma_0(\beta)]\hat{\phi} - \Gamma_1(\beta_s)\frac{T_s}{q_s}\frac{\delta\hat{B}_{\|}}{B_0}\right\} \tag{4.63}$$

10. Using the result from the quasineutrality condition (4.51), we can eliminate the $\hat{\phi}$ term to obtain

$$\frac{k_\perp^2 k_{\|}^2 c^2}{4\pi\omega^2}\left(\frac{\omega\hat{A}_{\|}}{k_{\|}c}\right) = -\sum_s \frac{q_s^2 n_s}{T_s}\left\{\Gamma_{0s}[1+\xi_s Z(\xi_s)]\left(\hat{\phi} - \frac{\omega\hat{A}_{\|}}{k_{\|}c}\right) + \Gamma_{1s}[1+\xi_s Z(\xi_s)]\left(\frac{T_s}{q_s}\frac{\delta\hat{B}_{\|}}{B}\right)\right\} \tag{4.64}$$

Perpendicular Ampere's Law

1. We begin with (4.4) and noting that

$$\hat{\mathbf{z}} \times \mathbf{v}_\perp = v_\perp(\sin\theta\hat{\mathbf{e}}_2 + \cos\theta\hat{\mathbf{e}}_1), \tag{4.65}$$

so that

$$\langle(\hat{\mathbf{z}} \times \mathbf{v}_\perp)g(\mathbf{R}, \mathcal{E}, \mu, t)\rangle_{\mathbf{r}} = \hat{g}(\mathcal{E}, \mu, t)e^{i\mathbf{k}\cdot\mathbf{r}}\frac{v_\perp}{2\pi}\oint d\theta(\sin\theta\hat{\mathbf{e}}_2 + \cos\theta\hat{\mathbf{e}}_1)e^{i(k_\perp v_\perp/\Omega)\cos\theta} \tag{4.66}$$

The first term integrates to zero, leaving

$$\langle(\hat{\mathbf{z}} \times \mathbf{v}_\perp)g(\mathbf{R}, \mathcal{E}, \mu, t)\rangle_{\mathbf{r}} = -i\hat{g}(\mathcal{E}, \mu, t)v_\perp J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}_1 \tag{4.67}$$

using the definition of $n = 1$ Bessel functions (A.17).

2. Focusing on the $\hat{\mathbf{e}}_1$ component, the velocity integral becomes

$$2\pi i\int_{-\infty}^{\infty}dv_\parallel\int_0^\infty v_\perp^2 dv_\perp J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{q^2F_0}{T}\left\{J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{\omega\hat{A}_\parallel}{k_\parallel c} + \right.$$

$$\left.\frac{\omega}{\omega - v_\parallel k_\parallel}\left[J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_\parallel}{B_0}\right]\right\}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4.68}$$

3. Using (4.40), the first term in this integral is

$$i\frac{v_{th}^2 k_\perp}{2\Omega}\frac{q^2 n}{T}\Gamma_1(\beta)\frac{\omega\hat{A}_\parallel}{k_\parallel c}. \tag{4.69}$$

Employing the plasma dispersion function (4.35) for the parallel integral and (4.40) for the perpendicular integral, the second term reduces to

$$-i\frac{v_{th}^2 k_\perp}{2\Omega}\frac{q^2 n}{T}\Gamma_1(\beta)\xi Z(\xi)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right). \tag{4.70}$$

Again using same approach for the parallel integral, but using (4.43) for the perpendicular integral, the third term becomes

$$-i\frac{v_{th}^2 k_\perp}{2\Omega}\frac{q^2 n}{T}\Gamma_2(\beta)\xi Z(\xi)\left(\frac{T}{q}\frac{\delta\hat{B}_\parallel}{B_0}\right). \tag{4.71}$$

4. Summing over terms and using the relations (1.4) and (1.5)to simplify, we find the perpendicular Ampere's Law produces

$$-\frac{B^2}{4\pi}\left(\frac{\delta\hat{B}_\parallel}{B}\right) = \sum_s q_s n_s\left[\Gamma_{1s}\left(\frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) - \Gamma_{1s}\xi_s Z(\xi_s)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) - \Gamma_{2s}\xi_s Z(\xi_s)\left(\frac{T_s}{q_s}\frac{\delta\hat{B}_\parallel}{B}\right)\right] \tag{4.72}$$

4.1.6 Solving for the Dispersion Relation

1. We assume a hydrogen plasma so that $n_i = n_e$ and $q_i = -q_e = e$. Using this simplification we can factor out n_i, q_i, and T_i from the sums over species s.

2. First, we note that

$$\frac{k_\perp^2 k_\parallel^2 c^2}{4\pi\omega^2}\frac{T_i}{q_i^2 n_i} = \frac{k_\perp^2\rho_i^2}{2}\frac{k_\parallel^2 v_A^2}{\omega^2}. \tag{4.73}$$

   Dividing (4.51) and (4.64) by $q_i^2 n_i/T_i$ and collecting like terms yields

$$\sum_s \frac{T_i}{T_s}[1+\Gamma_{0s}\xi_s Z(\xi_s)]\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{T_i}{T_s}(1-\Gamma_{0s})\left(\frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\xi_s)\left(\frac{T_i}{q_i}\frac{\delta\hat{B}_\parallel}{B}\right) = 0 \tag{4.74}$$

   and

$$\sum_s \frac{T_i}{T_s}\Gamma_{0s}[1+\xi_s Z(\xi_s)]\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \frac{k_\perp^2\rho_i^2}{2}\frac{k_\parallel^2 v_A^2}{\omega^2}\left(\frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{q_s}{q_i}\Gamma_{1s}[1+\xi_s Z(\xi_s)]\left(\frac{T_i}{q_i}\frac{\delta\hat{B}_\parallel}{B}\right) = 0 \tag{4.75}$$

3. Second, we note

$$\frac{B^2}{4\pi n_i q_i} = \frac{2}{\beta_i}\frac{T_i}{q_i}. \tag{4.76}$$

   Similarly, we can divide (4.72) by $q_i n_i$ to get

$$\sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\xi_s)\left(\hat{\phi} - \frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) - \sum_s \frac{q_s}{q_i}\Gamma_{1s}\left(\frac{\omega\hat{A}_\parallel}{k_\parallel c}\right) + \left[-\frac{2}{\beta_i} + \sum_s \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\xi_s)\right]\left(\frac{T_i}{q_i}\frac{\delta\hat{B}_\parallel}{B}\right) = 0 \tag{4.77}$$

4. We can write the coefficients for this set of equations more clearly by denoting the following sums as

$$A = \sum_s \frac{T_i}{T_s}[1+\Gamma_{0s}\xi_s Z(\xi_s)] \tag{4.78}$$

$$B = \sum_s \frac{T_i}{T_s}(1-\Gamma_{0s}) \tag{4.79}$$

$$C = \sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\xi_s) \tag{4.80}$$

$$D = \sum_s \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\xi_s) \tag{4.81}$$

$$E = \sum_s \frac{q_s}{q_i}\Gamma_{1s} \tag{4.82}$$

$$\alpha_s = \frac{k_\perp^2\rho_s^2}{2} \tag{4.83}$$

and

$$\overline{\omega} = \frac{\omega}{k_\parallel v_A} \tag{4.84}$$

5. Using these definitions, our system of equations can be written in matrix form as

$$\begin{pmatrix} A & B & C \\ A-B & \alpha_i/\overline{\omega}^2 & C+E \\ C & -E & D-2/\beta_i \end{pmatrix} \begin{pmatrix} \hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B} \end{pmatrix} = 0 \tag{4.85}$$

6. Setting the determinant of the matrix equal to zero yields the dispersion relation

$$\left[\frac{\alpha_i}{\overline{\omega}^2} - B + \frac{B^2}{A}\right]\left[\frac{2}{\beta_i} - D + \frac{C^2}{A}\right] - \left[E + \frac{BC}{A}\right]^2 = 0 \tag{4.86}$$

The physical interpretation of this dispersion relation, to be described in Section (5.1), is simple: the first term in (4.135) contains two terms, the first corresponding to the Alfvén wave solution and the second to the slow wave solution; the second term represents a coupling between the Alfvén and slow waves that occurs as $k_\perp \rho_i$ approaches unity.

4.2 Krook Collision Operator (Incorrect)

Beginning with the gyrokinetic equation

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{\mathbf{z}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} + \frac{c}{B_0}\left[\langle \chi \rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}} \right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{4.87}$$

we drop the nonlinear term (the Poisson Bracket) and choose a simple Krook collision operator of the form

$$\mathcal{C}(f_j) = -\sum_k \nu_{jk}(f_j - F_{0jk}) \tag{4.88}$$

where the term F_{0jk} represents a Maxwellian equilibrium. We consider only like-particle collisions and so reduce this collision operator to

$$\mathcal{C}(f_j) = -\nu_j(f_j - F_{0j}) \tag{4.89}$$

where we have defined $\nu_j \equiv \nu_{jj}$. The gyrokinetic distribution function is given by

$$f_s = F_{0s}(v,t) - \frac{q_s \phi(\mathbf{r},t)}{T_{0s}} F_{0s}(v,t) + h_s(\mathbf{R}_s, v, v_\perp, t) + \delta f_{2s} + \cdots . \tag{4.90}$$

Dropping second order terms, the zeroth order term of the collision operator cancels, leaving us with the first order term

$$\mathcal{C}(f_s) = -\nu_s\left(h_s - \frac{q_s \phi}{T_{0s}} F_{0s}\right). \tag{4.91}$$

Plugging this operator into the gyrokinetic equation, we find

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \nu_s \left(h_s - \frac{q_s \langle \phi \rangle_{\mathbf{R}_s}}{T_{0s}} F_{0s} \right) = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{4.92}$$

4.2.1 Solving for the Distribution Function

Decomposing the gyrokinetic distribution function into solutions of the form $h_s(\mathbf{R}, \mathbf{v}, t) = \hat{h}_s(\mathbf{v}) \exp[i(\mathbf{k} \cdot \mathbf{R} - \omega t)]$, we can solve for the distribution function

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s} \frac{\omega \langle \hat{\chi} \rangle_{\mathbf{R}} + i \nu_s \langle \hat{\phi} \rangle_{\mathbf{R}}}{\omega + i \nu_s - k_\parallel v_\parallel} \tag{4.93}$$

Plugging in the expression

$$\langle \hat{\chi} \rangle_{\mathbf{R}} = \left[J_0 (\frac{k_\perp v_\perp}{\Omega})(\hat{\phi} - \frac{v_\parallel \hat{A}_\parallel}{c}) + \frac{J_1 \left(\frac{k_\perp v_\perp}{\Omega} \right)}{\frac{k_\perp v_\perp}{\Omega}} \frac{m v_\perp^2}{q} \frac{\delta \hat{B}_\parallel}{B_0} \right] e^{i(\mathbf{k} \cdot \mathbf{R} - \omega t)}. \tag{4.94}$$

and

$$\langle \hat{\phi} \rangle_{\mathbf{R}} = J_0 (\frac{k_\perp v_\perp}{\Omega}) \hat{\phi} \tag{4.95}$$

the final solution for the distribution function is

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s} \left\{ J_0 \left(\frac{k_\perp v_\perp}{\Omega_s} \right) \frac{\omega \hat{A}_\parallel}{k_\parallel c} + \left(\frac{\omega + i \nu_s}{\omega + i \nu_s - k_\parallel v_\parallel} \right) J_0 \left(\frac{k_\perp v_\perp}{\Omega_s} \right) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) \right. \tag{4.96}$$

$$\left. + \left(\frac{\omega}{\omega + i \nu_s - k_\parallel v_\parallel} \right) \frac{J_1 \left(\frac{k_\perp v_\perp}{\Omega_s} \right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m v_\perp^2}{q} \frac{\delta \hat{B}_\parallel}{B_0} \right\}. \tag{4.97}$$

4.2.2 Results

$$\sum_s \frac{T_i}{T_s} [1 + \Gamma_{0s} \psi_s Z(\psi_s)] \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) + \sum_s \frac{T_i}{T_s} (1 - \Gamma_{0s}) \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) + \sum_s \frac{q_s}{q_i} \Gamma_{1s} \xi_s Z(\psi_s) \left(\frac{T_i}{q_i} \frac{\delta \hat{B}_\parallel}{B} \right) = 0 \tag{4.98}$$

$$\sum_s \frac{T_i}{T_s} \Gamma_{0s} [1 + \xi_s Z(\xi_s)] \left(1 + i \frac{\nu_s}{\omega} \right) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) + \frac{\alpha_i}{\overline{\omega}^2} \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) + \sum_s \frac{q_s}{q_i} \Gamma_{1s} [1 + \psi_s Z(\psi_s)] \left(\frac{T_i}{q_i} \frac{\delta \hat{B}_\parallel}{B} \right) = 0 \tag{4.99}$$

$$\sum_s \frac{q_s}{q_i} \Gamma_{1s} \psi_s Z(\psi_s) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) - \sum_s \frac{q_s}{q_i} \Gamma_{1s} \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c} \right) + \left[-\frac{2}{\beta_i} + \sum_s \frac{T_s}{T_i} \Gamma_{2s} \xi_s Z(\psi_s) \right] \left(\frac{T_i}{q_i} \frac{\delta \hat{B}_\parallel}{B} \right) = 0 \tag{4.100}$$

We can write the coefficients for this set of equations more clearly by denoting the following sums as

$$A' = \sum_s \frac{T_i}{T_s} [1 + \Gamma_{0s} \psi_s Z(\psi_s)] \tag{4.101}$$

$$B = \sum_s \frac{T_i}{T_s}(1 - \Gamma_{0s}) \tag{4.102}$$

$$C' = \sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\psi_s) \tag{4.103}$$

$$C'' = \sum_s \frac{q_s}{q_i}\Gamma_{1s}\psi_s Z(\psi_s) \tag{4.104}$$

$$D' = \sum_s \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\psi_s) \tag{4.105}$$

$$E = \sum_s \frac{q_s}{q_i}\Gamma_{1s} \tag{4.106}$$

$$F = \sum_s \left(\frac{T_i m_i}{T_s m_s}\right)^{1/2} \Gamma_{0s}[1 + \psi_s Z(\psi_s)]\psi_s \tag{4.107}$$

$$\alpha_s = \frac{k_\perp^2 \rho_s^2}{2} \tag{4.108}$$

$$\overline{\omega} = \frac{\omega}{k_\parallel v_A} \tag{4.109}$$

and

$$\xi_s = \frac{\omega}{k_\parallel v_{ts}} \tag{4.110}$$

$$\psi_s = \frac{\omega + i\nu_s}{k_\parallel v_{ts}} \tag{4.111}$$

Using these definitions, our system of equations can be written in matrix form as

$$\begin{pmatrix} A' & B & C' \\ F\sqrt{\beta_i}/\overline{\omega} & \alpha_i/\overline{\omega}^2 & C'' + E \\ C'' & -E' & D' - 2/\beta_i \end{pmatrix} \begin{pmatrix} \hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B} \end{pmatrix} = 0 \tag{4.112}$$

Finally, the dispersion relation can be found

$$\left[\frac{\alpha_i A'}{\overline{\omega}^2} - \frac{\sqrt{\beta_i}}{\overline{\omega}}BF\right]\left[\frac{2A'}{\beta_i} - A'D' + C'C''\right] - [A'E + BC'']\left[A'C'' + A'E - \frac{\sqrt{\beta_i}}{\overline{\omega}}C'F\right] = 0 \tag{4.113}$$

4.3 Krook Collision Operator (Correct)

This time we'll take the Krook collision operator to be

$$\mathcal{C}(f_s) = -\nu_s h_s \tag{4.114}$$

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \nu_s h_s = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{4.115}$$

4.3.1 Solving for the Distribution Function

Decomposing the gyrokinetic distribution function into solutions of the form $h_s(\mathbf{R}, \mathbf{v}, t) = \hat{h}_s(\mathbf{v}) \exp[i(\mathbf{k} \cdot \mathbf{R} - \omega t)]$, we can solve for the distribution function

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s} \frac{\omega \langle \hat{\chi} \rangle_{\mathbf{R}}}{\omega + i\nu_s - k_\parallel v_\parallel} \tag{4.116}$$

Plugging in the expression

$$\langle \hat{\chi} \rangle_{\mathbf{R}} = \left[J_0(\frac{k_\perp v_\perp}{\Omega})(\hat{\phi} - \frac{v_\parallel \hat{A}_\parallel}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} \frac{m v_\perp^2}{q} \frac{\delta \hat{B}_\parallel}{B_0} \right] e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)}. \tag{4.117}$$

the final solution for the distribution function is

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s} \left\{ \left(1 - \frac{i\nu_s}{\omega + i\nu_s - k_\parallel v_\parallel}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \frac{\omega \hat{A}_\parallel}{k_\parallel c} + \left(\frac{\omega + i\nu_s}{\omega + i\nu_s - k_\parallel v_\parallel}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) \right. \tag{4.118}$$

$$\left. + \left(\frac{\omega}{\omega + i\nu_s - k_\parallel v_\parallel}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m v_\perp^2}{q} \frac{\delta \hat{B}_\parallel}{B_0} \right\}. \tag{4.119}$$

4.3.2 Results

$$\sum_s \frac{T_i}{T_s}[1 + \Gamma_{0s}\psi_s Z(\psi_s)] \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{T_i}{T_s}(1 - \Gamma_{0s}) \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\psi_s) \left(\frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B}\right) = 0 \tag{4.120}$$

$$\sum_s \frac{T_i}{T_s}\Gamma_{0s}[1 + \xi_s Z(\xi_s)] \left(1 + i\frac{\nu_s}{\omega}\right) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) + \frac{\alpha_i}{\overline{\omega}^2} \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) + \sum_s \frac{q_s}{q_i}\Gamma_{1s}[1 + \psi_s Z(\psi_s)] \left(\frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B}\right) = 0 \tag{4.121}$$

$$\sum_s \frac{q_s}{q_i}\Gamma_{1s}\psi_s Z(\psi_s) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) - \sum_s \frac{q_s}{q_i}\Gamma_{1s} \left(\frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) + \left[-\frac{2}{\beta_i} + \sum_s \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\psi_s)\right] \left(\frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B}\right) = 0 \tag{4.122}$$

We can write the coefficients for this set of equations more clearly by denoting the following sums as

$$A' = \sum_s \frac{T_i}{T_s}[1 + \Gamma_{0s}\psi_s Z(\psi_s)] \tag{4.123}$$

$$B = \sum_s \frac{T_i}{T_s}(1 - \Gamma_{0s}) \tag{4.124}$$

$$C' = \sum_s \frac{q_s}{q_i}\Gamma_{1s}\xi_s Z(\psi_s) \tag{4.125}$$

$$C'' = \sum_s \frac{q_s}{q_i}\Gamma_{1s}\psi_s Z(\psi_s) \tag{4.126}$$

$$D' = \sum_s \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\psi_s) \tag{4.127}$$

$$E = \sum_s \frac{q_s}{q_i}\Gamma_{1s} \tag{4.128}$$

$$F = \sum_s \left(\frac{T_i m_i}{T_s m_s}\right)^{1/2} \Gamma_{0s}[1+\psi_s Z(\psi_s)]\psi_s \tag{4.129}$$

$$\alpha_s = \frac{k_\perp^2 \rho_s^2}{2} \tag{4.130}$$

$$\overline{\omega} = \frac{\omega}{k_\parallel v_A} \tag{4.131}$$

and

$$\xi_s = \frac{\omega}{k_\parallel v_{ts}} \tag{4.132}$$

$$\psi_s = \frac{\omega + i\nu_s}{k_\parallel v_{ts}} \tag{4.133}$$

Using these definitions, our system of equations can be written in matrix form as

$$\begin{pmatrix} A' & B & C' \\ F\sqrt{\beta_i}/\overline{\omega} & \alpha_i/\overline{\omega}^2 & C''+E \\ C'' & -E' & D'-2/\beta_i \end{pmatrix} \begin{pmatrix} \hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B} \end{pmatrix} = 0 \tag{4.134}$$

Finally, the dispersion relation can be found

$$\left[\frac{\alpha_i A'}{\overline{\omega}^2} - \frac{\sqrt{\beta_i}}{\overline{\omega}}BF\right]\left[\frac{2A'}{\beta_i} - A'D' + C'C''\right] - [A'E + BC'']\left[A'C'' + A'E - \frac{\sqrt{\beta_i}}{\overline{\omega}}C'F\right] = 0 \tag{4.135}$$

4.4 Lenard-Bernstein Collision Operator

One possible simplified model of the full Fokker-Planck equation is the one-dimensional Lenard-Bernstein equation (p.309, Stix, Waves in Plasmas)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} - \nu\frac{\partial}{\partial v}\left(vf + \frac{T}{m}\frac{\partial f}{\partial v}\right) = -\frac{q}{m}E(z,t)\frac{\partial f}{\partial v} \tag{4.136}$$

where ν is the collision frequency. (Note here that, as is the usual convention, Boltzmann's constant κ has been absorbed to give temperature T in units of energy.) This model was used for one-dimensional plasma oscillations in a Maxwellian plasma. Collisions in this model induce f to relax toward a stationary Maxwellian with an average particle energy $\langle mv^2/2\rangle = T/2$.

Linearizing and taking the Laplace-Fourier transform leads to

$$\left[-i\omega + ikv - \nu\left(1 + v\frac{\partial}{\partial v} + \frac{T}{m}\frac{\partial^2}{\partial v^2}\right)\right] f_1(\omega,k,v) = -\frac{q}{m}E(\omega,k)\frac{\partial f_0(v)}{\partial v} + f_1(t=0,k,v) \tag{4.137}$$

This equation may be solved for f_1 by splitting into two regions: an inner region near $v = \omega/k$ where $\partial f_0(v)/\partial v$ may be approximated as a constant, and an outer region $|\omega - kv| \gg \nu$ where the collision terms may be, to lowest order, neglected.

4.5 Linear Dispersion Relation with Hyperviscosity

We begin with the linearized, collisionless gyrokinetic equation with a hyperviscosity term of the form $+\nu_H\nabla_\perp^4 h_s$

$$\frac{\partial h_s}{\partial t} + v_\parallel\hat{\mathbf{z}}\cdot\frac{\partial h_s}{\partial \mathbf{R}_s} + \nu_H\nabla_\perp^4 h_s = \frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s}. \tag{4.138}$$

Decomposing the gyrokinetic distribution function into solutions of the form $h_s(\mathbf{R},\mathbf{v},t) = \hat{h}_s(\mathbf{v})\exp[i(\mathbf{k}\cdot\mathbf{R} - \omega t)]$, we can solve for the distribution function

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s}\frac{\omega\langle\hat{\chi}\rangle_{\mathbf{R}}}{\omega + i\nu_H k_\perp^4 - k_\parallel v_\parallel}. \tag{4.139}$$

Plugging in the expression

$$\langle\hat{\chi}\rangle_{\mathbf{R}} = \left[J_0(\frac{k_\perp v_\perp}{\Omega})(\hat{\phi} - \frac{v_\parallel \hat{A}_\parallel}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\frac{mv_\perp^2}{q}\frac{\delta\hat{B}_\parallel}{B_0}\right]e^{i(\mathbf{k}\cdot\mathbf{R}-\omega t)}. \tag{4.140}$$

and using the manipulation

$$\frac{\omega}{k_\parallel}\left(\frac{k_\parallel v_\parallel}{\omega + i\nu_H k_\perp^4 - k_\parallel v_\parallel}\right) = \frac{\omega}{k_\parallel}\left[\frac{\omega + i\nu_H k_\perp^4 - (\omega + i\nu_H k_\perp^4 - k_\parallel v_\parallel)}{\omega + i\nu_H k_\perp^4 - k_\parallel v_\parallel}\right] = \frac{\omega}{k_\parallel}\left(\frac{\omega + i\nu_H k_\perp^4}{\omega + i\nu_H k_\perp^4 - k_\parallel v_\parallel} - 1\right) \tag{4.141}$$

the final solution for the distribution function is

$$\hat{h}_s = \frac{q_s F_{0s}}{T_s} \left\{ J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \frac{\omega \hat{A}_\parallel}{k_\parallel c} + \left(\frac{\omega + i\nu_s}{\omega + i\nu_s - k_\parallel v_\parallel}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c}\right) \right. \tag{4.142}$$

$$\left. + \left(\frac{\omega}{\omega + i\nu_s - k_\parallel v_\parallel}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m v_\perp^2}{q} \frac{\delta \hat{B}_\parallel}{B_0} \right\}. \tag{4.143}$$

NOTE: There may be something wrong with the formula above.

4.5.1 Stupid and Wrong Moves

This is precisely the same as the solution without hyperviscosity if we replace ω with $\tilde{\omega} = \omega + i\nu_H k_\perp^4$. Hence, we can use the existing solution, needing only to include the hyperviscous dissipation in the final frequency so that $\omega = \tilde{\omega} - i\nu_H k_\perp^4$ where $\tilde{\omega}$ is the usual solution of the dispersion relation.

Chapter 5

Limits of the Gyrokinetic Linear Dispersion Relation

5.1 Analytical Limits of the Dispersion Relation

The complex eigenvalue solution $\overline{\omega}$ to (4.135) depends on three dimensionless parameters, the perpendicular scale compared to the ion gyroradius $k_\perp \rho_i$, the ion plasma beta β_i, and the ion to electron temperature ratio T_i/T_e. The linear, collisionless gyrokinetic dispersion relation can be simplified and solved analytically in certain limits of these parameters; the integrals of the Bessel functions over velocity space Γ_{0s}, Γ_{1s}, and Γ_{2s} and the plasma dispersion function $Z(\xi_s)$ can be approximated by simple analytical series for large and small arguments. The arguments of the integrals for ions and electrons are $\alpha_i = (k_\perp \rho_i)^2/2$ and $\alpha_e = (m_e/m_i)(T_e/T_i)(k_\perp \rho_i)^2/2$; for the plasma dispersion function they are $\xi_i = \overline{\omega}/\sqrt{\beta_i}$ and $\xi_e = (m_e/m_i)^{1/2}(T_i/T_e)^{1/2}\overline{\omega}/\sqrt{\beta_i}$. Natural parameters for expansion of these functions are α_i and $\sqrt{\beta_i}$. Below we explore the limits of large scale $\alpha_i \ll 1$, weak magnetization $\sqrt{\beta_i} \gg 1$ and strong magnetization $\sqrt{\beta_i} \ll 1$.

As we shall see in Section (5.1.1), the gyrokinetic dispersion relation separates into an Alfvén wave and a slow wave branch (the fast wave is ordered out by the gyrokinetic approximation). The slow waves are damped in this limit; based on this fact, we expect a nonlinear cascade of turbulent energy to smaller scales that reaches the scale of the ion gyroradius to be comprised primarily of Alfvén waves. Hence, the limits of weak and strong magnetization around the ion gyroradius scale, describe in Sections (5.1.2) and (5.1.3), will focus on the Alfvén wave solution to the dispersion relation.

5.1.1 Large Scale Limit, $k_\perp^2 \rho_i^2 \ll 1$

In the large-scale limit of $k_\perp^2 \rho_i^2 \ll 1$, or $\alpha_i \ll 1$, we can expand the functions Γ_{0s}, Γ_{1s}, and Γ_{2s} using an ascending series formula for the modified Bessel functions I_n (4.15). The resulting approximations for species s are $\Gamma_0(\alpha_s) \simeq 1 - \alpha_s$, $\Gamma_1(\alpha_s) \simeq 1 - 3\alpha_s/2$, and $\Gamma_2(\alpha_s) \simeq 2 - 3\alpha_s$. The dispersion relation in this large scale limit simplifies to

$$\left[\frac{1}{\overline{\omega}^2} - 1\right]\left[\frac{2}{\beta_i} - D + \frac{C^2}{A}\right] = 0 \tag{5.1}$$

The first factor leads to the familiar Alfvén wave dispersion relation for

$$\omega = k_{\|} v_A. \tag{5.2}$$

The second factor represents the slow wave solution of the dispersion relation. This solution depends on the ion plasma beta β_i, so we can further simplify the slow wave solution for collisionless gyrokinetics at large scale in limits of $\sqrt{\beta_i}$.

In the strongly magnetized limit, $\sqrt{\beta_i} \ll 1$, the argument of the plasma dispersion function for the ion terms is large $\xi_i \gg 1$; however, unless the temperature ratio $(T_i/T_e)^{1/2} \gg \sqrt{\beta_i}(m_i/m_e)^{1/2}$, the argument of the electron terms is not large. Taking the limit $(m_e/m_i)^{1/2}(T_i/T_e)^{1/2} \leq \sqrt{\beta_i} \ll 1$, we drop the electron terms of the plasma dispersion function. The plasma dispersion function can be expanded by (5.14) for $\xi_i \gg 1$ so that

$$\xi_i Z(\xi_i) \simeq i\sqrt{\pi}\xi_i \sigma e^{-\xi_i^2} - 1 + \frac{1}{2\xi_i^2} \tag{5.3}$$

Under this ordering, the first term in the slow wave portion of (5.1) is dominant leading to

$$\frac{T_i}{T_e} - \frac{k_{\|}^2 v_{th_i}^2}{\omega^2} + i\sqrt{\pi}\frac{\omega}{k_{\|} v_{th_i}} e^{-\frac{\omega^2}{k_{\|}^2 v_{th_i}^2}} = 0 \tag{5.4}$$

Assuming weak damping, to be checked later, we can solve for the real frequency and damping rate by expanding this equation about the the real frequency (Krall and Trivelpiece, Sec 8.6.2). The dispersion relation $D(\omega)$ is expanded about $\omega = \omega_r$ assuming the complex frequency $\omega = \omega_r + i\gamma$

$$D(\omega) = D(\omega_r) + i\gamma \frac{\partial D(\omega_r)}{\partial \omega_r} \tag{5.5}$$

Separating the dispersion relation into its real and imaginary parts $D(\omega) = D_r(\omega) + iD_i(\omega)$ and substituting, we find

$$D(\omega) = D_r(\omega_r) + iD_i(\omega_r) + i\gamma \frac{\partial D_r(\omega_r)}{\partial \omega_r} - \gamma \frac{\partial D_i(\omega_r)}{\partial \omega_r} \tag{5.6}$$

Setting the imaginary part to zero and solving for the damping rate yields

$$\gamma = -\frac{D_i(\omega_r)}{\frac{\partial D_r(\omega_r)}{\partial \omega_r}} \tag{5.7}$$

where ω_r solves the real part of the dispersion relation $D_r(\omega_r) = 0$. The real part of the dispersion relation yields

$$\omega = k_{\|} c_s \tag{5.8}$$

where $c_s^2 = T_e/m_i$; this is the familiar ion acoustic wave. Solving for the damping gives

$$\frac{\gamma}{\omega} = -\sqrt{\frac{\pi}{8}} \left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{1}{2}\frac{T_e}{T_i}} \tag{5.9}$$

This agrees with the solution for ion acoustic waves in Krall and Trivelpiece (Sec 8.6.3) in the limit $k^2 \lambda_{De}^2 \ll 1$.

5.1.2 Weakly Magnetized Limit, $\sqrt{\beta_i} \gg 1$

In the weakly magnetized limit, we simplify the gyrokinetic dispersion relation for $\sqrt{\beta_i} \gg 1$ allowing for $\alpha_i \sim 1$. Here we focus on the Alfvén wave solution to the dispersion relation since slow waves are heavily damped before reaching the scale of the ion gyroradius. For $T_i/T_e \gg (m_e/m_i)\alpha_i$, $\alpha_e \ll 1$ and the electron contribution to the velocity integrals of the Bessel functions can be simplified as in Section (5.1.1). For $\sqrt{\beta_i} \gg 1$, the plasma dispersion function for both ions and electrons can be expanded in the limit of small argument to give $\xi_s Z(\xi_s) \simeq i\sqrt{\pi}\xi_s$; in the limit that $(T_i/T_e)^{1/2} \ll (m_i/m_e)^{1/2}$, the electron terms are negligible compared to the ion terms and may be dropped

With these approximations, the terms of (4.135) of order $\mathcal{O}(\beta_i^{-1}$ and higher can be dropped, resulting in the equation

$$\overline{\omega}^2 \left\{ \Gamma_{2i}(1-\Gamma_{0i}) \left[\frac{T_i}{T_e} - \Gamma_{0i} + 1 - \Gamma_{1i} \right] - \Gamma_{0i}(1-\Gamma_{1i})^2 \right\} + i\overline{\omega}\sqrt{\frac{\beta_i}{\pi}}(1+\frac{T_i}{T_e})(1-\Gamma_{1i})^2 - \alpha_i \Gamma_{2i}(1+\frac{T_i}{T_e}) = 0 \tag{5.10}$$

Solving this quadratic equation yields the solution

$$\overline{\omega} = \frac{-i\sqrt{\frac{\beta_i}{\pi}}(1+\frac{T_i}{T_e})(1-\Gamma_{1i})^2 + \sqrt{-\frac{\beta_i}{\pi}(1+\frac{T_i}{T_e})^2(1-\Gamma_{1i})^4 + 4\alpha_i\Gamma_{2i}(1+\frac{T_i}{T_e})G}}{2G} \tag{5.11}$$

where

$$G = \Gamma_{2i}(1-\Gamma_{0i}) \left[\frac{T_i}{T_e} - \Gamma_{0i} + 1 - \Gamma_{1i} \right] - \Gamma_{0i}(1-\Gamma_{1i})^2 \tag{5.12}$$

5.1.3 Strongly Magnetized Limit, $\sqrt{\beta_i} \ll 1$

5.1.4 Limits of the Plasma Dispersion Function

These limits can be found on page 30 of the NRL Plasma Formulary

1. A power series representation of the plasma dispersion relation for a small argument $|\xi| \ll 1$ is

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - 2\xi \left(1 - \frac{2\xi^2}{3} + \frac{4\xi^4}{15} - \frac{8\xi^6}{105} + \cdots \right) \tag{5.13}$$

2. An asymptotic series representation of the plasma dispersion relation for a large argument $|\xi| \gg 1$ is

$$Z(\xi) = i\sqrt{\pi}\sigma e^{-\xi^2} - \frac{1}{\xi} \left(1 + \frac{1}{2\xi^2} + \frac{3}{4\xi^4} + \frac{15}{8\xi^6} + \cdots \right) \tag{5.14}$$

where

$$\sigma = \begin{cases} 0 & y > |x|^{-1} \\ 1 & |y| < |x|^{-1} \\ 2 & y < |x|^{-1} \end{cases} \tag{5.15}$$

and $\xi = x + iy$.

5.2 Gamma Function Definitions

Integrations over $v_\perp$ involve pairs of Bessel functions and can be written as modified Bessel functions. Three such integrals arise in the calculation of the linear gyrokinetic dispersion relation; we label them $\Gamma_0(\alpha)$, $\Gamma_1(\alpha)$, and $\Gamma_2(\alpha)$. These integrals are

$$\begin{aligned}
\Gamma_0(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/v_{th}^2} = I_0(\alpha)e^{-\alpha}, \\
\Gamma_1(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} \frac{2v_\perp^2}{v_{th}^2} \frac{J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} e^{-v_\perp^2/v_{th}^2} = [I_0(\alpha) - I_1(\alpha)]e^{-\alpha}, \\
\Gamma_2(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} \frac{4v_\perp^4}{v_{th}^4} \left[\frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\right]^2 e^{-v_\perp^2/v_{th}^2} = 2\Gamma_1(\alpha),
\end{aligned} \tag{5.16}$$

where I_0 and I_1 are the modified Bessel functions, the argument is $\alpha = \frac{k_\perp^2 \rho^2}{2}$.

In the large-scale limit $k_\perp^2 \rho_i^2 \ll 1$, or $\alpha_i \ll 1$, we can expand the functions $\Gamma_0(\alpha_s)$, $\Gamma_1(\alpha_s)$, and $\Gamma_2(\alpha_s)$ as follows: $\Gamma_0(\alpha_s) \simeq 1 - \alpha_s$, $\Gamma_1(\alpha_s) \simeq 1 - 3\alpha_s/2$, and $\Gamma_2(\alpha_s) \simeq 2 - 3\alpha_s$.

5.3 High β Expansion

In the high beta limit, we make two asymptotic expansions: one for $k_\perp \rho_i \sim \mathcal{O}(\beta_i^{-1/4})$ and another for $k_\perp \rho_i \sim \mathcal{O}(1)$. For both of these expansions, the plasma dispersion functions can be expanded for small arguments since $\sqrt{\beta_i} \gg 1$ and we assume $(T_{0i}/T_{0e})^{1/2} \ll (m_i/m_e)^{1/2}\sqrt{\beta_i}$.

5.3.1 $k_\perp \rho_i \sim \mathcal{O}(\beta_i^{-1/4})$ Expansion

For $k_\perp \rho_i \sim \mathcal{O}(\beta_i^{-1/4})$, we will find that $\overline{\omega} \sim \mathcal{O}(1)$. In this limit, we can use the small argument limit of the perpendicular velocity integrals (A.1) to find the order of the dispersion relation coefficients $B \sim \mathcal{O}(\beta_i^{-1/2})$ and $E \sim \mathcal{O}(\beta_i^{-1/2})$. Keeping only terms of order $\mathcal{O}(\beta_i^{-1})$, the dispersion relation simplifies to

$$\alpha_i D(\overline{\omega}^2 - 1) = \frac{9}{4}\overline{\omega}^2 \alpha_i^2. \tag{5.17}$$

We can write $D = i\sqrt{\pi} G_1 \frac{\overline{\omega}}{\sqrt{\beta_i}}$, where $G_1 = 2[\Gamma_{1i} + (m_e/m_i)^{1/2}(T_{0e}/T_{0i})^{1/2}]$. Solving for $\overline{\omega}$ yields

$$\overline{\omega} = -i\frac{9\alpha_i}{8G_1}\sqrt{\frac{\beta_i}{\pi}} \pm \sqrt{1 - \frac{81\beta_i \alpha_i^2}{64\pi G_1^2}}. \tag{5.18}$$

Let us consider the limits of this solution. The first limit occurs when

$$\frac{81\beta_i \alpha_i^2}{64\pi G_1^2} \ll 1 \tag{5.19}$$

The $\alpha_i \ll 1$ limit gives $G_1 \simeq 2$ and substituting $\alpha_i = k_\perp^2 \rho_i^2/2$ we find this limit is equivalent to

$$k_\perp \rho_i \ll \frac{8\pi^{1/4}}{3\beta_I^{1/4}} \tag{5.20}$$

or more simply $k_\perp \rho_i \ll \beta_i^{-1/4}$. In this limit, we can expand the square root to obtain

$$\overline{\omega} = \pm 1 - i\frac{9\alpha_i}{8G_1}\sqrt{\frac{\beta_i}{\pi}} = \pm 1 - i\frac{9}{16}\frac{k_\perp^2\rho_i^2}{2}\sqrt{\frac{\beta_i}{\pi}}. \tag{5.21}$$

This solution reproduces the large-wavelength limit of the Alfvén wave with weak damping and verifies our assumption that $\overline{\omega} \sim \mathcal{O}(1)$.

The second limit is just the opposite, occurring for

$$k_\perp \rho_i \gg \frac{8\pi^{1/4}}{3\beta_I^{1/4}} \tag{5.22}$$

or more simply $k_\perp \rho_i \gg \beta_i^{-1/4}$. In this limit, the solution is

$$\overline{\omega} = \begin{cases} -i\frac{4G_1}{9\alpha_i}\sqrt{\frac{\pi}{\beta_i}} \\ -i\frac{9\alpha_i}{4G_1}\sqrt{\frac{\beta_i}{\pi}} \end{cases} \tag{5.23}$$

Here, we find that the Alfvén waves solution become purely imaginary, with a weakly (upper) and strongly (lower) damped solution.

5.3.2 $k_\perp \rho_i > 1$ Expansion

In the limit $k_\perp \rho_i > 1$, we will find that $\overline{\omega} \sim \mathcal{O}(\beta_i^{-1/2})$. Keeping only terms of order $\mathcal{O}(1)$, the dispersion relation reduces to

$$\overline{\omega}^2 \beta_i E^2 + \alpha_i \beta_i D - 2\alpha_i = 0. \tag{5.24}$$

Solving for the frequency produces the result

$$\overline{\omega} = -i\frac{\alpha_i G_1}{2E^2}\sqrt{\frac{\pi}{\beta_i}} \pm \sqrt{\frac{2\alpha_i}{\beta_i E^2} - \frac{\alpha_i^2 G_1^2 \pi}{4E^4 \beta_i}}. \tag{5.25}$$

This solution confirms our assumption that $\overline{\omega} \sim \mathcal{O}(\beta_i^{-1/2})$.

The lower limit of this solution occurs for

$$\frac{\alpha_i^2 G_1^2 \pi}{4E^4 \beta_i} \gg \frac{2\alpha_i}{\beta_i E^2}, \tag{5.26}$$

equivalent to

$$k_\perp \rho_i \ll \frac{2\sqrt{\pi}}{3}, \tag{5.27}$$

or more simply $k_\perp \rho_i \ll 1$. In this case, we find the solutions become

$$\overline{\omega} = \begin{cases} -i\frac{1}{\sqrt{\pi\beta_i}} \\ -i\frac{4G_1}{9\alpha_i}\sqrt{\frac{\pi}{\beta_i}} \end{cases} \tag{5.28}$$

The lower solution matches the weakly damped (upper) root above in the overlap region $\beta_i^{-1/4} \ll k_\perp\rho_i \ll 1$. The upper solution corresponds to the slow wave root.

The upper limit of this solution, which we expect to correspond to kinetic Alfvén waves, is $k_\perp\rho_i \ll \frac{2\sqrt{\pi}}{3}$. In this case, we find

$$\overline{\omega} = \pm\sqrt{\frac{2\alpha_i}{E^2\beta_i}} - i\frac{\alpha_i G_1}{2E^2}\sqrt{\frac{\pi}{\beta_i}} \tag{5.29}$$

Assuming $\alpha_i \gg 1$ and $\alpha_e \ll 1$, we expand

$$G_1 = 2\left[\frac{1}{\sqrt{\pi}(2\alpha_i)^{3/2}} + \left(\frac{m_e}{m_i}\frac{T_{0e}}{T_{0i}}\right)^{1/2}\right] \tag{5.30}$$

and $E = -1$. Substituting these values and simplifying finds the solution

$$\overline{\omega} = \pm\sqrt{\frac{k_\perp^2\rho_i^2}{\beta_i}} - i\left(\frac{k_\perp^2\rho_i^2}{2}\right)\sqrt{\frac{\pi}{\beta_i}}\left[\frac{1}{\sqrt{\pi}(k_\perp\rho_i)^3} + \left(\frac{m_e}{m_i}\frac{T_{0e}}{T_{0i}}\right)^{1/2}\right] \tag{5.31}$$

Hence, in this limit the solution becomes the $\beta_i \gg 1$ limit of kinetic Alfvén waves.

Chapter 6

Analysis in Linear Gyrokinetics

6.1 Density Fluctuations in Gyrokinetics

The results for pertubations electric fields, magnetic fields, velocity, and density are summarized below

$$\frac{\hat{E}_\parallel}{k_\parallel} = -iX \tag{6.1}$$

$$\frac{\hat{E}_{\perp x}}{k_\parallel} = -i\frac{k_\perp}{k_\parallel}(X+Y) \tag{6.2}$$

$$\frac{\hat{E}_{\perp y}}{k_\parallel} = \frac{\overline{\omega}}{k_\perp \rho_i}\frac{2}{\sqrt{\beta_i}}Z \tag{6.3}$$

$$\frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B_0} = Z \tag{6.4}$$

$$\frac{T_i}{q_i}\frac{\delta \hat{B}_\perp}{B_0} = \frac{k_\perp \rho_i}{\overline{\omega}}\frac{\sqrt{\beta_i}}{2}Y \tag{6.5}$$

$$\frac{T_i}{q_i}\frac{\delta \hat{v}_{\parallel s}}{v_A} = -\frac{q_s}{q_i}\overline{\omega}\left[(A_s - B_s)X + (C_s + E_s)Z\right] \tag{6.6}$$

$$\frac{T_i}{q_i}\frac{\delta \hat{v}_{\perp s}}{v_A} = i\frac{q_i}{q_s}k_\perp \rho_i \frac{\sqrt{\beta_i}}{2}\left[E_s Y - C_s X + D_s Z\right] \tag{6.7}$$

$$\frac{T_i}{q_i}\frac{\delta \hat{n}_s}{n_s} = \frac{q_s}{q_i}\left[A_s X + B_s Y + C_s Z\right] \tag{6.8}$$

where we have used the definitions

$$X = \hat{\phi} - \frac{\omega \hat{A}_\parallel}{k_\parallel c} \tag{6.9}$$

$$Y = \frac{\omega \hat{A}_\parallel}{k_\parallel c} \tag{6.10}$$

$$Z = \frac{T_i}{q_i} \frac{\delta \hat{B}_\parallel}{B_0}. \tag{6.11}$$

and also the definitions

$$A_s = \frac{T_i}{T_s}[1 + \Gamma_{0s}\xi_s Z(\xi_s)], \tag{6.12}$$

$$B_s = \frac{T_i}{T_s}(1 - \Gamma_{0s}), \tag{6.13}$$

$$C_s = \frac{q_i}{q_s}\Gamma_{1s}\xi_s Z(\xi_s), \tag{6.14}$$

$$D_s = \frac{T_s}{T_i}\Gamma_{2s}\xi_s Z(\xi_s), \tag{6.15}$$

$$E_s = \frac{q_i}{q_s}\Gamma_{1s}, \tag{6.16}$$

6.1.1 Electric and Magnetic Field Fluctuations in Gyrokinetics

Given the definitions above, we can solve for the magnitudes of the x, y, and $\parallel$ components of the fields. This can be used to compare to the *in situ* field measurements from solar wind satellite data. The results are

$$|E_{\perp x\mathbf{k}}|\, \rho_i = k_\perp \rho_i |X + Y| \tag{6.17}$$

$$|E_{\perp y\mathbf{k}}|\, \rho_i = \frac{2\varpi}{\sqrt{\beta_i}} \left(\frac{k_\parallel}{k_\perp} \right) |Z| \simeq 0 \tag{6.18}$$

$$\left|E_{\parallel\mathbf{k}}\right| \rho_i = k_\parallel \rho_i |X| \tag{6.19}$$

$$\frac{T_i}{q_i} \frac{|B_{\perp x\mathbf{k}}|}{B_0} = \left(\frac{k_\parallel}{k_\perp} \right) |Z| \simeq 0 \tag{6.20}$$

$$\frac{T_i}{q_i} \frac{|B_{\perp y\mathbf{k}}|}{B_0} = \frac{k_\perp \rho_i}{\varpi} \frac{\sqrt{\beta_i}}{2} |Y| \tag{6.21}$$

$$\frac{T_i}{q_i} \frac{\left|B_{\parallel\mathbf{k}}\right|}{B_0} = |Z| \tag{6.22}$$

Note also that we can rewrite

$$\frac{T_i}{q_i B_0} = \frac{\rho_i}{2} \frac{v_{ti}}{c} \tag{6.23}$$

6.2 Eigenfunction Construction in Linear, Collisionless Gyrokinetics

The dispersion relation is given by

$$\begin{pmatrix} A & B & C \\ A-B & \alpha_i/\overline{\omega}^2 & C+E \\ C & -E & D-2/\beta_i \end{pmatrix} \begin{pmatrix} \hat{\phi}-\frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{\omega \hat{A}_\parallel}{k_\parallel c} \\ \frac{T_i}{q_i}\frac{\delta \hat{B}_\parallel}{B} \end{pmatrix} = 0 \tag{6.24}$$

6.2.1 Alfven Solution

We choose $Y = 1$ and solve for X and Z to obtain

$$X = \frac{CE - B(2/\beta_i - D)}{C^2 + A(2/\beta_i - D)} Y \tag{6.25}$$

$$Z = \frac{-(AE + BC)}{C^2 + A(2/\beta_i - D)} Y \tag{6.26}$$

Solving it in a more illuminating way gives the alternative version

$$X = \frac{C\left(\alpha_i A/\overline{\omega}^2 - AB + B^2\right) - B(AE + BC)}{A(AE + BC)} Y \tag{6.27}$$

$$Z = \frac{\alpha_i A/\overline{\omega}^2 - AB + B^2}{-(AE + BC)} Y \tag{6.28}$$

6.2.2 Slow Wave Solution

For the slow wave you can either choose $Z = 1$ and solve for X and Y to obtain

$$X = \frac{CE - B(2/\beta_i - D)}{-(AE + BC)} Z \tag{6.29}$$

$$Y = \frac{C^2 + A(2/\beta_i - D)}{-(AE + BC)} Z \tag{6.30}$$

Alternatively, you can choose $X = 1$ and solve for Y and Z to obtain

$$Z = \frac{-(AE + BC)}{CE - B(2/\beta_i - D)} X \tag{6.31}$$

$$Y = \frac{C^2 + A(2/\beta_i - D)}{CE - B(2/\beta_i - D)} X \tag{6.32}$$

6.3 Derivation of the Ion and Electron Heating in Linear, Collisionless Gyrokinetics

We specify a real forcing by an antenna in vector potential

$$\mathbf{A}_{ant} = \sum_{j=0}^{\infty} \left(\hat{\mathbf{A}}_{ant_j} e^{i(\mathbf{k}_j \cdot \mathbf{r} - \omega_j t)} + \hat{\mathbf{A}}^*_{ant_j} e^{-i(\mathbf{k}_j \cdot \mathbf{r} - \omega_j t)} \right). \tag{6.33}$$

Note that in the GS2 code, the forcing is done in $A_\parallel$ only, so we will specify here to the case that $\mathbf{A}_{ant} = A_{ant\parallel}\hat{\mathbf{z}}$. The Fourier transformed vector potential, specified in the terms of $\hat{A}_\parallel$ and $\delta\hat{B}_\parallel$, is given by

$$\hat{\mathbf{A}} = \hat{A}_\parallel \hat{\mathbf{z}} - i\frac{\delta\hat{B}_\parallel}{k_\perp}\hat{\mathbf{y}} \tag{6.34}$$

Including the antenna forcing in the Fourier transformed Ampere's Law yields

$$k_\perp^2 \left(\hat{A}_\parallel + \hat{A}_{ant\parallel} \right) \hat{\mathbf{z}} + i\mathbf{k}\delta\hat{B}_\parallel \times \hat{\mathbf{z}} = \frac{4\pi}{c}\delta\mathbf{J} \tag{6.35}$$

The gyrokinetic potential then becomes

$$\langle \hat{\chi} \rangle_{\mathbf{R}} = \left[J_0(\frac{k_\perp v_\perp}{\Omega}) \left(\hat{\phi} - \frac{v_\parallel(\hat{A}_\parallel + \hat{A}_{ant\parallel})}{c} \right) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} \frac{mv_\perp^2}{q} \frac{\delta\hat{B}_\parallel}{B_0} \right] e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)}. \tag{6.36}$$

The linearized gyrokinetic equation (keeping collisions) is given by

$$\frac{\partial g}{\partial t} + v_\parallel \hat{\mathbf{z}} \cdot \frac{\partial g}{\partial \mathbf{R}} - \langle \mathcal{C}_L(g, F_0) \rangle_{\mathbf{R}} - \frac{q}{T}\frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} F_0 = 0 \tag{6.37}$$

We model the collisions as a small damping term $\langle \mathcal{C}_L(g, F_0) \rangle_{\mathbf{R}} = -\epsilon g_s$; collisions are included to get the causality right, but we will take $\epsilon \to 0$ to recover the collisionless result. Fourier tranforming the gyrokinetic equation, assuming all quantities vary as $e^{-i(\mathbf{k}\cdot\mathbf{R} - \omega t)}$, we can solve for the distribution function

$$\hat{g}_s = \frac{q_s F_{0s}}{T_s} \frac{\omega}{\omega - k_\parallel v_\parallel + i\epsilon} \langle \hat{\chi} \rangle_{\mathbf{R}} \tag{6.38}$$

Substituting in the gyrokinetic potential (6.36), and representing the parallel component of vector potential by $\hat{A}_{tot\parallel} = \hat{A}_\parallel + \hat{A}_{ant\parallel}$, gives the result

$$\hat{g}_s = \frac{q_s F_{0s}}{T_s} \left\{ J_0(\gamma_s)\frac{\omega \hat{A}_{tot\parallel}}{k_\parallel c} + \frac{\omega}{\omega - k_\parallel v_\parallel} \left[J_0(\gamma_s) \left(\hat{\phi} - \frac{(\omega + i\epsilon)\hat{A}_{tot\parallel}}{k_\parallel c} \right) + \frac{J_1(\gamma_s)}{\gamma_s} \frac{m_s v_\perp^2}{q_s} \frac{\delta\hat{B}_\parallel}{B_0} \right] \right\} \tag{6.39}$$

where

$$\gamma_s = \frac{k_\perp v_\perp}{\Omega_s} \tag{6.40}$$

and we have used the relation

$$\left(\frac{\omega}{\omega - v_\parallel k_\parallel + i\epsilon}\right) v_\parallel = \left(\frac{\omega}{\omega - v_\parallel k_\parallel}\right)\frac{\omega + i\epsilon}{k_\parallel} - \frac{\omega}{k_\parallel}$$

to simplify the result.

We have shown that the heating in collisionless gyrokinetics is given by

$$\frac{3}{2}n_{0s}\frac{\partial T_{0s}}{\partial \tau} + \frac{\partial}{\partial t}\left[\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left(\frac{1}{2}mv^2\delta f_{2s} + q_s\phi g_s\right) - \int \frac{d^3\mathbf{r}}{V}\frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\right] = \int \frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v} q_s \frac{\partial \langle\chi\rangle_\mathbf{R}}{\partial t} g_s. \tag{6.41}$$

The term in brackets on the left-hand side produces no heating over the medium time average, so to calculate the heating we want to use the term on the right-hand side. After the medium time average

$$\overline{a}(t) = \frac{1}{\Delta T}\int_{t-\Delta T/2}^{t+\Delta T/2} a(t')dt' \tag{6.42}$$

where we average over a period ΔT long compared to the fluctuation timescale but short compared to the heating timescale, $1/\omega \ll \Delta T \ll 1/(\omega\epsilon^2)$, we find

$$\frac{3}{2}n_{0s}\frac{\partial \overline{T}_{0s}}{\partial \tau} = \int_{t-\Delta T/2}^{t+\Delta T/2}\frac{dt'}{\Delta T}\int \frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v} q_s \frac{\partial \langle\chi\rangle_\mathbf{R}}{\partial t'} g_s. \tag{6.43}$$

We represent the gyrokinetic potential by a Fourier series

$$\langle\chi\rangle_\mathbf{R} = \sum_{j=0}^{\infty}\left(\langle\hat{\chi}\rangle_{\mathbf{R}_j} e^{i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t)} + \langle\hat{\chi}\rangle^*_{\mathbf{R}_j} e^{-i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t)}\right) \tag{6.44}$$

as well as the distribution function

$$g_s = \sum_{j=0}^{\infty}\left(\hat{g}_{s_j} e^{i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t)} + \hat{g}^*_{s_j} e^{-i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t)}\right). \tag{6.45}$$

Using (6.44) and (6.45) to substitute into (6.43) gives

$$\frac{3}{2}n_{0s}\frac{\partial \overline{T}_{0s}}{\partial \tau} = iq_s \sum_{j=0}^{\infty}\sum_{j'=0}^{\infty}\int \frac{dt'}{\Delta T}\int \frac{d^3\mathbf{R}}{V} \tag{6.46}$$

$$\int d^3\mathbf{v}\left(-\omega_j\langle\hat{\chi}\rangle_{\mathbf{R}_j} e^{i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t')} + \omega_j\langle\hat{\chi}\rangle^*_{\mathbf{R}_j} e^{-i(\mathbf{k}_j\cdot\mathbf{R}-\omega_j t')}\right)\left(\hat{g}_{s_{j'}} e^{i(\mathbf{k}_{j'}\cdot\mathbf{R}-\omega_{j'} t')} + \hat{g}^*_{s_{j'}} e^{-i(\mathbf{k}_{j'}\cdot\mathbf{R}-\omega_{j'} t')}\right) \tag{6.47}$$

The integration over space gives delta functions $\delta(\mathbf{k}_j - \mathbf{k}_{j'})$ multiplying $\langle\hat{\chi}\rangle_{\mathbf{R}_j}\hat{g}^*_{s_{j'}} + \langle\hat{\chi}\rangle^*_{\mathbf{R}_j}\hat{g}_{s_{j'}}$ and $\delta(\mathbf{k}_j + \mathbf{k}_{j'})$ multiplying $\langle\hat{\chi}\rangle_{\mathbf{R}_j}\hat{g}_{s_{j'}} + \langle\hat{\chi}\rangle^*_{\mathbf{R}_j}\hat{g}^*_{s_{j'}}$. The first delta function eliminates one of the sums since the delta function is only non-zero when $j = j'$; the second delta function is always zero (except for $j = 0$, which we will see later does not contribute) and eliminates these terms. The time integration can then be performed trivially because the condition $j = j'$ removes the exponential time dependence. After time and space integrations, and using (6.38) to express $\hat{g}_{s_j}$ in terms of $\langle\hat{\chi}\rangle_{\mathbf{R}_j}$, we obtain

$$\frac{3}{2}n_{0s}\frac{\partial\overline{T}_{0s}}{\partial\tau} = i\frac{q_s^2}{T_s}\sum_{j=0}^{\infty}\omega_j^2\int d^3\mathbf{v}F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2\left(\frac{\omega_j}{\omega_j - k_\parallel v_\parallel + i\epsilon} - \frac{\omega_j}{\omega_j - k_\parallel v_\parallel - i\epsilon}\right) \tag{6.48}$$

The integration over $v_\parallel$ can be done by contour integration under the pole at $v_\parallel = \omega_j/k_\parallel \pm i\epsilon/k_\parallel$. For small values of ϵ, the contour integration yields the result

$$\int_C dv_\parallel F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2\left(\frac{\omega_j}{\omega_j - k_\parallel v_\parallel + i\epsilon} - \frac{\omega_j}{\omega_j - k_\parallel v_\parallel - i\epsilon}\right) \tag{6.49}$$

$$= \int_{-\infty}^{\infty} dv_\parallel\frac{\omega_j F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2}{\omega_j - k_\parallel v_\parallel + i\epsilon} - \int_{-\infty}^{\infty} dv_\parallel\frac{\omega_j F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2}{\omega_j - k_\parallel v_\parallel - i\epsilon} - i\pi\left(\omega_j F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2\right)_{v_\parallel = \omega_j/k_\parallel - i\epsilon/k_\parallel} \tag{6.50}$$

As we take the damping due to collisionality to zero, $\epsilon \to 0$, the principal values cancel leaving only the residue at the pole. Hence, we can express the result of this contour integration as a delta function in $v_\parallel$ by $-i\pi\delta(v_\parallel - \omega_j/k_\parallel)$. Therefore we are left with only the perpendicular velocity integrals to perform

$$\frac{3}{2}n_{0s}\frac{\partial\overline{T}_{0s}}{\partial\tau} = \pi\frac{q_s^2}{T_s}\sum_{j=1}^{\infty}\omega_j^2\int d^3\mathbf{v}F_{0s}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2\delta\left(v_\parallel - \frac{\omega_j}{k_\parallel}\right) \tag{6.51}$$

Note the $j = 0$ term corresponds to $k = 0$, and since $\omega(k = 0) = 0$, this term does not contribute to the sum so we may drop it. Simplifying the relation above, we have

$$\frac{3}{2}\frac{\partial\overline{T}_{0s}}{\partial\tau} = \frac{2\sqrt{\pi}q_s^2}{v_{th_s}^3 T_s}\sum_{j=1}^{\infty}\omega_j^2 e^{-\frac{\omega_j^2}{k_\parallel^2 v_{th_s}^2}}\int v_\perp dv_\perp e^{-\frac{v_\perp^2}{v_{th_s}^2}}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2 \tag{6.52}$$

Using (6.36) with $v_\parallel = \omega_j/k_\parallel$, the perpendicular velocity integral becomes

$$\int_0^{\infty} v_\perp dv_\perp e^{-\frac{v_\perp^2}{v_{th_s}^2}}\left|\langle\hat{\chi}\rangle_{\mathbf{R}_j}\right|^2 = \frac{v_{th_s}^2}{2}\left[\Gamma_{0s}|X|^2 + \frac{T_s q_i}{T_i q_s}\Gamma_{1s}(XZ^* + X^*Z) + \left(\frac{T_s q_i}{T_i q_s}\right)^2\Gamma_{2s}|Z|^2\right] \tag{6.53}$$

where we have used the definitions

$$X = \hat{\phi} - \frac{\omega_j\hat{A}_{tot\parallel}}{k_\parallel c} \tag{6.54}$$

and

$$Z = \frac{T_i}{q_i}\frac{\delta\hat{B}_\parallel}{B_0}. \tag{6.55}$$

Our final result for the heating of species s is

$$\frac{3}{2}\frac{\partial\overline{T}_{0s}}{\partial\tau} = \frac{\sqrt{\pi}}{v_{th_s}}\frac{q_s^2}{T_i}\sum_{j=1}^{\infty}\omega_j^2 e^{-\frac{\omega_j^2}{k_\parallel^2 v_{th_s}^2}}\left[\frac{T_i}{T_s}\Gamma_{0s}|X|^2 + \frac{q_i}{q_s}\Gamma_{1s}(XZ^* + X^*Z) + \frac{T_s}{T_i}\Gamma_{2s}|Z|^2\right]. \tag{6.56}$$

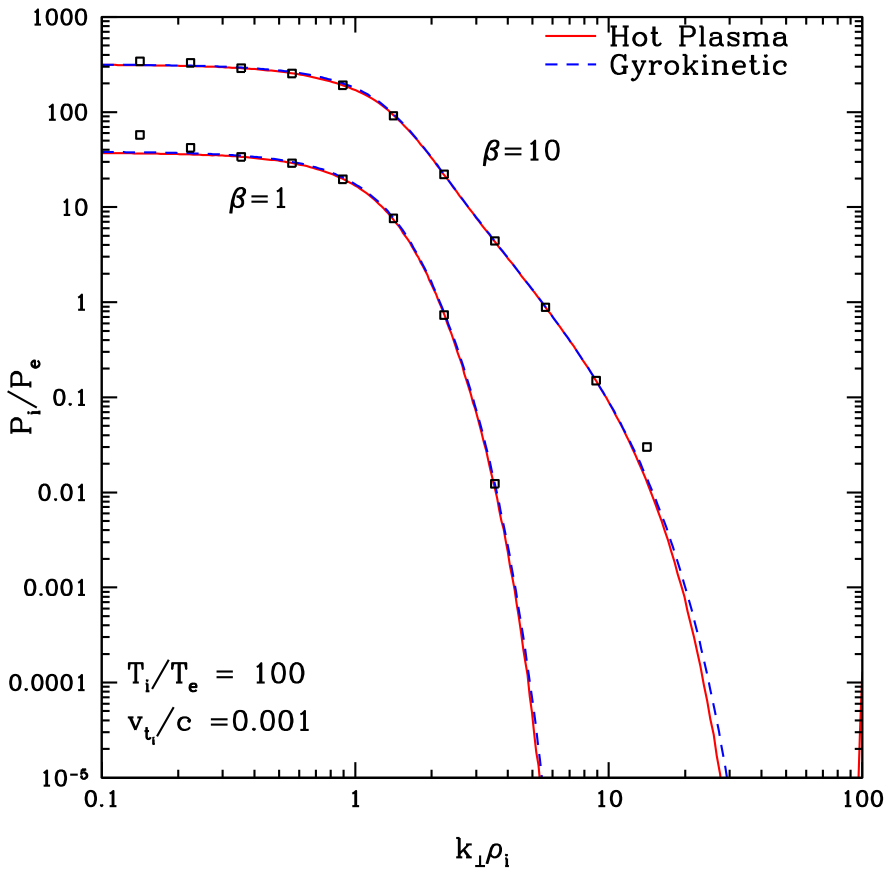

Figure 6.1: Comparison of Hot Plasma Dispersion Relation and Gyrokinetic Dispersion relation results for P_i/P_e.

In terms of $\overline{\omega} = \frac{\omega}{k_\parallel v_A}$, the ratio of ion to electron power is given by

$$\frac{P_i}{P_e} = \left(\frac{T_e m_i}{T_i m_e}\right)^{1/2} \frac{e^{-\frac{\overline{\omega}_j^2}{\beta_i}} \left[\Gamma_{0s}|X|^2 + \Gamma_{1s}(XZ^* + X^*Z) + \Gamma_{2s}|Z|^2\right]}{e^{-\frac{\overline{\omega}_j^2}{\beta_i}\frac{T_i m_e}{T_e m_i}} \left[\frac{T_i}{T_e}\Gamma_{0s}|X|^2 - \Gamma_{1s}(XZ^* + X^*Z) + \frac{T_e}{T_i}\Gamma_{2s}|Z|^2\right]}. \tag{6.57}$$

6.4 Laplace-Fourier Solutions to Linear, Collisionless Systems

6.4.1 Driven Linearized Electrostatic Vlasov Equation

To begin, we consider the electrostatic Vlasov System given by the Vlasov Equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} \mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \tag{6.58}$$

Poisson's Equation,

$$\nabla \mathbf{E} = 4\pi \sum_s q_s \int d^3\mathbf{v} f_s \tag{6.59}$$

and the electrostatic approximation of Faraday's Law,

$$\nabla \times \mathbf{E} = 0. \tag{6.60}$$

Expanding the distribution function and the electric field into equilibrium and small fluctuating components using

$$f_s = F_{0s}(v) + \delta f_s(\mathbf{r}, \mathbf{v}, t) \tag{6.61}$$

$$\mathbf{E} = \mathbf{E}_0 + \delta \mathbf{E} \tag{6.62}$$

where we assume a field-free plasma with a uniform Maxwellian equilibrium

$$\mathbf{E}_0 = 0 \tag{6.63}$$

$$F_{0s}(v) = \frac{n_{0s}}{\pi^{3/2} v_{ts}^3} \exp\left(-\frac{v^2}{v_{ts}^2}\right), \tag{6.64}$$

using Faraday's Law to write the electrostatic field as a gradient of a potential $\delta \mathbf{E} = -\nabla\phi$, and assuming charge neutrality of the equilibrium $\sum_s q_s n_{0s} = 0$, we obtain the linearized electrostatic Vlasov system

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \nabla \delta f_s + \frac{q_s F_{0s}}{T_s} \mathbf{v} \cdot \nabla\phi = 0 \tag{6.65}$$

$$-\nabla^2 \phi = 4\pi \sum_s q_s \int d^3\mathbf{v} \delta f_s. \tag{6.66}$$

Now, we must choose how to drive this system by adding an antenna term. Without knowing the correct way to drive this system, let us consider adding two possible terms:

1. Adding a term $-q_s F_{0s}/T_s \mathbf{v} \cdot \nabla \phi_a$ to the right-hand side of equation (6.65).

2. Adding a term $-\nabla^2 \phi_a$ to the left-hand side of equation (6.66).

We can consider three cases: case (1) first term only, case (2) second term only, or case (3) both terms. In order to determine the solution for all of these cases together, we add in accounting factors σ_1 and σ_2 to keep track of which source terms yield which terms in the final Laplace-Fourier solution. Thus, our system is defined by the equations

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \nabla \delta f_s + \frac{q_s F_{0s}}{T_s} \mathbf{v} \cdot \nabla \phi = -\sigma_1 \frac{q_s F_{0s}}{T_s} \mathbf{v} \cdot \nabla \phi_a \tag{6.67}$$

$$-\nabla^2(\phi + \sigma_2 \phi_a) = 4\pi \sum_s q_s \int d^3\mathbf{v} \delta f_s. \tag{6.68}$$

Taking the Fourier transform in all spatial coordinates produces

$$\frac{\partial \delta f_{\mathbf{k}s}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}s} + \frac{q_s F_{0s}}{T_s} i\mathbf{k} \cdot \mathbf{v} \phi_{\mathbf{k}} = -\sigma_1 \frac{q_s F_{0s}}{T_s} i\mathbf{k} \cdot \mathbf{v} \phi_{\mathbf{k}a} \tag{6.69}$$

$$\mathbf{k}^2(\phi_{\mathbf{k}} + \sigma_2 \phi_{\mathbf{k}a}) = 4\pi \sum_s q_s \int d^3\mathbf{v} \delta f_{\mathbf{k}s}. \tag{6.70}$$

Now let us choose a driving term of the form

$$\phi_{\mathbf{k}a} = \phi_{\mathbf{k}0} e^{-i\omega_0 t} \tag{6.71}$$

which has a Laplace transform

$$\hat{\phi}_{\mathbf{k}a} = \frac{\phi_{\mathbf{k}0}}{p + i\omega_0}. \tag{6.72}$$

Taking the Laplace transform of the Fourier transformed equations above yields the Laplace-Fourier solution for the distribution function

$$\delta \hat{f}_{\mathbf{k}s} = \frac{\delta f_{\mathbf{k}s}(t=0)}{p + i\mathbf{k} \cdot \mathbf{v}} - \frac{q_s F_{0s}}{T_s} \frac{i\mathbf{k} \cdot \mathbf{v}}{p + i\mathbf{k} \cdot \mathbf{v}} \left[\hat{\phi}_{\mathbf{k}} + \sigma_1 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0} \right] \tag{6.73}$$

We can then plug this solution into the Laplace transformed Poisson's equation

$$\mathbf{k}^2(\hat{\phi}_{\mathbf{k}} + \sigma_2 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0}) = 4\pi \sum_s q_s \int d^3\mathbf{v} \delta \hat{f}_{\mathbf{k}s} \tag{6.74}$$

and solve for $\hat{\phi}_{\mathbf{k}}$.

Before doing so, however, we specify the initial condition $\delta f_{\mathbf{k}s}(t=0) = 0$ to simplify the results. We need to find the zeroth velocity moment of the distribution function,

$$q_s \int d^3\mathbf{v} \delta \hat{f}_{\mathbf{k}s} = -\frac{q_s^2 n_{0s}}{T_s} \left(\hat{\phi}_{\mathbf{k}} + \sigma_1 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0} \right) \int \frac{d^3\mathbf{v}}{\pi^{3/2} v_{ts}^3} \frac{i\mathbf{k} \cdot \mathbf{v}}{p + i\mathbf{k} \cdot \mathbf{v}} e^{-\frac{v^2}{v_{ts}^2}} \tag{6.75}$$

Without loss of generality, we choose $\mathbf{k} = k\hat{\mathbf{z}}$. Dividing the top and bottom of the integrand by $-ikv_{ts}$, defining $x \equiv \frac{v_z}{v_{ts}}$ and $\xi_s \equiv \frac{p}{-ikv_{ts}}$, the v_z component of the integral can be cast into the form

$$\int_{-\infty}^{+\infty} \frac{dx}{\pi^{1/2}} \frac{-x}{\xi_s - x} e^{-x^2} = 1 + \xi_s Z(\xi_s) \tag{6.76}$$

where $Z(\xi_s)$ is the plasma dispersion function; the v_x and v_y components simply give unity. The zeroth velocity moment of the distribution function becomes

$$q_s \int d^3\mathbf{v} \delta \hat{f}_{\mathbf{k}s} = -\frac{q_s^2 n_{0s}}{T_s}\left(\hat{\phi}_{\mathbf{k}} + \sigma_1 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0}\right)[1 + \xi_s Z(\xi_s)]. \tag{6.77}$$

Therefore, Poisson's equation, after using the definitions $\omega_{ps}^2 = 4\pi n_{0s} q_s^2/m_s$ and $v_{ts}^2 = 2T_s/m_s$, becomes

$$\hat{\phi}_{\mathbf{k}} + \sigma_2 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0} = -\left(\hat{\phi}_{\mathbf{k}} + \sigma_1 \frac{\phi_{\mathbf{k}0}}{p + i\omega_0}\right) \sum_s \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)]. \tag{6.78}$$

The solution for $\hat{\phi}_{\mathbf{k}}$ is

$$\hat{\phi}_{\mathbf{k}}(p) = -\sigma_2 \frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)\left\{1 + \sum_s \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)]\right\}} - \sigma_1 \frac{\phi_{\mathbf{k}0} \sum_s \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)]}{(p + i\omega_0)\left\{1 + \sum_s \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)]\right\}} \tag{6.79}$$

If we define the plasma susceptibility for species s as

$$\chi_s = \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)] \tag{6.80}$$

and the dispersion relation as

$$D(p) = 1 + \sum_s \frac{2\omega_{ps}^2}{k^2 v_{ts}^2}[1 + \xi_s Z(\xi_s)] = 1 + \sum_s \chi_s, \tag{6.81}$$

the solution can be written more simply as

$$\hat{\phi}_{\mathbf{k}}(p) = -\sigma_2 \frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)D(p)} - \sigma_1 \frac{\phi_{\mathbf{k}0} \sum_s \chi_s}{(p + i\omega_0)D(p)}. \tag{6.82}$$

Now let us invesigate the consquences for each of the cases for driving discussed above. Let's write dispersion relation in terms of two solutions $\pm\omega - i\gamma$ giving $D(p) = (p + i\omega + \gamma)(p - i\omega + \gamma)/p^2$ where we assume that real frequency $\omega \geq 0$ and damping rate $\gamma \geq 0$.

Case(3): Both Terms

In this case, $\sigma_1 = \sigma_2 = 1$, so we have

$$\hat{\phi}_{\mathbf{k}}(p) = -\frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)}. \tag{6.83}$$

The inverse Laplace transform, using the Residue Theorem, gives the solution

$$\phi_{\mathbf{k}}(t) = -\phi_{\mathbf{k}0} e^{-i\omega_0 t} \tag{6.84}$$

so the distribution function becomes

$$\delta \hat{f}_{\mathbf{k}s} = 0. \tag{6.85}$$

Hence, the antenna potential is directly opposed and there is no plasma response. This is clearly a lousy choice for an antenna.

Case(1): First term

In this case, $\sigma_1 = 1$ and $\sigma_2 = 0$, so we have

$$\hat{\phi}_{\mathbf{k}}(p) = -\frac{\phi_{\mathbf{k}0} \sum_s \chi_s(p)}{(p + i\omega_0)D(p)} = -\frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)} + \frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)D(p)} \tag{6.86}$$

where the second form was obtained using $D(p) = 1 + \sum_s \chi_s$. The inverse Laplace transform, using the Residue Theorem, gives the solution

$$\phi_{\mathbf{k}}(t) = -\phi_{\mathbf{k}0} e^{-i\omega_0 t} + \frac{\phi_{\mathbf{k}0} e^{-i\omega_0 t}}{D(p = -i\omega_0)} \mp \frac{(\pm\omega - i\gamma)^2 \phi_{\mathbf{k}0} e^{-i(\pm\omega - i\gamma)t}}{2\omega[\omega_0 - (\pm\omega - i\gamma)]} \tag{6.87}$$

Plugging the solution for $\hat{\phi}_{\mathbf{k}}(p)$ back into the distribution function yields

$$\delta\hat{f}_{\mathbf{k}s} = -\frac{q_s F_{0s}}{T_s} \frac{i\mathbf{k}\cdot\mathbf{v}}{p + i\mathbf{k}\cdot\mathbf{v}} \frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)D(p)} \tag{6.88}$$

Case(2): Second term

In this case, $\sigma_1 = 0$ and $\sigma_2 = 1$, so we have

$$\hat{\phi}_{\mathbf{k}}(p) = -\frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)D(p)}. \tag{6.89}$$

The inverse Laplace transform, using the Residue Theorem, gives the solution

$$\phi_{\mathbf{k}}(t) = -\frac{\phi_{\mathbf{k}0} e^{-i\omega_0 t}}{D(p = -i\omega_0)} \pm \frac{(\pm\omega - i\gamma)^2 \phi_{\mathbf{k}0} e^{-i(\pm\omega - i\gamma)t}}{2\omega[\omega_0 - (\pm\omega - i\gamma)]} \tag{6.90}$$

Plugging the solution for $\hat{\phi}_{\mathbf{k}}(p)$ back into the distribution function yields

$$\delta\hat{f}_{\mathbf{k}s} = \frac{q_s F_{0s}}{T_s} \frac{i\mathbf{k}\cdot\mathbf{v}}{p + i\mathbf{k}\cdot\mathbf{v}} \frac{\phi_{\mathbf{k}0}}{(p + i\omega_0)D(p)} \tag{6.91}$$

Conclusion

Overall, these Cases (1) and (2) give the same set of equations if you plug the solutions into (6.67) and (6.68), different only by an overall negative sign. Thus, these cases are completely equivalent. Note, however, that when plugging the solutions back into the equations, the resulting equations look like Case (2), suggesting this is the more natural way of writing the driven system of equations. I guess the equivalence of these systems is somewhat obvious, as can be seen by a simple transformation of the variable ϕ to transform from one set of equations to the other.

6.4.2 Driven Linear Collisionless Gyrokinetics with $\delta B_\parallel = 0$

We derive the Laplace solution for driven linear collisionless gyrokinetics with the simplification that we constrain $\delta B_\parallel = 0$. The system of equations we solve is

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} + \frac{q_s F_{0s}}{T_s} v_\parallel \frac{\partial \langle \phi \rangle_{\mathbf{R}_s}}{\partial z} = -\frac{q_s F_{0s}}{T_s} \frac{v_\parallel}{c} \frac{\partial \langle A_\parallel \rangle_{\mathbf{R}_s}}{\partial t} \tag{6.92}$$

$$\sum_s \frac{q_s^2 n_s}{T_s} \left(\phi - \langle\langle \phi \rangle\rangle_s\right) = \sum_s q_s \int d^3\mathbf{v} \langle g_s \rangle_{\mathbf{r}} \tag{6.93}$$

$$-\frac{c}{4\pi} \nabla_\perp^2 (A_\parallel + A_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle g_s \rangle_{\mathbf{r}} \tag{6.94}$$

where we choose the antenna driving term to be

$$A_{\parallel a} = A_{\parallel 0} e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} \tag{6.95}$$

Fourier transforming these equations in space yields

$$\frac{\partial g_{\mathbf{k}s}}{\partial t} + i k_\parallel v_\parallel g_{\mathbf{k}s} + i k_\parallel v_\parallel \frac{q_s F_{0s}}{T_s} v_\parallel J_{0s} \phi_{\mathbf{k}} = -\frac{q_s F_{0s}}{T_s} \frac{v_\parallel}{c} J_{0s} \frac{\partial A_{\parallel \mathbf{k}}}{\partial t} \tag{6.96}$$

$$\sum_s \frac{q_s^2 n_s}{T_s} \left(1 - \Gamma_{0s}\right) \phi_{\mathbf{k}} = \sum_s q_s \int d^3\mathbf{v} J_{0s} g_{\mathbf{k}s} \tag{6.97}$$

$$\frac{c k_\perp^2}{4\pi} (A_{\parallel \mathbf{k}} + A_{\parallel \mathbf{k} a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel J_{0s} g_{\mathbf{k}s} \tag{6.98}$$

Next we must Laplace transform in time. Note that the driving term is transformed by

$$\hat{A}_{\parallel \mathbf{k} a} = \int_0^\infty A_{\parallel \mathbf{k} 0} e^{-i\omega_0 t} e^{-pt} = \frac{A_{\parallel \mathbf{k} 0}}{p + i\omega_0} \tag{6.99}$$

Performing the Laplace transform in time gives

$$p \hat{g}_{\mathbf{k}s} - g_{\mathbf{k}s}(0) + i k_\parallel v_\parallel \hat{g}_{\mathbf{k}s} + i k_\parallel v_\parallel \frac{q_s F_{0s}}{T_s} v_\parallel J_{0s} \hat{\phi}_{\mathbf{k}} = -\frac{q_s F_{0s}}{T_s} \frac{v_\parallel}{c} J_{0s} \left[p \hat{A}_{\parallel \mathbf{k}} - A_{\parallel \mathbf{k}}(0)\right] \tag{6.100}$$

$$\sum_s \frac{q_s^2 n_s}{T_s} \left(1 - \Gamma_{0s}\right) \hat{\phi}_{\mathbf{k}} = \sum_s q_s \int d^3\mathbf{v} J_{0s} \hat{g}_{\mathbf{k}s} \tag{6.101}$$

$$\frac{c k_\perp^2}{4\pi} \left(\hat{A}_{\parallel \mathbf{k}} + \frac{A_{\parallel \mathbf{k} 0}}{p + i\omega_0}\right) = \sum_s q_s \int d^3\mathbf{v} v_\parallel J_{0s} \hat{g}_{\mathbf{k}s} \tag{6.102}$$

We can now solve for the Laplace-Fourier transformed distribution function

$$\hat{g}_{\mathbf{k}s} = \frac{g_{\mathbf{k}s}(0)}{p + i k_\parallel v_\parallel} - \frac{q_s F_{0s}}{T_s} J_{0s} \left[\frac{i k_\parallel v_\parallel \hat{\phi}_{\mathbf{k}}}{p + i k_\parallel v_\parallel} + \frac{v_\parallel}{c} \frac{p \hat{A}_{\parallel \mathbf{k}}}{p + i k_\parallel v_\parallel} - \frac{v_\parallel}{c} \frac{A_{\parallel \mathbf{k}}(0)}{p + i k_\parallel v_\parallel}\right] \tag{6.103}$$

Since this is a driven system, we can choose zero initial conditions to simplify the solution as much as possible, $g_{\mathbf{k}s}(0) = 0$ and $A_{\|\mathbf{k}}(0) = 0$. This simplifies the distribution function solution to

$$\hat{g}_{\mathbf{k}s} = -\frac{q_s F_{0s}}{T_s} J_{0s} \frac{ik_\| v_\|}{p + ik_\| v_\|} \left[\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right] \tag{6.104}$$

Performing the integrals over velocity, using (A.27) and (A.28) for the parallel velocity integrals and (A.1) for the perpendicular velocity integrals, and plugging into Maxwell's equations gives

$$\sum_s \frac{q_s^2 n_s}{T_s} \left[(1 + \Gamma_{0s}\xi_s Z_s)\left(\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right) + (1 - \Gamma_{0s})\frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right] = 0 \tag{6.105}$$

$$-\frac{k_\perp^2 k_\|^2 c^2}{4\pi p^2}\left[\frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c} + \frac{ipA_{\|\mathbf{k}0}/(k_\| c)}{p + i\omega_0}\right] + \left(\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right)\sum_s \frac{q_s^2 n_s}{T_s}\Gamma_{0s}(1 + \xi_s Z_s) = 0. \tag{6.106}$$

Assuming a hydrogenic plasma, $n_i = n_e$ and $q_i = -q_e$, this can be put into a more recognizable form

$$\left(\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right)\sum_s \frac{T_i}{T_s}(1 + \Gamma_{0s}\xi_s Z_s) + \left(\frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right)\sum_s \frac{T_i}{T_s}(1 - \Gamma_{0s}) = 0 \tag{6.107}$$

$$\left(\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right)\sum_s \frac{T_i}{T_s}\Gamma_{0s}(1 + \xi_s Z_s) + \frac{\alpha_i}{\overline{\omega}^2}\left(\frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c}\right) = -\frac{\alpha_i}{\overline{\omega}^2}\frac{ipA_{\|\mathbf{k}0}/(k_\| c)}{p + i\omega_0} \tag{6.108}$$

where we have defined

$$\xi_s \equiv \frac{p}{-ik_\| v_{ts}}, \tag{6.109}$$

$$\alpha_i \equiv \frac{k_\perp^2 \rho_i^2}{2}, \tag{6.110}$$

and

$$\overline{\omega} \equiv \frac{ip}{k_\| v_A}. \tag{6.111}$$

The following definitions simplify the final result:

$$\tilde{E}_\| = \hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c} \tag{6.112}$$

$$\tilde{A}_\| = \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c} \tag{6.113}$$

$$\tilde{S} = \frac{ipA_{\|\mathbf{k}0}/(k_\| c)}{p + i\omega_0} \tag{6.114}$$

$$A = \sum_s \frac{T_i}{T_s}(1 + \Gamma_{0s}\xi_s Z_s) \tag{6.115}$$

$$B = \sum_s \frac{T_i}{T_s}(1 - \Gamma_{0s}). \tag{6.116}$$

Hence, the final set of equation is given by

$$\begin{pmatrix} A & B \\ A - B & \alpha_i/\overline{\omega}^2 \end{pmatrix} \begin{pmatrix} \tilde{E}_\parallel \\ \tilde{A}_\parallel \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha_i/\overline{\omega}^2 \tilde{S} \end{pmatrix} \tag{6.117}$$

Solutions for $\tilde{A}_\parallel$ and $\tilde{E}_\parallel$ are given by

$$\tilde{A}_\parallel = \frac{-\alpha_i A/\overline{\omega}^2 \tilde{S}}{\alpha_i A/\overline{\omega}^2 - AB + B^2} \tag{6.118}$$

$$\tilde{E}_\parallel = \frac{\alpha_i B/\overline{\omega}^2 \tilde{S}}{\alpha_i A/\overline{\omega}^2 - AB + B^2} \tag{6.119}$$

and the solution for the distribution function is given by

$$\hat{g}_{\mathbf{k}s} = -\frac{q_s F_{0s}}{T_s} J_{0s} \frac{ik_\parallel v_\parallel}{p + ik_\parallel v_\parallel} \tilde{E}_\parallel \tag{6.120}$$

Now we will focus on the solution for $A_\parallel(t)$. The Laplace-Fourier solution is

$$\hat{A}_{\parallel\mathbf{k}} = \frac{\alpha_i A k_\parallel^2 v_A^2 A_{\parallel\mathbf{k}0}}{p^2(p + i\omega_0)D(p)} \tag{6.121}$$

where we have defined the dispersion relation as

$$D(p) = \frac{\alpha_i A}{\overline{\omega}^2} - AB + B^2. \tag{6.122}$$

To put this into a more useful form, we can define

$$Q^2(p) \equiv \frac{\alpha_i A k_\parallel^2 v_A^2}{AB - B^2} \tag{6.123}$$

where we note that $A = A(p)$. The Laplace-Fourier solution may then be written

$$\hat{A}_{\parallel\mathbf{k}} = \frac{-Q^2 A_{\parallel\mathbf{k}0}}{(p^2 + Q^2)(p + i\omega_0)}. \tag{6.124}$$

Now the expression $p^2 + Q^2$ is just an alternate form of the dispersion relation; the solutions to $p^2 + Q^2 = 0$ are the eigenfrequencies of the system. The system with $\delta B_\parallel = 0$ has two solutions, the Alfven waves. We can rewrite the expression $p^2 + Q^2 = (p + i\omega_1)(p + i\omega_2)$, where ω_1 and ω_2 are the eigenfrequencies; we know for this system these solutions typically have the form $\omega_1 = \omega_r - i\gamma$ and $\omega_2 = -\omega_r - i\gamma$. Now we can write the system so the inverse Laplace transform is easily found by application of the residue theorem,

$$\hat{A}_{\parallel\mathbf{k}} = \frac{-Q^2 A_{\parallel\mathbf{k}0}}{(p + i\omega_0)(p + i\omega_1)(p + i\omega_2)}. \tag{6.125}$$

The solution is

$$A_{\|\mathbf{k}}(t) = A_{\|\mathbf{k}0}\left[\frac{Q^2(p=-i\omega_0)e^{-i\omega_0 t}}{(\omega_1-\omega_0)(\omega_2-\omega_0)} + \frac{Q^2(p=-i\omega_1)e^{-i\omega_1 t}}{(\omega_0-\omega_1)(\omega_2-\omega_1)} + \frac{Q^2(p=-i\omega_2)e^{-i\omega_2 t}}{(\omega_0-\omega_2)(\omega_1-\omega_2)}\right] \tag{6.126}$$

Note that since $\omega_1 = \omega_r - i\gamma$ and $\omega_2 = -\omega_r - i\gamma$, the second and third terms will decay with time. Also, because ω_1 and ω_2 are the solutions to $p^2 + Q^2 = 0$, we can replace $Q^2(p = -i\omega_1) = \omega_1^2$ and $Q^2(p = -i\omega_2) = \omega_2^2$ if we desire.

The solution above is as accurate as the replacement $p^2 + Q^2 = (p + i\omega_1)(p + i\omega_2)$. For long times, we can get rid of this inaccuracy by putting $p^2 + Q^2$ back into the solution for the first term to give an alternative version

$$A_{\|\mathbf{k}}(t) = A_{\|\mathbf{k}0}\left[\frac{Q^2(p=-i\omega_0)e^{-i\omega_0 t}}{\omega_0^2 - Q^2(p=-i\omega_0)} + \frac{Q^2(p=-i\omega_1)e^{-i\omega_1 t}}{(\omega_0-\omega_1)(\omega_2-\omega_1)} + \frac{Q^2(p=-i\omega_2)e^{-i\omega_2 t}}{(\omega_0-\omega_2)(\omega_1-\omega_2)}\right] \tag{6.127}$$

Comparison of this solution with the results of `GS2` is given in Figure 6.2.

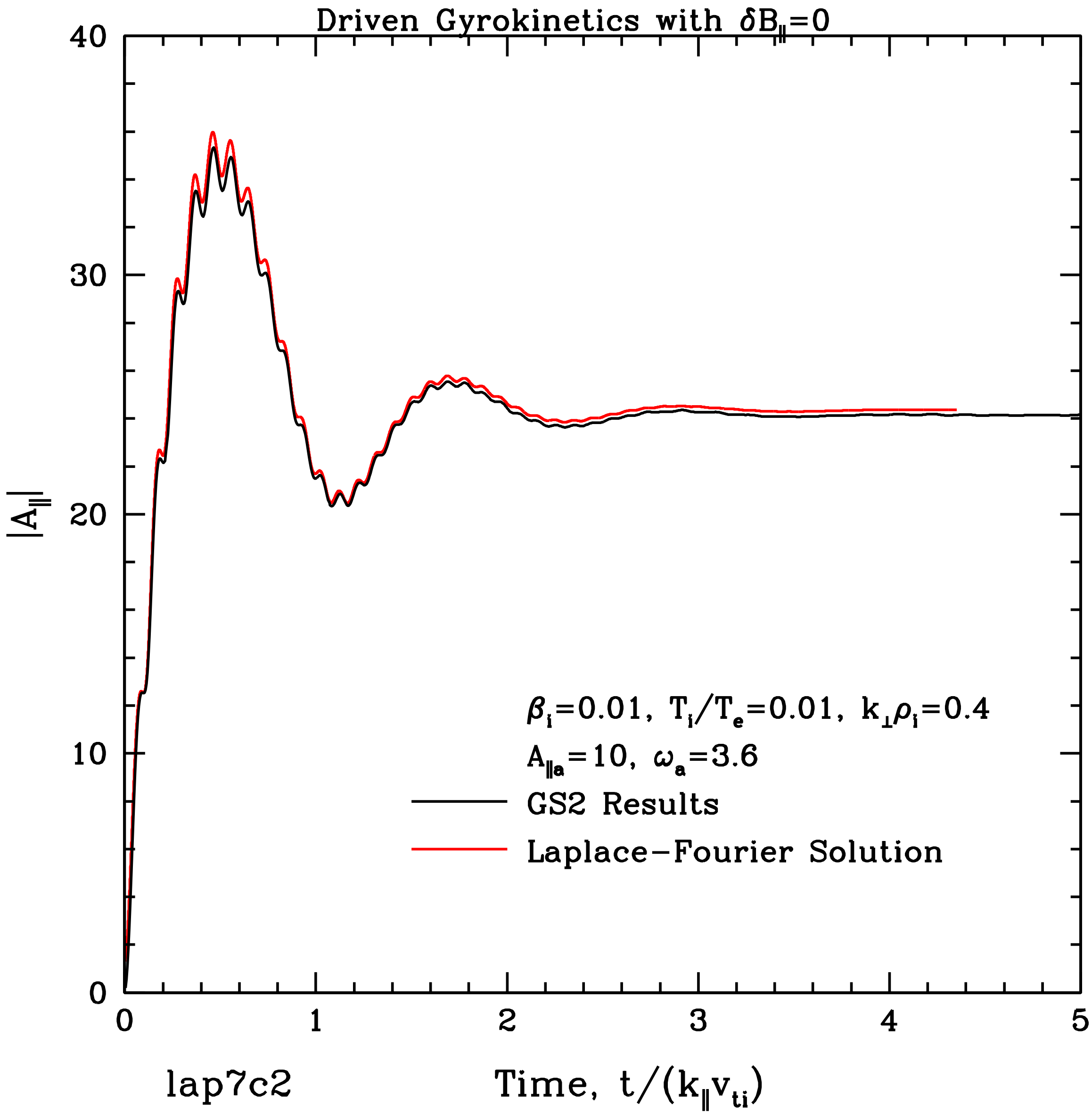

Figure 6.2: For a linear system driven by an antenna of amplitude $A_{\|0} = 10$ and driving frequency $\omega_0 = 3.6$ with parameters $\beta_i = 0.01$, $T_i/T_e = 0.01$, and $k_\perp\rho_i = 0.4$, the comparison of the analytical Laplace-Fourier solution with GS2 results is shown. Note that the eigenfrequnecies are given by $\pm 3.0705 - i0.0074885$ (normalized to $k_\| v_A$).

6.5 Turbulent Cascade Models using Linear Gyrokinetics

Chapter 7

Conserved Quantities in Gyrokinetics

7.1 Conserved Quantities in Gyrokinetics

First, we multiply the gyrokinetic equation by $T_{0s}h_s/F_{0s}$ and integrate over all guiding-center space $\mathbf{R}$ and velocity

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} h_s \frac{\partial h_s}{\partial t} + \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} h_s v_\parallel \frac{\partial h_s}{\partial z} + \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} h_s \frac{c}{B_0} \left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right]$$
$$- \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} = 0 \tag{7.1}$$

The second term is odd in $v_\parallel$ and disappears when integrating over velocity. The third term disappears because

$$\int \frac{d^3\mathbf{R}_s}{V} h_s \left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] = 0 \tag{7.2}$$

leaving us with

$$\frac{\partial}{\partial t} \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \frac{h_s^2}{2} = \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} \tag{7.3}$$

Now, we sum over species and split gyrokinetic potential $\chi = \phi - \mathbf{v}\cdot\mathbf{A}/c$ on the right-hand side to get

$$\sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} = \sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \left(\frac{\partial\langle\phi\rangle_{\mathbf{R}_s}}{\partial t} - \frac{\partial\langle\frac{v_\parallel A_\parallel}{c}\rangle_{\mathbf{R}_s}}{\partial t} - \frac{\partial\langle\frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\rangle_{\mathbf{R}_s}}{\partial t} \right) \tag{7.4}$$

and we will treat each term on the right-hand side individually.

The first term can be written

$$\sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial\langle\phi\rangle_{\mathbf{R}_s}}{\partial t} = \sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}} \frac{\partial\phi}{\partial t} = \int \frac{d^3\mathbf{r}}{V} \frac{\partial\phi}{\partial t} \sum_s \int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}} \tag{7.5}$$

Using the quasineutrality condition

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi = \sum_s \int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}}, \tag{7.6}$$

this term can be written

$$\int \frac{d^3\mathbf{r}}{V} \frac{\partial \phi}{\partial t} \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi = \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \frac{\phi^2}{2} \tag{7.7}$$

The second term can be written

$$-\sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial \langle \frac{v_\parallel A_\parallel}{c} \rangle_{\mathbf{R}_s}}{\partial t} = -\sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \langle h_s \rangle_{\mathbf{r}} \frac{v_\parallel}{c} \frac{\partial A_\parallel}{\partial t}$$
$$= -\int \frac{d^3\mathbf{r}}{V} \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \sum_s \int d^3\mathbf{v} q_s v_\parallel \langle h_s \rangle_{\mathbf{r}} \tag{7.8}$$

Using the parallel Ampere's Law

$$-\frac{c}{4\pi} \nabla_\perp^2 A_\parallel = \sum_s \int d^3\mathbf{v} q_s v_\parallel \langle h_s \rangle_{\mathbf{r}}. \tag{7.9}$$

we obtain

$$\int \frac{d^3\mathbf{r}}{V} \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \frac{c}{4\pi} \nabla_\perp^2 A_\parallel = \frac{1}{4\pi} \int \frac{d^3\mathbf{r}}{V} \nabla_\perp^2 A_\parallel \frac{\partial A_\parallel}{\partial t} \tag{7.10}$$

Integrating by parts once in space gives

$$\frac{1}{4\pi} \int \frac{d^3\mathbf{r}}{V} \nabla_\perp^2 A_\parallel \frac{\partial A_\parallel}{\partial t} = -\frac{1}{4\pi} \int \frac{d^3\mathbf{r}}{V} \nabla_\perp A_\parallel \frac{\partial \nabla_\perp A_\parallel}{\partial t} = -\frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \frac{|\nabla_\perp A_\parallel|^2}{8\pi} \tag{7.11}$$

The third term can be written

$$-\sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s h_s \frac{\partial \langle \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c} \rangle_{\mathbf{R}_s}}{\partial t} = -\sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s \langle h_s \rangle_{\mathbf{r}} \frac{\partial \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}}{\partial t}$$
$$= -\int \frac{d^3\mathbf{r}}{V} \frac{1}{c} \frac{\partial \mathbf{A}_\perp}{\partial t} \cdot \sum_s \int d^3\mathbf{v} q_s \mathbf{v}_\perp \langle h_s \rangle_{\mathbf{r}} \tag{7.12}$$

The perpendicular Ampere's Law is (is this correct?)

$$\nabla_\perp \delta B_\parallel = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} \langle \hat{\mathbf{z}} \times \mathbf{v}_\perp h_s \rangle_{\mathbf{r}}. \tag{7.13}$$

Crossing this equation with $\hat{\mathbf{z}}$ gives

$$-\frac{c}{4\pi} \hat{\mathbf{z}} \times \nabla_\perp \delta B_\parallel = \sum_s \int d^3\mathbf{v} q_s \langle \mathbf{v}_\perp h_s \rangle_{\mathbf{r}}. \tag{7.14}$$

Substituting in using this form of Ampere's law gives

$$-\int \frac{d^3\mathbf{r}}{V} \frac{1}{c} \frac{\partial \mathbf{A}_\perp}{\partial t} \cdot \left(-\frac{c}{4\pi} \hat{\mathbf{z}} \times \nabla_\perp \delta B_\parallel\right) = +\frac{1}{4\pi} \int \frac{d^3\mathbf{r}}{V} \hat{\mathbf{z}} \times \nabla_\perp \delta B_\parallel \cdot \frac{\partial \mathbf{A}_\perp}{\partial t} \tag{7.15}$$

Now, we note that

$$\hat{\mathbf{z}} \times \nabla_\perp \delta B_\parallel = \frac{\partial \delta B_\parallel}{\partial x}\hat{\mathbf{y}} - \frac{\partial \delta B_\parallel}{\partial y}\hat{\mathbf{x}}. \tag{7.16}$$

Therefore, we can integrate by parts in space to find

$$+\frac{1}{4\pi}\int \frac{d^3\mathbf{r}}{V}\left(\frac{\partial \delta B_\parallel}{\partial x}\frac{\partial A_{\perp y}}{\partial t} - \frac{\partial \delta B_\parallel}{\partial y}\frac{\partial A_{\perp x}}{\partial t}\right) = +\frac{1}{4\pi}\int \frac{d^3\mathbf{r}}{V}\delta B_\parallel \frac{\partial}{\partial t}\left(\frac{\partial A_{\perp y}}{\partial x} - \frac{\partial A_{\perp x}}{\partial y}\right)$$
$$= -\frac{1}{4\pi}\int \frac{d^3\mathbf{r}}{V}\delta B_\parallel \frac{\partial \delta B_\parallel}{\partial t} = -\frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\frac{|\delta B_\parallel|^2}{8\pi} \tag{7.17}$$

after noting that

$$\delta B_\parallel = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{A} = \frac{\partial A_{\perp x}}{\partial y} - \frac{\partial A_{\perp y}}{\partial x} \tag{7.18}$$

Putting all of these pieces together, we find

$$\frac{\partial}{\partial t}\sum_s \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2} = \frac{\partial}{\partial t}\int \frac{d^3\mathbf{r}}{V}\left(\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\frac{\phi^2}{2} - \frac{|\nabla_\perp A_\parallel|^2}{8\pi} - \frac{|\delta B_\parallel|^2}{8\pi}\right). \tag{7.19}$$

This can be written alternatively as

$$\frac{\partial}{\partial t}\left[\sum_s\left(\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2} - \int \frac{d^3\mathbf{r}}{V}\frac{q_s^2 n_{0s}}{T_{0s}}\frac{\phi^2}{2}\right) + \int \frac{d^3\mathbf{r}}{V}\frac{|\delta \mathbf{B}|^2}{8\pi}\right] = 0 \tag{7.20}$$

7.1.1 Question

Is this true?

$$\int \frac{d^3\mathbf{r}}{V}\frac{q_s^2 n_{0s}}{T_{0s}}\frac{\phi^2}{2} = \int \frac{d^3\mathbf{R}_s}{V}\frac{q_s^2 n_{0s}}{T_{0s}}\frac{\langle\phi\rangle_{\mathbf{R}_s}^2}{2} \tag{7.21}$$

7.1.2 Demonstrating Equivalence to Power Balance Form

It is important to note here that

$$\sum_s \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left(\frac{q_s F_{0s}}{T_{0s}}\phi\langle h_s\rangle_\mathbf{r} - \frac{q_s^2 F_{0s}^2}{T_{0s}^2}\phi^2\right) = \int \frac{d^3\mathbf{r}}{V}\phi\sum_s\int d^3\mathbf{v} q_s\langle h_s\rangle_\mathbf{r} - \sum_s\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_{0s}}\phi^2. \tag{7.22}$$

But note from quasineutrality that

$$\sum_s \int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_{0s}}\phi = \sum_s \int d^3\mathbf{v} q_s\langle h_s\rangle_\mathbf{r}, \tag{7.23}$$

so we obtain

$$\int \frac{d^3\mathbf{r}}{V}\phi\sum_s\int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_{0s}}\phi - \sum_s\int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{q_s^2 F_{0s}}{T_{0s}}\phi^2 = 0. \tag{7.24}$$

Therefore, we can subtract this to the expression from before to give

$$\frac{\partial}{\partial t}\left[\sum_s \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2} - \sum_s \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left(-q_s\phi\langle h_s\rangle_{\mathbf{r}} + \frac{q_s^2F_{0s}}{T_{0s}}\phi^2 - \frac{q_s^2F_{0s}}{T_{0s}}\frac{\phi^2}{2}\right)\right.$$
$$\left. + \int \frac{d^3\mathbf{r}}{V}\frac{|\delta\mathbf{B}|^2}{8\pi}\right] = 0 \tag{7.25}$$

Therefore, we arrive at the same conserved quantity, the energy, as appears in the Power Balance Equation

$$\frac{\partial}{\partial t}\sum_s\left[\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2} - \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left(-q_s\phi\langle h_s\rangle_{\mathbf{r}} + \frac{q_s^2F_{0s}}{T_{0s}}\frac{\phi^2}{2}\right) + \int \frac{d^3\mathbf{r}}{V}\frac{|\delta\mathbf{B}|^2}{8\pi}\right] = 0 \tag{7.26}$$

7.2 Moments of the Gyrokinetic Equation

First, we multiply the gyrokinetic equation by $T_{0s}h_s/F_{0s}$ and integrate over all guiding-center space $\mathbf{R}$ and velocity

$$\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}h_s\frac{\partial h_s}{\partial t}+\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}h_s v_\parallel\frac{\partial h_s}{\partial z}+\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}h_s\frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s},h_s\right]$$
$$-\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s h_s\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}=0 \tag{7.27}$$

The second term is odd in $v_\parallel$ and disappears when integrating over velocity. The third term disappears because

$$\int \frac{d^3\mathbf{R}_s}{V}h_s\left[\langle\chi\rangle_{\mathbf{R}_s},h_s\right]=0 \tag{7.28}$$

leaving us with

$$\frac{\partial}{\partial t}\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2}-\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s h_s\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}=0 \tag{7.29}$$

Next, we multiply the gyrokinetic equation by $q_s\langle\chi\rangle_{\mathbf{R}_s}$ and integrate over all guiding-center space $\mathbf{R}$ and velocity

$$\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}\frac{\partial h_s}{\partial t}+\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}v_\parallel\frac{\partial h_s}{\partial z}+\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}\frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s},h_s\right]$$
$$-\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{q_s^2F_{0s}}{T_{0s}}\langle\chi\rangle_{\mathbf{R}_s}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}=0 \tag{7.30}$$

The third term disappears when integrating over all space, leaving

$$\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}\frac{\partial h_s}{\partial t}+\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}v_\parallel\frac{\partial h_s}{\partial z}-\frac{\partial}{\partial t}\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{q_s^2F_{0s}}{T_{0s}}\frac{\langle\chi\rangle_{\mathbf{R}_s}^2}{2}=0 \tag{7.31}$$

Next, we subtract eq (7.31) from eq (7.29) to obtain

$$\frac{\partial}{\partial t}\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\left(\frac{T_{0s}}{F_{0s}}\frac{h_s^2}{2}-q_s h_s\langle\chi\rangle_{\mathbf{R}_s}+\frac{q_s^2F_{0s}}{T_{0s}}\frac{\langle\chi\rangle_{\mathbf{R}_s}^2}{2}\right)=-\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}q_s\langle\chi\rangle_{\mathbf{R}_s}v_\parallel\frac{\partial h_s}{\partial z} \tag{7.32}$$

7.3 The Gyrokinetic $\nabla\cdot\mathbf{J}=0$

We can find a gyrokinetic form of $\nabla\cdot\mathbf{J}=0$ the zeroth moment of the gyrokinetic equation multiplied by q_s, summed over species, integrated over velocity, and ring-averaged at constant position $\mathbf{r}$,

$$\sum_s\left\langle\int d^3\mathbf{v}q_s\left[\frac{\partial h_s}{\partial t}+v_\parallel\frac{\partial h_s}{\partial z}+\frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}},h_s\right]-\frac{q_s}{T_{0s}}F_{0s}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}\right]\right\rangle_{\mathbf{r}}=0. \tag{7.33}$$

Rearranging and taking the ring average into the integrand, we find

$$\sum_s \int d^3\mathbf{v} q_s v_\parallel \frac{\partial \langle h_s \rangle_{\mathbf{r}}}{\partial z} = -\frac{\partial}{\partial t} \sum_s \int d^3\mathbf{v} q_s \langle h_s \rangle_{\mathbf{r}} - \sum_s \int d^3\mathbf{v} \frac{q_s c}{B_0} \langle [\langle \chi \rangle_{\mathbf{R}}, h_s] \rangle_{\mathbf{r}}$$
$$+\frac{\partial}{\partial t} \sum_s \int d^3\mathbf{v} \frac{q_s^2 F_{0s}}{T_{0s}} \langle \langle \chi \rangle_{\mathbf{R}_s} \rangle_{\mathbf{r}} \tag{7.34}$$

The first term on the right-hand side can be replaced by the quasineutrality condition

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi = \sum_s \int d^3\mathbf{v} q_s \langle h_s \rangle_{\mathbf{r}} \tag{7.35}$$

giving

$$\sum_s \int d^3\mathbf{v} q_s v_\parallel \frac{\partial \langle h_s \rangle_{\mathbf{r}}}{\partial z} = -\frac{\partial}{\partial t} \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi - \sum_s \int d^3\mathbf{v} \frac{q_s c}{B_0} \langle [\langle \chi \rangle_{\mathbf{R}}, h_s] \rangle_{\mathbf{r}}$$
$$+\frac{\partial}{\partial t} \sum_s \int d^3\mathbf{v} \frac{q_s^2 F_{0s}}{T_{0s}} \langle \langle \chi \rangle_{\mathbf{R}_s} \rangle_{\mathbf{r}} \tag{7.36}$$

To connect with eq (7.32) of the previous section, we must multiply by χ and integrate over all positions $\mathbf{r}$ to get

$$\sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s v_\parallel \chi \frac{\partial \langle h_s \rangle_{\mathbf{r}}}{\partial z} = -\sum_s \int \frac{d^3\mathbf{r}}{V} \frac{q_s^2 n_{0s}}{T_{0s}} \chi \frac{\partial \phi}{\partial t} - \sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{q_s c}{B_0} \chi \langle [\langle \chi \rangle_{\mathbf{R}}, h_s] \rangle_{\mathbf{r}}$$
$$+\sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{q_s^2 F_{0s}}{T_{0s}} \chi \frac{\partial \langle \langle \chi \rangle_{\mathbf{R}_s} \rangle_{\mathbf{r}}}{\partial t} \tag{7.37}$$

The first term on the right-hand side can be rewritten by expanding the gyrokinetic potential $\chi = \phi - \mathbf{v} \cdot \mathbf{A}/c$

$$-\sum_s \int \frac{d^3\mathbf{r}}{V} \frac{q_s^2 n_{0s}}{T_{0s}} \chi \frac{\partial \phi}{\partial t} = -\sum_s \int \frac{d^3\mathbf{r}}{V} \frac{q_s^2 n_{0s}}{T_{0s}} \left(\phi \frac{\partial \phi}{\partial t} - \frac{v_\parallel A_\parallel}{c} \frac{\partial \phi}{\partial t} - \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c} \frac{\partial \phi}{\partial t} \right) \tag{7.38}$$

We can eliminate the second term on the right-hand side by noting that

$$\int \frac{d^3\mathbf{r}}{V} \chi \langle [\langle \chi \rangle_{\mathbf{R}}, h_s] \rangle_{\mathbf{r}} = \int \frac{d^3\mathbf{R}_s}{V} \langle \chi \rangle_{\mathbf{R}} [\langle \chi \rangle_{\mathbf{R}}, h_s] = 0 \tag{7.39}$$

and a similar trick on the third term gives

$$\int \frac{d^3\mathbf{r}}{V} \chi \frac{\partial \langle \langle \chi \rangle_{\mathbf{R}_s} \rangle_{\mathbf{r}}}{\partial t} = \int \frac{d^3\mathbf{R}_s}{V} \langle \chi \rangle_{\mathbf{R}_s} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} = \frac{\partial}{\partial t} \int \frac{d^3\mathbf{R}_s}{V} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}^2}{2}. \tag{7.40}$$

To connect with the right-hand side of eq (7.32) we perform the same trick yet again on the left-hand side here

$$\sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} q_s v_\parallel \chi \frac{\partial \langle h_s \rangle_{\mathbf{r}}}{\partial z} = \sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s v_\parallel \langle \chi \rangle_{\mathbf{R}_s} \frac{\partial h_s}{\partial z}. \tag{7.41}$$

Thus, the equation that we use to connect to eq (7.32) is

$$\sum_s \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s v_\parallel \left\langle \chi \right\rangle_{\mathbf{R}_s} \frac{\partial h_s}{\partial z} = -\sum_s \int \frac{d^3\mathbf{r}}{V} \frac{q_s^2 n_{0s}}{T_{0s}} \chi \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{q_s^2 F_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}^2}{2} \tag{7.42}$$

Chapter 8

Connection of Gyrokinetics to Reduced MHD

8.1 Summary

The key points made in this paper are:

1. When a hyperviscosity term of the form $\nu_H \nabla^4 h_s$ is added to the gyrokinetic equation, the resulting vorticity equation is the reduced MHD limit is given by

$$\frac{\partial}{\partial t}\nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel + \left(1 + \frac{T_{0i}}{T_{0e}}\right)\frac{2}{\rho_i^2}\nu_H \nabla_\perp^4 \Phi = 0. \tag{8.1}$$

   This does not agree with the form we expect for hyperviscosity in reduced MHD.

2. If, on the other hand, a hyperviscosity term of the form $\nu_H \nabla^4 g_s$ is used, the reduced MHD limit gives

$$\frac{\partial}{\partial t}\nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel + \nu_H \nabla_\perp^4 \nabla_\perp^2 \Phi = 0, \tag{8.2}$$

   precisely the form one desires. Hence, it appears one needs to use g_s in the hyperviscous term to correctly model hyperviscous damping.

3. The magnetic flux equation, in the reduced MHD limit, resulting from the use of a hyperviscosity term of the form $\nu_H \nabla^4 g_s$, is

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial t}\left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) + \nu_H \nabla^4 \left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) = 0. \tag{8.3}$$

   Hence, the hyperdiffusive term on Ψ is of order

$$k_\perp^2 \frac{c^2}{\omega_{pe}^2} = \frac{k_\perp^2 \rho_i^2}{\beta_i}\frac{m_e}{m_i} \ll 1. \tag{8.4}$$

   Thus, the hyperviscous term gives only weak hyperresisitivity.

4. To achieve a magnetic flux equation with hyperresistivity of the form

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \eta_H \nabla^4 \Psi = 0, \tag{8.5}$$

one should add the right-hand side of the gyrokinetic equation the term

$$-\frac{q_s}{T_{0s}} \eta_H \nabla^4 \left\langle \frac{v_\parallel A_\parallel}{c} \right\rangle_{\mathbf{R}_s} F_{0s}. \tag{8.6}$$

5. Unfortunately, when the hyperviscous term of the form $\nu_H \nabla^4 g_s$ is used, the hyperviscous heating is expressed in a non-positive definite form.

6. Finally, numerical tests with `GS2` show that when $\nu_H \nabla^4 h_s$ is the hyperviscous term, the effective damping is of order $(m_i/m_e)\nu_h k_\perp^4$, a mass ratio higher than expected. When the term is $\nu_H \nabla^4 g_s$, the damping rate is consistent with predictions.

8.2 Gyrokinetic Connection to Reduced MHD with Diffusive terms

Here I will describe how to derive the RMHD equations as a limit of the Gyrokinetic equations including hyperdiffusive terms.

8.2.1 The Vorticity Equation

We begin with the gyrokinetic equation including the hyperviscous term $\nu_H \nabla^4 h_s$

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left[\langle \chi \rangle_{\mathbf{R}_s}, h_s \right] - \left\langle \left(\frac{\partial h_s}{\partial t} \right)_{\mathrm{coll}} \right\rangle_{\mathbf{R}_s} + \nu_H \nabla^4 h_s = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{8.7}$$

To derive the vorticity equation, we want to use the form of the equation as implemented in the `GS2` using g_s where

$$g_s \equiv h_s - \frac{q_s \langle \phi \rangle}{T_s} F_{0s} + \frac{q_s \langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp \rangle}{c T_s} F_{0s} \tag{8.8}$$

We define two quantities to simplify this calculation (for the Fourier components of the gyrokinetic potential)

$$\tilde{\phi} \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s}) \hat{\phi} + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m_s v_\perp^2}{q_s} \frac{\delta \hat{B}_\parallel}{B_0} \tag{8.9}$$

and

$$A \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s}) \frac{v_\parallel A_\parallel}{c} \tag{8.10}$$

so that we have

$$\langle \chi \rangle = \tilde{\phi} - A \tag{8.11}$$

The gyrokinetic equation written as a function of g (except for the collision operator) then becomes

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} + \frac{q_s}{T_s} v_\parallel F_{0s} \frac{\partial \tilde{\phi}}{\partial z} + \frac{c}{B} \left[\tilde{\phi} - A, h_s \right] - \langle \mathcal{C}(h_s) \rangle + \nu_H \nabla^4 g_s + \frac{q_s \nu_H}{T_s} F_{0s} \nabla^4 \tilde{\phi} = -\frac{q_s}{T_s} F_{0s} \frac{\partial A}{\partial t} \tag{8.12}$$

Sum Velocity Integrated Gyrokinetic Equations

To derive the vorticity equation, the first step is to multiply the gyrokinetic equation by q_s, ring average at constant position $\mathbf{r}$, integrate the equation over velocity, and sum over species. We will consider each term in turn.

To simplify the first term, we use the quasineutrality condition in terms of g_s,

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\hat{\phi} = \sum_s q_s \int_{\mathbf{v}} \langle h_s\rangle_{\mathbf{r}} = \sum_s q_s \int_{\mathbf{v}} \langle g_s\rangle_{\mathbf{r}} + \sum_s \int_{\mathbf{v}} \frac{q_s^2}{T_s} J_0^2 \hat{\phi} F_{0s} + \sum_s \int_{\mathbf{v}} \frac{q_s m_s v_\perp^2}{T_s}\frac{J_0 J_1}{\gamma_s} F_{0s}\frac{\delta \hat{B}_\parallel}{B_0} \tag{8.13}$$

Performing the integration over velocity for the potential terms and solving for the integral of g_s gives

$$\sum_s q_s \int_{\mathbf{v}} \langle g_s\rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(1-\Gamma_{0s})\hat{\phi} - \sum_s q_s n_{0s}\Gamma_{1s}\frac{\delta \hat{B}_\parallel}{B_0} \tag{8.14}$$

The first term in the gyrokinetic equation then becomes

$$\frac{\partial}{\partial t}\sum_s \int_{\mathbf{v}} q_s \langle g_s\rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(1-\Gamma_{0s})\frac{\partial \hat{\phi}}{\partial t} - \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\Gamma_{1s}\frac{\partial}{\partial t}\left(\frac{T_s}{q_s}\frac{\delta \hat{B}_\parallel}{B_0}\right) \tag{8.15}$$

For the second term, we will use the Parallel Ampere's Law in terms of g_s given by

$$-\frac{c}{4\pi}\nabla_\perp^2 A_\parallel = \sum_s \int d^3\mathbf{v} q_s v_\parallel \langle h_s\rangle_{\mathbf{r}} = \sum_s \int d^3\mathbf{v} q_s v_\parallel \langle g_s\rangle_{\mathbf{r}} + \sum_s \int d^3\mathbf{v}\frac{q_s^2}{T_{0s}} F_{0s} v_\parallel \left\langle \left\langle \phi - \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} \tag{8.16}$$

The last term on the right had side is zero after velocity integration because it is odd in $v_\parallel$. Hence, the second term becomes

$$\frac{\partial}{\partial z}\sum_s \int_{\mathbf{v}} q_s v_\parallel \langle g_s\rangle_{\mathbf{r}} = -\frac{c}{4\pi}\frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel \tag{8.17}$$

The third term is

$$\frac{\partial}{\partial z}\sum_s \int_{\mathbf{v}} \frac{q_s^2 F_{0s}}{T_{0s}} v_\parallel \left\langle \left\langle \phi - \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} = 0 \tag{8.18}$$

because the entire term is odd in $v_\parallel$.

We will skip the fourth and fifth terms for the moment, since the nonlinearity and collisionality are not essential to understanding the manifestation of hyperdiffusivity in the reduced MHD equations.

The sixth term can be treated in much the same way as the the first, using quasineutrality to simplify the term The first term in the gyrokinetic equation then becomes

$$\nu_H \nabla^4 \sum_s \int_{\mathbf{v}} q_s \langle g_s\rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(1-\Gamma_{0s})\nu_H \nabla^4 \hat{\phi} - \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\Gamma_{1s}\nu_H \nabla^4\left(\frac{T_s}{q_s}\frac{\delta \hat{B}_\parallel}{B_0}\right) \tag{8.19}$$

The seventh term of the left-hand side just becomes

$$\nu_H \nabla^4 \sum_s \int_{\mathbf{v}} \frac{q_s^2}{T_{0s}} F_{0s} \langle \tilde{\phi} \rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \Gamma_{0s} \nu_H \nabla^4 \hat{\phi} + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \Gamma_{1s} \nu_H \nabla^4 \left(\frac{T_s}{q_s} \frac{\delta \hat{B}_\parallel}{B_0} \right) \tag{8.20}$$

The term on the right hand side is

$$\frac{\partial}{\partial t} \sum_s \int_{\mathbf{v}} \frac{q_s^2}{T_{0s}} F_{0s} \frac{v_\parallel}{c} \left\langle \langle A_\parallel \rangle_{\mathbf{R}_s} \right\rangle_{\mathbf{r}} = 0 \tag{8.21}$$

Putting all of these terms together, we see that the seventh term cancels the terms in the sixth term that have Γ_{ns}, leaving us with

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \left[(1 - \Gamma_{0s}) \frac{\partial \hat{\phi}}{\partial t} - \Gamma_{1s} \frac{\partial}{\partial t} \left(\frac{T_s}{q_s} \frac{\delta \hat{B}_\parallel}{B_0} \right) \right] - \frac{c}{4\pi} \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \nu_H \nabla^4 \hat{\phi} = 0 \tag{8.22}$$

Take $k_\perp \rho_i \ll 1$ Limit

In the MHD limit $k_\perp \rho_i \ll 1$, we can expand $\Gamma_{0i} \simeq 1 - \frac{k_\perp^2 \rho_i^2}{2}$, $\Gamma_{1i} \simeq 1 - \frac{3k_\perp^2 \rho_i^2}{4}$, $\Gamma_{0e} \simeq 1$, and $\Gamma_{1e} \simeq 1$. We will also drop the $\delta \hat{B}_\parallel$ terms because we know that in the MHD limit there is not parallel magnetic field perturbation due to Alfven waves. This leaves us the result

$$\frac{q_i^2 n_{0i}}{T_{0i}} \frac{k_\perp^2 \rho_i^2}{2} \frac{\partial \hat{\phi}}{\partial t} - \frac{c}{4\pi} \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \nu_H \nabla^4 \hat{\phi} = 0 \tag{8.23}$$

Rearranging the equation and multiplying by c/B_0 gives

$$\frac{\partial}{\partial t} k_\perp^2 \frac{c\hat{\phi}}{B_0} - \frac{2c^2 T_{0i}}{4\pi B_0 q_i^2 n_{0i} \rho_i^2} \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel + \left(1 + \frac{T_{0i}}{T_{0e}}\right) \frac{2}{\rho_i^2} \nu_H \nabla^4 \frac{c\hat{\phi}}{B_0} = 0 \tag{8.24}$$

Noting that

$$\frac{2c^2 T_{0i}}{4\pi B_0 q_i^2 n_{0i} \rho_i^2} = \frac{B_0}{4\pi n_{0i} m_i} \tag{8.25}$$

and using the definition of the reduced MHD stream and flux functions

$$\Phi = \frac{c\hat{\phi}}{B_0} \tag{8.26}$$

$$\Psi = \frac{A_\parallel}{\sqrt{4\pi n_{0i} m_i}} \tag{8.27}$$

and converting back from Fourier to real space produces

$$\frac{\partial}{\partial t} \nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel + \left(1 + \frac{T_{0i}}{T_{0e}}\right) \frac{2}{\rho_i^2} \nu_H \nabla_\perp^4 \Phi = 0 \tag{8.28}$$

THIS IS WRONG! The diffusive term is down by two factors of $k_\perp$, so the hyperviscosity implemented in this way is simply wrong.

Correcting the Problem

The key is to implement hyperviscosity in the gyrokinetic equation using a hyperviscous term of the form $\nu_H \nabla^4 g_s$ instead of $\nu_H \nabla^4 h_s$. In this case, the seventh term in equation (8.12) is missing and so does not cancel with parts of the sixth term. If we follow this path, then the result of the ring average at constant position $\mathbf{r}$, integrate the equation over velocity, and sum over species gives

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \left[(1-\Gamma_{0s}) \frac{\partial \hat{\phi}}{\partial t} - \Gamma_{1s} \frac{\partial}{\partial t} \left(\frac{T_s}{q_s} \frac{\delta \hat{B}_\parallel}{B_0} \right) \right] - \frac{c}{4\pi} \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel \tag{8.29}$$

$$+ \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \left[(1-\Gamma_{0s}) \nu_H \nabla^4 \hat{\phi} - \Gamma_{1s} \nu_H \nabla^4 \left(\frac{T_s}{q_s} \frac{\delta \hat{B}_\parallel}{B_0} \right) \right] = 0 \tag{8.30}$$

Now, taking the MHD limit $k_\perp \rho_i \ll 1$ and dropping the $\delta \hat{B}_\parallel$ terms gives, after the same manipulations as above

$$\frac{\partial}{\partial t} \nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z} \nabla_\perp^2 A_\parallel + \nu_H \nabla_\perp^4 \nabla_\perp^2 \Phi = 0 \tag{8.31}$$

8.2.2 The Magnetic Flux Equation

We begin with the gyrokinetic equation including the hyperviscous term $\nu_H \nabla^4 g_s$

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left[\langle \chi \rangle_{\mathbf{R}_s}, h_s \right] - \left\langle \left(\frac{\partial h_s}{\partial t} \right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} + \nu_H \nabla^4 g_s = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{8.32}$$

We'll drop the nonlinear and collision terms to simplify matters (since the hyperdiffusive term is unaffected by the presence of these terms).

Sum Velocity Integrated Gyrokinetic Equations

To derive the magnetic flux equation, the first step is to mulitply the gyrokinetic equation by $q_s v_\parallel$, ring average at constant position $\mathbf{r}$, integrate the equation over velocity, and sum over species. We will consider each term in turn.

For the first term, we use the parallel Ampere's Law to obtain,

$$\frac{\partial}{\partial t} \sum_s \int_{\mathbf{v}} q_s v_\parallel \langle h_s \rangle_{\mathbf{r}} = -\frac{c}{4\pi} \frac{\partial}{\partial t} \nabla_\perp^2 A_\parallel \tag{8.33}$$

For the second term, we will define the pressure as a moment of the full distribution function given by

$$P_s \equiv \int_{\mathbf{v}} \frac{m_s v^2}{2} f_s \tag{8.34}$$

We can similarly define the parallel pressure $P_{\parallel s}$ as the moment using $\frac{m_s v_\parallel^2}{2}$. Expanding the distribution function into its consituent pieces.

$$f_s = F_{0s} \left(1 - \frac{q_s \phi}{T_{0s}} \right) + \langle h_s \rangle_{\mathbf{r}} \tag{8.35}$$

Noting that

$$\int_{\mathbf{v}} v_\parallel^2 F_{0s} = \frac{n_{0s}T_{0s}}{m_s}, \tag{8.36}$$

we find that the parallel pressure is given by

$$P_{\parallel s} = \frac{1}{2}n_{0s}T_{0s} - \frac{1}{2}q_s\phi n_{0s} + \delta P_{\parallel s} \tag{8.37}$$

where we have defined the non-adiabatic part of the parallel pressure by

$$\delta P_{\parallel s} \equiv \int_{\mathbf{v}} \frac{m_s v_\parallel^2}{2}\langle h_s\rangle_{\mathbf{r}} \tag{8.38}$$

In terms of this definition, the second term becomes

$$\frac{\partial}{\partial z}\sum_s \int_{\mathbf{v}} q_s v_\parallel^2 \langle h_s\rangle_{\mathbf{r}} = \frac{\partial}{\partial z}\sum_s \frac{2q_s\delta P_{\parallel s}}{m_s} \tag{8.39}$$

The third term is the hyperviscous term $\nu_H\nabla^4 g_s$. First, we note that when multiplied by $v_\parallel$ and integrated velocity, the difference between the $\nu_H\nabla^4 g_s$ and $\nu_H\nabla^4 h_s$ goes away.

$$\int_{\mathbf{v}} v_\parallel \langle g_s\rangle_{\mathbf{r}} = \int_{\mathbf{v}} v_\parallel \langle h_s\rangle_{\mathbf{r}} \tag{8.40}$$

So, using parallel Ampere's Law, we find,

$$\nu_H\nabla^4\sum_s \int_{\mathbf{v}} q_s v_\parallel \langle g_s\rangle_{\mathbf{r}} = -\frac{c}{4\pi}\nu_H\nabla^4\nabla_\perp^2 A_\parallel \tag{8.41}$$

Finally, the term on the right-hand side gives

$$\frac{\partial}{\partial t}\sum_s \int_{\mathbf{v}} \frac{q_s^2}{T_{0s}}F_{0s}v_\parallel \left\langle\left\langle \phi - \frac{v_\parallel A_\parallel}{c} - \frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} \tag{8.42}$$

The first and third terms are odd in $v_\parallel$ and so yield zero upon integration over velocity. Manipulating what remains gives

$$-\frac{c}{4\pi}\frac{\partial}{\partial t}\sum_s \frac{4\pi q_s^2 n_{0s}}{m_s c^2}\int_{\mathbf{v}} \frac{m_s F_{0s}}{n_{0s}T_{0s}}v_\parallel^2 \left\langle\langle A_\parallel\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} = -\frac{c}{4\pi}\frac{\partial}{\partial t}\sum_s \frac{\omega_{ps}^2}{c^2}\int_{\mathbf{v}} \frac{m_s F_{0s}}{n_{0s}T_{0s}}v_\parallel^2 \left\langle\langle A_\parallel\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} \tag{8.43}$$

Eliminate $\delta P_{\parallel s}$

To eliminate the parallel pressure term, we use the fact that in the limit $k_\perp\rho_s \ll 1$, the lowest order solution $h_s^{(0)}$ admits solutions in which the velocity dependence can be separated from the space and time dependence,

$$h_s^{(0)} = H_s(\mathbf{R},t)F_{0s}(v). \tag{8.44}$$

This is examined in detail in Alex's paper on gyrokinetics and Reduced MHD. Substituting this solution into the quasineutrality condition

$$\sum_s q_s \delta n_s = \sum_s q_s \left(-\frac{q_s n_{0s}}{T_{0s}} \phi + \int d^3\mathbf{v} \langle h_s \rangle_{\mathbf{r}} \right) = 0 \tag{8.45}$$

gives the solution

$$\sum_s q_s n_{0s} H_s(\mathbf{R}, t) = \sum_s q_s n_{0s} \left(\frac{\delta n_s}{n_{0s}} + \frac{q_s \phi}{T_{0s}} \right) = 0 \tag{8.46}$$

We can then find the solution [NOTE: Can we really remove this from the sum? I am not happy about this step, but perhaps there is an unstated ordering assumption in Alex's paper allowing him to take this step.]

$$h_s^{(0)} = \left(\frac{\delta n_s}{n_{0s}} + \frac{q_s \phi}{T_{0s}} \right) F_{0s}(v) \tag{8.47}$$

which gives solution for the total distribution function

$$f_s = \left(1 + \frac{\delta n_s}{n_{0s}} \right) F_{0s}(v) \tag{8.48}$$

and the solution for the parallel pressure is then

$$P_{\parallel s} = \frac{1}{2} n_{0s} T_{0s} - \frac{1}{2} \delta n_s T_{0s}. \tag{8.49}$$

Hence, using equation (8.37), we find

$$\delta P_{\parallel s} = \frac{1}{2} n_{0s} T_{0s} \left(\frac{\delta n_s}{n_{0s}} + \frac{q_s \phi}{T_{0s}} \right) \tag{8.50}$$

Now, we can rewrite the second term in terms of ϕ and δn_s as

$$\frac{\partial}{\partial z} \sum_s \frac{2 q_s \delta P_{\parallel s}}{m_s} = \frac{c^2}{4\pi} \sum_s \frac{\omega_{ps}^2}{c^2} \frac{\partial}{\partial z} \left(\phi + \frac{T_{0s}}{q_s} \frac{\delta n_s}{n_{0s}} \right) \tag{8.51}$$

Putting everything together

Putting all of these terms together and multiplying by $\frac{4\pi}{c}$ gives

$$-\frac{\partial}{\partial t} \nabla_\perp^2 A_\parallel + c \sum_s \frac{\omega_{ps}^2}{c^2} \frac{\partial}{\partial z} \left(\phi + \frac{T_{0s}}{q_s} \frac{\delta n_s}{n_{0s}} \right) - \nu_H \nabla^4 \nabla_\perp^2 A_\parallel = -\frac{\partial}{\partial t} \sum_s \frac{\omega_{ps}^2}{c^2} \int_{\mathbf{v}} \frac{m_s F_{0s}}{n_{0s} T_{0s}} v_\parallel^2 \left\langle \langle A_\parallel \rangle_{\mathbf{R}_s} \right\rangle_{\mathbf{r}} \tag{8.52}$$

Take $k_\perp \rho_i \ll 1$ Limit

We know that by Fourier analyzing in space will give us the result

$$\int_{\mathbf{R}} \int_{\mathbf{v}} \left\langle \langle A_\parallel \rangle_{\mathbf{R}_s} \right\rangle_{\mathbf{r}} = \int_{\mathbf{v}} \sum_{\mathbf{k}} \Gamma_{0s} \hat{A}_\parallel, \tag{8.53}$$

so we will put in the Γ_{0s} (with a sign change for the Fourier transform) to account for the double ring average even though we are being a bit sloppy with notation. In the limit $k_\perp \rho_i \ll 1$, $\Gamma_{0s} \simeq 1$, and we can take $\sum_s \omega_{ps}^2 = \omega_{pe}^2 \left(1 + \frac{m_e}{m_i}\right) = \omega_{pe}^2$. We also neglect the density perturbation δn_s in the MHD limit since Alfven waves are incompressible. Hence, we find

$$\frac{\partial A_\parallel}{\partial t} - c\frac{\partial \phi}{\partial z} + \frac{\partial}{\partial t}\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 A_\parallel + \nu_H \nabla^4 \frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 A_\parallel = 0 \tag{8.54}$$

Multiplying by $(4\pi n_{0i} m_i)^{-1/2}$ and using the definitions of the stream and flux functions equations (8.26)–(9.5), we find

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial t}\left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) + \nu_H \nabla^4 \left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) = 0 \tag{8.55}$$

If we order the damping rate due to hyperdiffusivity $\mathcal{O}(\nu_H k_\perp^4) = \mathcal{O}(\omega)$, as is appropriate at the of the cascade where we want the damping to be strong, then the hyperdiffusive term, compared to the $\frac{\partial \Psi}{\partial t}$ term, is of order

$$k_\perp^2 \frac{c^2}{\omega_{pe}^2} = \frac{k_\perp^2 \rho_i^2}{\beta_i}\frac{m_e}{m_i} \ll 1 \tag{8.56}$$

Therefore, except for at very low $\beta_i \sim \mathcal{O}(\frac{m_e}{m_i})$, the diffusive effect (hyperresistivity) of the hyperviscosity term on $A_\parallel$ from the hyperviscosity term is negligible.

Discussion of Hyperresistivity

To achieve an effective hyperresistivity, we would like a term that produces a Reduced MHD flux equation of the form

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \eta_H \nabla^4 \Psi = 0 \tag{8.57}$$

To achieve such an effect in the magnetic flux equation, we can put in a term on the right-hand side of the form

$$\frac{q_s}{T_{0s}} \eta_H \nabla^4 \langle \chi \rangle_{\mathbf{R}_s} F_{0s} \tag{8.58}$$

In this case, only the $A_\parallel$ part of the gyrokinetic potential will affect the final form of the magnetic flux equation because the ϕ and $\delta B_\parallel$ terms are odd when multiplied by $v_\parallel$. Whether to include these terms or not, it is necessary to view how this addition affects the vorticity equation. With only the $A_\parallel$ term, the vorticity equation is unchanged because the extra term goes away when integrated over velocity. It seems most likely that you want only to add the terms necessary to achieve this hyperresistivity, so the desired term to add to the right-hand side of the gyrokinetic equation would be

$$-\frac{q_s}{T_{0s}}\eta_H \nabla^4 \left\langle \frac{v_\| A_\|}{c} \right\rangle_{\mathbf{R}_s} F_{0s} \tag{8.59}$$

One could also add a term to the left-hand side, much like the hyperviscosity, of the form

$$\frac{\omega_{pe}^2}{c^2}\eta_H \nabla_\perp^2 g_s. \tag{8.60}$$

This will also produce the desired term in the magnetic flux equation, but it will also have an effect of normal viscosity, with a huge coefficient out front because of the $\frac{\omega_{pe}^2}{c^2}$ factor. Hence, it will likely create a large Laplacian viscosity, and undesired effect.

8.3 Corrected Hyperviscous Heating

In this section we will derive the form of the hyperviscous heating for the case when the hyperviscous term is $\nu_H \nabla^4 g_s$ instead of $\nu_H \nabla^4 h_s$.

8.3.1 Entropy Equation with Hyperviscosity

We begin with the gyrokinetic equation including ther hyperviscous term

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle \chi \rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} + \nu_H \nabla^4 g_s = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{8.61}$$

We multiply the gyrokinetic equation by $T_{0s} h_s / F_{0s}$ and integrate over space and velocity to obtain the **entropy equation**

$$\int \frac{d^3\mathbf{R}}{V} \int d^3\mathbf{v} q_s \frac{\partial \langle \chi \rangle}{\partial t} h_s - \frac{d}{dt} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s C(h_s) \rangle_{\mathbf{R}_s}$$
$$- \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \nu_H h_s \nabla_\perp^4 g_s = 0 \tag{8.62}$$

where we have applied the gyrokinetic approximation $k_\parallel \ll k_\perp$ to the hyperviscous operator $\nabla^4 \to \nabla_\perp^4$. We need to manipulate the last term into a suitable form for calculating the energy.

Relation between g_s and h_s

First, note that the definition of g_s in terms of h_s is

$$g_s \equiv h_s - \frac{q_s \langle \phi \rangle_{\mathbf{R}_s}}{T_{0s}} F_{0s} + \frac{q_s \langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp \rangle_{\mathbf{R}_s}}{T_{0s} c} F_{0s} \tag{8.63}$$

When Fourier decomposed, note that we can write each of these pieces as

$$g_s = \sum_{\mathbf{k}} \hat{g}_{s\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{8.64}$$

$$h_s = \sum_{\mathbf{k}} \hat{h}_{s\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{8.65}$$

$$\langle \phi \rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \hat{\phi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{8.66}$$

$$\left\langle \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c} \right\rangle_{\mathbf{R}_s} = -\sum_{\mathbf{k}} \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m_s v_\perp^2}{q_s} \frac{\delta \hat{B}_{\parallel\mathbf{k}}}{B_0} e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{8.67}$$

Therefore, the Fourier components are related by

$$\hat{g}_{s\mathbf{k}} \equiv \hat{h}_{s\mathbf{k}} - \frac{q_s J_{0s} \hat{\phi}_{\mathbf{k}}}{T_{0s}} F_{0s} + \frac{J_{1s}}{\gamma_s} \frac{m_s v_\perp^2}{T_{0s}} \frac{\delta \hat{B}_{\parallel\mathbf{k}}}{B_0} F_{0s} \tag{8.68}$$

where we have used the notation $\gamma_s = \frac{k_\perp v_\perp}{\Omega_s}$ and $J_{ns} = J_n(\gamma_s)$.

Manipulating the Hyperviscous Heating Term

Noting here that

$$\nabla_\perp^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \tag{8.69}$$

we can perform an integration by parts in space twice on each term to obtain

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}\nu_H}{F_{0s}} h_s \nabla_\perp^4 g_s \tag{8.70}$$

$$\begin{aligned}
&= \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}\nu_H}{F_{0s}} h_s \nabla_\perp^4 \left(h_s - \frac{q_s\langle\phi\rangle_{\mathbf{R}_s}}{T_{0s}} F_{0s} + \frac{q_s\langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp\rangle_{\mathbf{R}_s}}{T_{0s}c} F_{0s} \right) \\
&= \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}\nu_H}{F_{0s}} \left[\left(\frac{\partial^2 h_s}{\partial^2 x}\right)^2 + 2\left(\frac{\partial^2 h_s}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 h_s}{\partial^2 y}\right)^2 \right] \\
&- \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s \nu_H \left[\frac{\partial^2\langle\phi\rangle}{\partial^2 x}\frac{\partial^2 h_s}{\partial^2 x} + 2\frac{\partial^2\langle\phi\rangle}{\partial x\partial y}\frac{\partial^2 h_s}{\partial x\partial y} + \frac{\partial^2\langle\phi\rangle}{\partial^2 y}\frac{\partial^2 h_s}{\partial^2 y} \right] \\
&+ \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{q_s\nu_H}{c} \left[\frac{\partial^2\langle \mathbf{v}_\perp\cdot\mathbf{A}_\perp\rangle}{\partial^2 x}\frac{\partial^2 h_s}{\partial^2 x} + 2\frac{\partial^2\langle \mathbf{v}_\perp\cdot\mathbf{A}_\perp\rangle}{\partial x\partial y}\frac{\partial^2 h_s}{\partial x\partial y} + \frac{\partial^2\langle \mathbf{v}_\perp\cdot\mathbf{A}_\perp\rangle}{\partial^2 y}\frac{\partial^2 h_s}{\partial^2 y} \right]
\end{aligned}$$

Fourier Decomposition of Heating Term

Here we will show how this heating term is expressed in the Fourier decomposition. Using the Fourier series expansions of h_s^* and ϕ, as given by equations (8.65)–(8.66), for one of the terms we find

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s\nu_H \frac{\partial^2\langle\phi\rangle}{\partial^2 x}\frac{\partial^2 h_s^*}{\partial^2 x} = \int d^3\mathbf{v} \int \frac{d^3\mathbf{R}_s}{V} q_s\nu_H \sum_{\mathbf{k}} \mathbf{k}_x^2 J_{0s}\hat{\phi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \sum_{\mathbf{k}'} \mathbf{k}_x'^2 \hat{h}_{s\mathbf{k}'}^* e^{-i\mathbf{k}'\cdot\mathbf{R}_s}$$

$$= \int d^3\mathbf{v} \sum_{\mathbf{k}}\sum_{\mathbf{k}'} q_s\nu_H \mathbf{k}_x^2 \mathbf{k}_x'^2 J_{0s}\hat{\phi}_{\mathbf{k}}\hat{h}_{s\mathbf{k}'}^* \int \frac{d^3\mathbf{R}_s}{V} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_s} \tag{8.71}$$

Noting that

$$\int d^3\mathbf{r} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta(\mathbf{k}-\mathbf{k}'), \tag{8.72}$$

we can use the delta function to eliminate one of the sums to find

$$\frac{1}{V}\sum_{\mathbf{k}} \int d^3\mathbf{v} q_s\nu_H \mathbf{k}_x^4 J_{0s}\hat{\phi}_{\mathbf{k}}\hat{h}_{s\mathbf{k}}^*. \tag{8.73}$$

Putting all of the terms together gives

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}\nu_H}{F_{0s}} h_s \nabla_\perp^4 g_s \tag{8.74}$$

$$= \sum_{\mathbf{k}} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \nu_H k_\perp^4 \left[|h_{\mathbf{k}}|^2 - J_{0s}\frac{q_s\hat{\phi}_{\mathbf{k}}}{T_{0s}}\hat{h}_{s\mathbf{k}}^* F_{0s} - \frac{J_{1s}}{\gamma_s}\frac{m_s v_\perp^2}{T_{0s}}\frac{\delta\hat{B}_{\parallel\mathbf{k}}}{B_0}\hat{h}_{s\mathbf{k}}^* F_{0s} \right]$$

Normalization

We multiply each term of the equation by $\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}$. The first term gives

$$\begin{aligned}
&\left[\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}\right]\sum_{\mathbf{k}}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\nu_H k_\perp^4 h_{s\mathbf{k}}^2 \\
&= \sum_{\mathbf{k}}\int \frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}}\frac{T_s}{T_0}\left(\frac{\nu_H a_0}{\rho_0^4 v_{t0}}\right)(k_\perp^4\rho_0^4)\left(\frac{h_{s\mathbf{k}}}{F_{0s}}\frac{a_0}{\rho_0}\right)^2 \\
&= \sum_{\mathbf{k}}\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\hat{\nu}_H\hat{k}_\perp^4\hat{h}_{s\mathbf{k}}^2.
\end{aligned} \tag{8.75}$$

The second term yields

$$\begin{aligned}
&-\left[\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}\right]\sum_{\mathbf{k}}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\nu_H k_\perp^4 J_{0s}\frac{q_s\hat{\phi}_{\mathbf{k}}}{T_s}\hat{h}_{s\mathbf{k}}F_{0s} \\
&= -\sum_{\mathbf{k}}\int \frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}}\frac{q_s}{q_0}\left(\frac{\nu_H a_0}{\rho_0^4 v_{t0}}\right)(k_\perp^4\rho_0^4)\left(\frac{q_0\hat{\phi}_{\mathbf{k}}}{T_0}\frac{a_0}{\rho_0}\right)\left(\frac{h_{s\mathbf{k}}}{F_{0s}}\frac{a_0}{\rho_0}\right) \\
&= -\sum_{\mathbf{k}}\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{q}_s\hat{\nu}_H\hat{k}_\perp^4\hat{J}_{0s}\hat{\phi}_{\mathbf{k}}\hat{h}_{s\mathbf{k}}.
\end{aligned} \tag{8.76}$$

The third term produces

$$\begin{aligned}
&-\left[\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}\right]\sum_{\mathbf{k}}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\nu_H k_\perp^4 \frac{J_{1s}}{\gamma_s}\frac{m_s v_\perp^2}{T_s}\frac{\delta\hat{B}_{\parallel\mathbf{k}}}{B_0}\hat{h}_{s\mathbf{k}}F_{0s} \\
&= -\sum_{\mathbf{k}}\int \frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}}\left(\frac{\nu_H a_0}{\rho_0^4 v_{t0}}\right)(k_\perp^4\rho_0^4)\frac{2T_s}{T_0}\frac{m_s v_\perp^2}{2T_s}\frac{J_{1s}}{\gamma_s}\left(\frac{\delta\hat{B}_{\parallel\mathbf{k}}}{B_0}\frac{a_0}{\rho_0}\right)\left(\frac{h_{s\mathbf{k}}}{F_{0s}}\frac{a_0}{\rho_0}\right) \\
&= -\sum_{\mathbf{k}}\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\hat{\nu}_H\hat{k}_\perp^4\hat{J}_{1s}2\hat{v}_{\perp s}^2\delta\hat{B}_{\parallel\mathbf{k}}\hat{h}_{s\mathbf{k}}.
\end{aligned} \tag{8.77}$$

Pulling everything together we find

$$\sum_{\mathbf{k}}\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{\nu}_H\hat{k}_\perp^4\hat{n}_s\hat{T}_s\left(\hat{h}_{s\mathbf{k}}^2-\hat{J}_{0s}\frac{\hat{q}_s\hat{\phi}_{\mathbf{k}}}{\hat{T}_s}\hat{h}_{s\mathbf{k}}-\hat{J}_{1s}2\hat{v}_{\perp s}^2\delta\hat{B}_{\parallel\mathbf{k}}\hat{h}_{s\mathbf{k}}\right) \tag{8.78}$$

8.3.2 Numerical Tests

Let us consider a generalized equation for the energy evolution

$$\frac{dE}{dt} = -2\gamma E - \nu_h k_\perp^4 E + P_a \tag{8.79}$$

where γ is the natural collisionless damping rate, $\nu_h k_\perp^4$ is the effective hyperviscous damping rate, and P_a is the antenna power driving the simulation.

When driven at a constant frequency and amplitude, the system eventually reaches a steady state in which $\frac{dE}{dt} = 0$. In this case, the amplitude of the energy in steady-state is given by

$$E = \frac{P_a}{2\gamma + \nu_h k_\perp^4} \tag{8.80}$$

The total heating due to collisionless damping is then given by

$$P_\gamma = 2\gamma E = \frac{2\gamma P_a}{2\gamma + \nu_h k_\perp^4} \tag{8.81}$$

and that due to hyperviscous damping is

$$P_{\nu_h} = 2\gamma E = \frac{\nu_h k_\perp^4 P_a}{2\gamma + \nu_h k_\perp^4} \tag{8.82}$$

Taking the ratio of the hyperviscous to the total heating, the unknown antenna power drops out, leaving the theoretical value

$$\frac{P_{\nu_h}}{P_{tot}} = \frac{\nu_h k_\perp^4}{2\gamma + \nu_h k_\perp^4} \tag{8.83}$$

Figure 8.1 plots $P_e/(P_i + P_e)$, roughly the ratio $\frac{P_{\nu_h}}{P_{tot}}$ (since electron heating is primarily hyperviscous and ion heating is primarily due to collisionless damping), against the theoretical relation given by equation (8.83) above. When the hyperviscous term used is of the form $\nu_h k_\perp^4 h_s$, the effective hyperviscous damping rate is increased by a factor of the mass ratio m_i/m_e compared to equation (8.83); to match the results, the formula must be amended to

$$\frac{P_{\nu_h}}{P_{tot}} = \frac{(m_i/m_e)\nu_h k_\perp^4}{2\gamma + (m_i/m_e)\nu_h k_\perp^4}. \tag{8.84}$$

When the hyperviscous term used is of the form $\nu_h k_\perp^4 g_s$, the results agree fairly well with equation (8.83).

Here are some additional notes and considerations:

- Note that the antenna power is unknown because, although the amplitude of the antenna is known, the power transmitted into the plasma depends on the plasma response which is not a straightforward function of the impedance matching.
- Since the linear modes are damped by collisionles damping frequency γ, should the energy effectively be damped by 2γ?

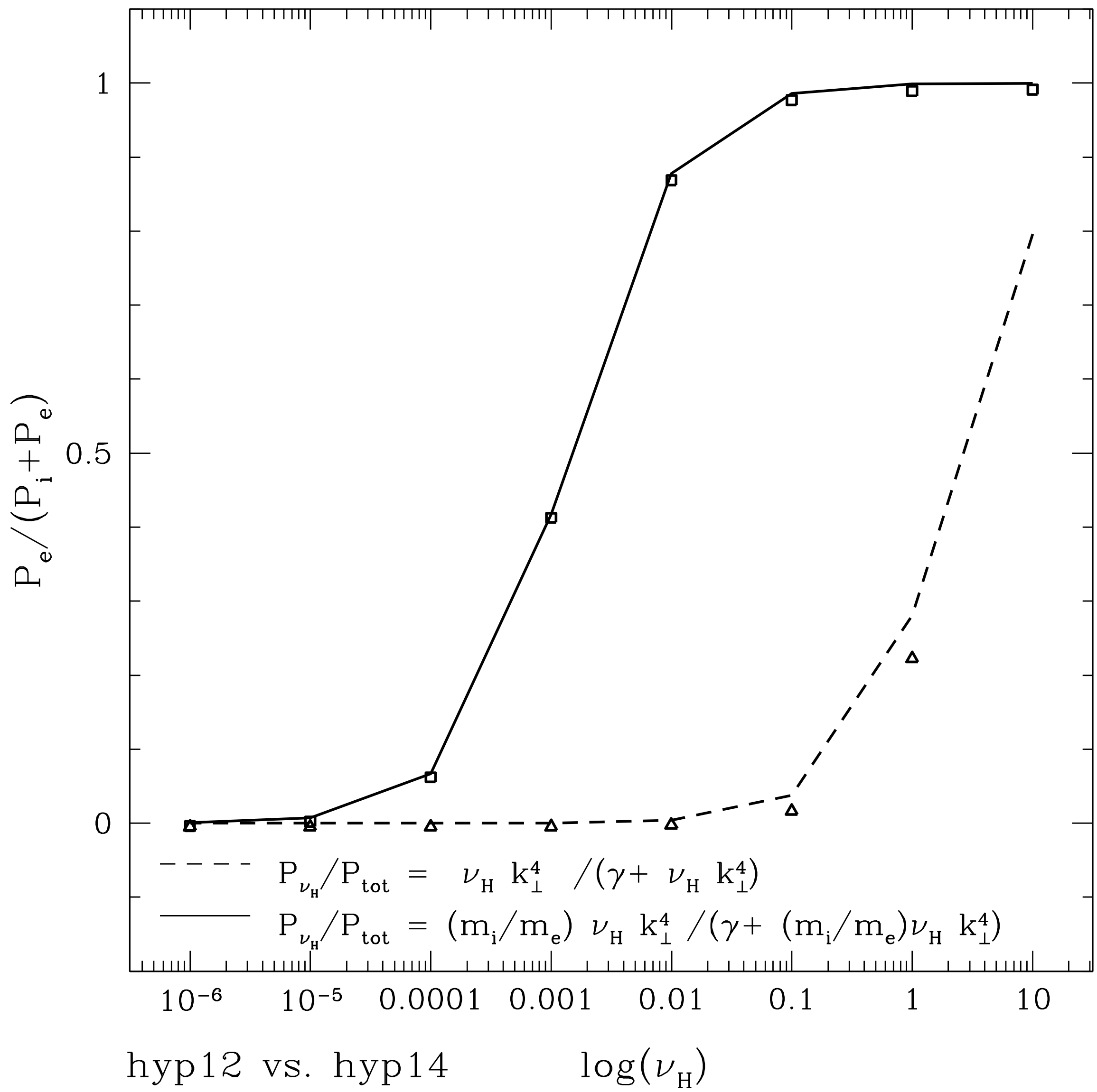

Figure 8.1: Plot of $P_e/(P_i + P_e)$ vs. the theoretical relation given by equation (8.83). For a hyperviscosity term of the form $\nu_h k_\perp^4 h_s$ (squares), the hyperviscous heating rate must be increased by the mass ratio m_i/m_e matches the numerical results. For a hyperviscosity term of the form $\nu_h k_\perp^4 g_s$ (triangles), the results agree much more closely with equation (8.83).

- Check the ion hyperviscosity to see if it improves the fit.

- If the antenna power is written $P_a = \frac{k_\perp^2 A_{\|0}}{\Delta t}$, how do we determine this timescale for power input? More generally, knowing the linear damping rates and antenna amplitude, can we predict the steady state amplitude of the energy? This is complicated by the fact that there are more than one energy in the system, and so perhaps it is not so simple.

Chapter 9

Hyperdiffusivity in Gyrokinetics

9.1 The Need for Non-Physical Damping in Nonlinear Runs

Nonlinear simulations of the cascade of turbulent energy in collisionless magnetized plasmas require the addition of a non-physical damping term to dissipated at the smallest scales in the system, thereby avoiding a build-up of energy, or bottleneck, in the smallest scale modes in the simulation domain. This is clear even in runs at high β_i, where the damping on ions is very strong at $k_\perp \rho_i \sim 1$, as shown in Figure 9.1. Note here that the hyperdiffusive damping was much too strong in run `trans3`.

Therefore, we need to implement some non-physical damping mechanism to remove energy rapidly at the smallest scales in the box. There appear to be two avenues to accomplish this:

1. Hyperdiffusivity: Adding a hyperdiffusive term to the gyrokinetic equation. A successful choice for the hyperdiffusive operator will have two properties: it must reduce, in the MHD limit, to some form recognizable as hyperviscosity or hyperresistivity; and, heating by this term must be positive definite.

2. Hypercollisionality: Adding a hypercollisional term where the coefficient is wavenumber dependent.

These two possibities are explored in the remainder of this document.

9.2 Hyperdiffusivity

In this section, we look at the possibility of adding some hyperdiffusive operator $\mathcal{H}$ to the gyrokinetic equation to achieve a wavenumber dependent damping.

9.2.1 Early Attempts at Hyperdiffusivity

Early attempts at including hyperdiffusivity terms tried adding a term of the form $\mathcal{H} = \nu_{Hs} \nabla_\perp^4$ to the gyrokinetic equation for each species s. I summarize here the attempts to use this approach:

1. When a hyperviscosity term of the form $\nu_H \nabla^4 h_s$ is added to the gyrokinetic equation, the resulting vorticity equation is the reduced MHD limit is given by

$$\frac{\partial}{\partial t}\nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel + \left(1 + \frac{T_{0i}}{T_{0e}}\right)\frac{2}{\rho_i^2}\nu_H \nabla_\perp^4 \Phi = 0. \tag{9.1}$$

   This does not agree with the form we expect for hyperviscosity in reduced MHD.

2. If, on the other hand, a hyperviscosity term of the form $\nu_H \nabla^4 g_s$ is used, the reduced MHD limit gives

$$\frac{\partial}{\partial t}\nabla_\perp^2 \Phi - v_A \frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel + \nu_H \nabla_\perp^4 \nabla_\perp^2 \Phi = 0, \tag{9.2}$$

   precisely the form one desires. Hence, it appears one needs to use g_s in the hyperviscous term to correctly model hyperviscous damping.

3. The magnetic flux equation, in the reduced MHD limit, resulting from the use of a hyperviscosity term of the form $\nu_H \nabla^4 g_s$, is

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial t}\left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) + \nu_H \nabla^4 \left(\frac{c^2}{\omega_{pe}^2}\nabla_\perp^2 \Psi\right) = 0. \tag{9.3}$$

   Hence, the hyperdiffusive term on Ψ is of order

$$k_\perp^2 \frac{c^2}{\omega_{pe}^2} = \frac{k_\perp^2 \rho_i^2}{\beta_i}\frac{m_e}{m_i} \ll 1. \tag{9.4}$$

   Thus, the hyperviscous term gives only weak hyperresisitivity.

4. To achieve a magnetic flux equation with hyperresistivity of the form

$$\frac{\partial \Psi}{\partial t} - v_A \frac{\partial \Phi}{\partial z} + \eta_H \nabla^4 \Psi = 0, \tag{9.5}$$

   one should add the right-hand side of the gyrokinetic equation the term

$$-\frac{q_s}{T_{0s}}\eta_H \nabla^4 \left\langle \frac{v_\parallel A_\parallel}{c} \right\rangle_{\mathbf{R}_s} F_{0s}. \tag{9.6}$$

5. Unfortunately, when the hyperviscous term of the form $\nu_H \nabla^4 g_s$ is used, the hyperviscous heating is expressed in a non-positive definite form.

6. Finally, numerical tests with `GS2` show that when $\nu_H \nabla^4 h_s$ is the hyperviscous term, the effective damping is of order $(m_i/m_e)\nu_h k_\perp^4$, a mass ratio higher than expected. When the term is $\nu_H \nabla^4 g_s$, the damping rate is consistent with predictions.

9.2.2 More Recent Thoughts on Hyperdiffusivity

In considering the above information, Bill noticed that if you write the gyrokinetic equation in this form,

$$\frac{\partial}{\partial t}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right) + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] = \mathcal{C}(h_s), \tag{9.7}$$

and then multiply by $T_s h_s / F_{0s}$ and integrate over all space and velocity, the second and third terms on the left-hand side integrate to zero over all space, leaving

$$\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} h_s \frac{\partial}{\partial t}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right) = \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} \left\langle h_s \mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s} \tag{9.8}$$

The general form of this seems to be

$$\frac{\partial}{\partial t}\mathcal{E} = \mathcal{C}_{\mathcal{E}} \tag{9.9}$$

where $\mathcal{C}_{\mathcal{E}}$ is the collisional heating rate.

This inspires us to try to add a hyperdiffusive term to the right-hand side of the gyrokinetic equation in this manner

$$\frac{\partial}{\partial t}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right) + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] = \mathcal{C}(h_s) + \mathcal{H}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right). \tag{9.10}$$

Doing this, the resulting entropy balance equation is given by

$$\begin{aligned}
&\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} h_s \frac{\partial}{\partial t}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right) \\
&= \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} \left\langle h_s \mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s} + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} h_s \mathcal{H}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right)
\end{aligned} \tag{9.11}$$

The limits of this operator in reduced MHD should give equations like (9.2) and (9.5), but I have not gone through this in detail.

9.2.3 Positive-Definite Properties of this approach

Here we investigate whether the hyperdiffusive heating term,

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} h_s \nu_H \nabla_\perp^4 \left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle_{\mathbf{R}_s}\right) \tag{9.12}$$

is positive definite. We go through this examination term by term below

Integrating by parts twice shows that the first term is positive definite for each species

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} h_s \nu_H \nabla_\perp^4 h_s = \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} \nu_H \left|\nabla_\perp^2 h_s\right|^2 \geq 0. \tag{9.13}$$

Using the property of ring averages that, when integrated over all space, one can interchange averages,

$$\int \frac{d^3\mathbf{R}_s}{V} f(\mathbf{R}_s)\langle g(\mathbf{r})\rangle_{\mathbf{R}_s} = \int \frac{d^3\mathbf{r}}{V}\langle f(\mathbf{R}_s)\rangle_{\mathbf{r}} g(\mathbf{r}), \tag{9.14}$$

we can write the second as

$$\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v} q_s h_s \nu_H \nabla_\perp^4 \langle\phi\rangle_{\mathbf{R}_s} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}} \nu_H \nabla_\perp^4 \phi \tag{9.15}$$

To demonstrate positive-definiteness, we must now sum over species, and use the quasineutrality condition to obtain

$$\int \frac{d^3\mathbf{r}}{V}\nu_H \nabla_\perp^4 \phi \left(\sum_s \int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}}\right) = \int \frac{d^3\mathbf{r}}{V}\nu_H \nabla_\perp^4 \phi \sum_s \frac{q_s^2 n_s}{T_s}\phi \tag{9.16}$$

Finally integration by parts in space twice gives the final result

$$\int \frac{d^3\mathbf{r}}{V}\nu_H \nabla_\perp^4 \phi \sum_s \frac{q_s^2 n_s}{T_s}\phi = \nu_H \sum_s \frac{q_s^2 n_s}{T_s}\int \frac{d^3\mathbf{r}}{V}\left|\nabla_\perp^2 \phi\right|^2 \geq 0. \tag{9.17}$$

Similar manipulations, also requiring a sum over species but using the parallel component of Ampere's Law, give for the third term

$$\begin{aligned}\sum_s \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v} q_s h_s \nu_H \nabla_\perp^4 \left\langle \frac{v_\parallel A_\parallel}{c}\right\rangle_{\mathbf{R}_s} &= \sum_s \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \langle h_s\rangle_{\mathbf{r}} \nu_H \nabla_\perp^4 \frac{v_\parallel A_\parallel}{c} \\ = \int \frac{d^3\mathbf{r}}{V}\nu_H \nabla_\perp^4 \frac{A_\parallel}{c}\left(\frac{-c}{4pi}\nabla_\perp^2 A_\parallel\right) &= \frac{\nu_H}{4\pi}\int \frac{d^3\mathbf{r}}{V}\left|\nabla_\perp^3 A_\parallel\right|^2 \geq 0. \end{aligned}\tag{9.18}$$

The fourth term requires use of the identity

$$\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v} q_s h_s \nu_H \nabla_\perp^4 \left\langle \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s} = \int \frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v} q_s \langle \mathbf{v}_\perp h_s\rangle_{\mathbf{r}} \cdot \nu_H \nabla_\perp^4 \frac{\mathbf{A}_\perp}{c}. \tag{9.19}$$

Summing over species once more, using the perpendicualr component of Ampere's law, and integrating by parts once, we find

$$\begin{aligned}\int \frac{d^3\mathbf{r}}{V}\left(\sum_s \int d^3\mathbf{v} q_s \langle \mathbf{v}_\perp h_s\rangle_{\mathbf{r}}\right)\cdot \nu_H \nabla_\perp^4 \frac{\mathbf{A}_\perp}{c} &= \int \frac{d^3\mathbf{r}}{V}\left(\frac{-c}{4\pi}\nabla_\perp^2 \mathbf{A}_\perp\right)\cdot \nu_H \nabla_\perp^4 \frac{\mathbf{A}_\perp}{c} \\ = \frac{\nu_H}{4\pi}\int \frac{d^3\mathbf{r}}{V}\left|\nabla_\perp^3 \mathbf{A}_\perp\right|^2 \geq 0. \end{aligned}\tag{9.20}$$

Collecting all of the terms, we obtain the final result

$$\begin{aligned}&\sum_s \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}} h_s \nu_H \nabla_\perp^4 \left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle_{\mathbf{R}_s}\right) \qquad (9.21)\\ = \;\; &\nu_H \sum_s \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\left|\nabla_\perp^2 h_s\right|^2 + \frac{\nu_H}{4\pi}\int \frac{d^3\mathbf{r}}{V}\left[\left(\sum_s \frac{4\pi q_s^2 n_s}{T_s}\right)\left|\nabla_\perp^2 \phi\right|^2 + \left|\nabla_\perp^3 A_\parallel\right|^2 + \left|\nabla_\perp^3 \mathbf{A}_\perp\right|^2\right]\end{aligned}$$

This form is positive definite, but only when summed over species. Since the entropy balance should hold for each species individually, entropy production from each species should be positive-definite. But we see that this operator only becomes positive definite after summing over species; there may exist non-physical transfer of energy between species (mediated by the field). Hence, although this approach may suffice, it may experience severe problems, particularly in determining the heating for each species.

9.3 Hypercollisionality

Steve prefers using a hypercollisionality, basically using our present collision operator but with a wavenumber dependent coefficient. Using the pitch-angle scattering component of the collision operator as an example, the operator has the form

$$\mathcal{C}(f_e) = \nu_e(E)\frac{1}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f_e}{\partial \xi}\right] \tag{9.22}$$

where the coefficient has the form

$$\nu_e(E) = \nu_{ei}\left(\frac{v_{te}}{v}\right)^3\left[Z_{eff} + H_{ee}\left(\frac{v}{v_{te}}\right)\right] \tag{9.23}$$

and

$$\nu_{ei} = \frac{4\pi n_e e^4 \log\lambda}{(2T_e)^{3/2} m_e^{1/2}} \tag{9.24}$$

In this case, the gyrokinetic equation would become

$$\frac{\partial}{\partial t}\left(h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle\right) + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] = \mathcal{C}(h_s) + \mathcal{C}_n(h_s), \tag{9.25}$$

where the hypercollisional term $\mathcal{C}_n$ is given by

$$\mathcal{C}_n(f_e) = \nu_e(E)(k_\perp \rho_i)^n \frac{1}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f_e}{\partial \xi}\right]. \tag{9.26}$$

This form has the advantage that the heating is certainly positive definite. As well, the collisionality coefficient for each species can differ, presumably providing the freedom of adjusting the effective magnetic Prandtl number of the simulation.

9.4 Hyperdiffusive Formulation using G_s

In this section, we investigate the use of a new generalized hyperdiffusive formulation.

9.4.1 Gyrokinetic Equation

We begin with the gyrokinetic equation

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \mathcal{C}(h_s) = \frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s}. \tag{9.27}$$

Now, let us define a new distribution function G_s defined by

$$G_s \equiv h_s - \frac{q_s F_{0s}}{T_s}\langle\chi\rangle \tag{9.28}$$

Writing the gyrokinetic equation in terms of G_s gives

$$\frac{\partial G_s}{\partial t} + v_\parallel \frac{\partial G_s}{\partial z} + v_\parallel \frac{q_s F_{0s}}{T_s}\frac{\partial\langle\chi\rangle}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, G_s\right] - \mathcal{C}(h_s) = 0. \tag{9.29}$$

We define a new linear hyperdiffusive operator,

$$\mathcal{H} = \nu_H \nabla^4 \tag{9.30}$$

and operate on G_s, adding it to the gyrokinetic equation to give

$$\frac{\partial G_s}{\partial t} + v_\parallel \frac{\partial G_s}{\partial z} + v_\parallel \frac{q_s F_{0s}}{T_s}\frac{\partial\langle\chi\rangle}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, G_s\right] - \langle\mathcal{C}(h_s)\rangle_{\mathbf{R}_s} + \mathcal{H}(G_s) = 0. \tag{9.31}$$

9.4.2 Deriving the Entropy-Balance Equation

Now we multiply (9.31) by $T_s G_s / F_{0s}$ and integrate over all space and velocity. Then second and fourth terms contribute nothing upon integration over all space, leaving us with

$$\begin{aligned}&\frac{d}{dt}\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{2F_{0s}}G_s^2 + \int\frac{d^3\mathbf{R}}{V}\int d^3\mathbf{v}v_\parallel G_s q_s\frac{\partial\left\langle\chi\right\rangle_{\mathbf{R}_s}}{\partial z} - \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\left\langle G_s\mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s}\\&+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}G_s\mathcal{H}(G_s) = 0\end{aligned} \tag{9.32}$$

To eliminate the problematic second term in this equation, we now multiply (9.31) by $q_s\left\langle\chi\right\rangle_{\mathbf{R}_s}$ and integrate over all space and velocity. The third and fourth terms integrate over space to zero, and we integrate by parts in space on the second term to obtain

$$\begin{aligned}&\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\left\langle\chi\right\rangle_{\mathbf{R}_s}\frac{\partial G_s}{\partial t} - \int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\left\langle q_s\chi\mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s}\\&+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}q_s\left\langle\chi\right\rangle_{\mathbf{R}_s}\mathcal{H}(G_s) = 0\end{aligned} \tag{9.33}$$

Adding (9.32) and (9.33) results in

$$\frac{d}{dt}\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{2F_{0s}}G_s^2+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}q_s\left\langle\chi\right\rangle_{\mathbf{R}_s}\frac{\partial G_s}{\partial t}-\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\left\langle\left(G_s+\frac{q_sF_{0s}}{T_s}\chi\right)\mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s}$$
$$+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_s}{F_{0s}}\left(G_s+\frac{q_sF_{0s}}{T_s}\left\langle\chi\right\rangle_{\mathbf{R}_s}\right)\mathcal{H}(G_s)=0 \tag{9.34}$$

9.5 A Few Additional Remarks on Hyperviscosity

1. If the reduced MHD hyperviscosity limits are wrong, does this mean the hyperviscosity as usually framed in reduced MHD is a bad model?
2. Does hyperdiffusivity in the $\nu_H\nabla^4G_s$ have zero mean even if the $\nu_H\int h_s\nabla^4\langle\chi\rangle$ piece goes less than zero?

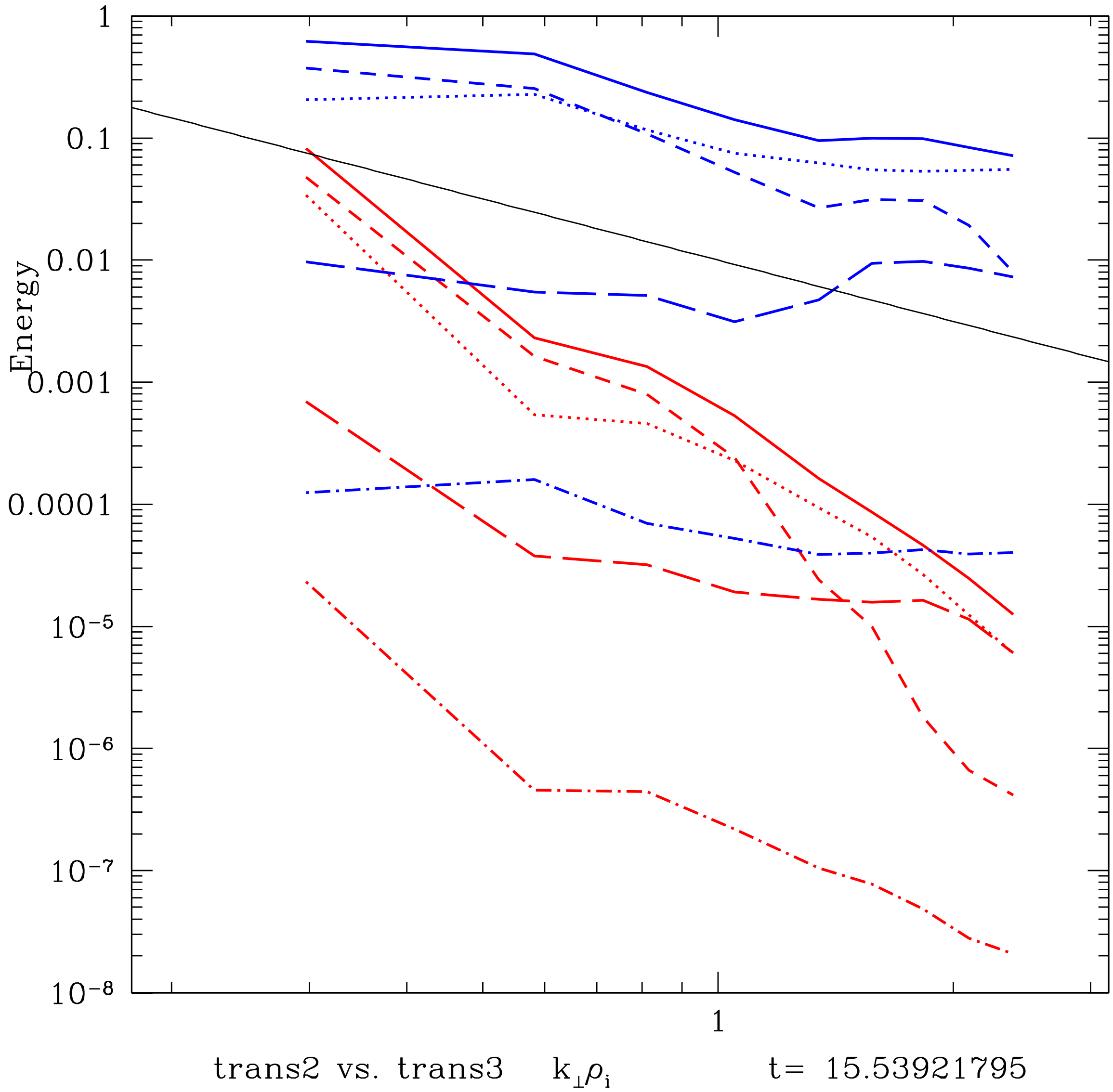

Figure 9.1: Plot of two energy spectra vs. wavenumber for two runs with $\beta_i = 100$, `trans2` (blue, without hyperdiffusivity) and `trans3` (red, with hyperdiffusivity). Different components of the fluctuating energy δW are ploteed: total energy (solid), δf_i (dotted), $E_{A_\parallel}$ (short-dashed), $E_{\delta B_\parallel}$ (long-dashed), and δf_e (dash-dotted). The thin black line denotes $(k_\perp \rho_i)^{-5/3}$. At the highest wavenumbers in the box, there is still significant energy in the cascade for trans2, even for a run in which the ion damping at $k_\perp \rho_i \sim 1$.

Chapter 10

Collisions in Gyrokinetics

10.1 Collisional Theory

From Schekochihin et al. (2006), the gyroaveraged form of the collision operator is given by

$$\langle C_{ss}(h_s)\rangle_s = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s}\nu_D^{ss}(v)\left\{\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi} - \frac{v^2(1+\xi^2)}{4v_{t_s}^2}k_\perp^2\rho_s^2 h_{\mathbf{k}s} + 2\frac{v_\perp J_1(\alpha_s)U_\perp(h_{\mathbf{k}s}) + v_\parallel J_0(\alpha_s)U_\parallel(h_{\mathbf{k}s})}{v_{t_s}^2}f_{0s}\right\} \tag{10.1}$$

where

$$\alpha_s = \frac{k_\perp v_\perp}{\Omega_s}. \tag{10.2}$$

This form arises from taking $\left.\frac{\partial}{\partial\xi}\right|_{\mathbf{R}_s}$ instead of $\left.\frac{\partial}{\partial\xi}\right|_{\mathbf{r}}$.

`Question: Is there a non-zero boundary term on the second term? I don't think so.`

10.1.1 Properties of the Collision Operator

Conservation of number, momentum, energy.

10.2 Conservative Krook Collision Operator

We begin with the Vlasov Equation for an electromagnetic plasma species s

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = 0. \tag{10.3}$$

Adding collisions to the right-hand side turn this into the Fokker-Planck equation

$$\frac{df_s}{dt} = \mathcal{C}(f_s). \tag{10.4}$$

This collision operator $\mathcal{C}$ should satisfy three constraints:

1. Conservation of number by species
$$\int d^3\mathbf{v}\mathcal{C}(f_s) = 0 \tag{10.5}$$

2. Conservation of total momentum
$$\sum_s \int d^3\mathbf{v} m_s \mathbf{v}\mathcal{C}(f_s) = 0 \tag{10.6}$$

3. Conservation of total energy
$$\sum_s \int d^3\mathbf{v}\frac{m_s v^2}{2}\mathcal{C}(f_s) = 0 \tag{10.7}$$

Can we construct a simple operator that satisfies these properties?

A general Maxwellian fluid can be described uniquely by three parameters: the number density n_s, mean flow velocity $\mathbf{U}_s$, and energy T_s. The form of Maxellian for a species s is given by

$$F_{Ms} = \frac{n_s m_s}{(2\pi T_s)^{3/2}}\exp\left[-\frac{m_s(\mathbf{v}-\mathbf{U}_s)^2}{2T_s}\right] \tag{10.8}$$

We see that these three parameters are related simply to the velocity moments of the distribution function

$$n_s = \int d^3\mathbf{v} f_{Ms}, \tag{10.9}$$

$$n_s\mathbf{U}_s = \int d^3\mathbf{v}\mathbf{v} f_{Ms}, \tag{10.10}$$

and

$$n_s\left(\frac{3T_s}{m_s} + U_s^2\right) = \int d^3\mathbf{v} v^2 f_{Ms}. \tag{10.11}$$

10.2.1 Construction of the Krook Collision Operator

We propose a simple Krook collision operator that, by construction, satisfies the conservation of number, momentum, and energy. We propose to use

$$\mathcal{C}(f_s) = -\sum_r \nu_{sr}(f_s - \overline{F}_r) \tag{10.12}$$

where we have defined

$$\overline{F}_r = \frac{\overline{n}_r m_r^{3/2}}{(2\pi\overline{T}_r)^{3/2}} \exp\left[-\frac{m_r(\mathbf{v} - \overline{\mathbf{U}}_r)^2}{2\overline{T}_r}\right] \tag{10.13}$$

such that

$$\overline{n}_r = \int d^3\mathbf{v} f_r, \tag{10.14}$$

$$\overline{n}_r\overline{\mathbf{U}}_r = \int d^3\mathbf{v}\mathbf{v} f_r, \tag{10.15}$$

and

$$\overline{n}_r\left(\frac{3\overline{T}_r}{2} + \frac{m_r\overline{U}_r^2}{2}\right) = \int d^3\mathbf{v}\frac{m_r v^2}{2} f_r. \tag{10.16}$$

Let us now check that this operator satisfies the conservation properties (10.5)–(10.7).

Beginning with the conservation of number (10.5), we find

$$\sum_s \int d^3\mathbf{v}\mathcal{C}(f_s) = -\sum_s\sum_r \nu_{sr}\left(\int d^3\mathbf{v} f_s - \int d^3\mathbf{v}\overline{F}_r\right) \tag{10.17}$$

The first integral is simply $\int d^3\mathbf{v} f_s = \overline{n}_s$. Noting that

$$\frac{1}{\sqrt{\pi}}\int \frac{dv_i}{v_{tr}} e^{-(v_i - \overline{U}_{ri})^2/v_{tr}^2} = 1 \tag{10.18}$$

the second integral simplifies to $\int d^3\mathbf{v}\overline{F}_r = \overline{n}_r$. Thus, we obtain

$$\sum_s \int d^3\mathbf{v}\mathcal{C}(f_s) = -\sum_s\sum_r \nu_{sr}\left(\overline{n}_s - \overline{n}_r\right). \tag{10.19}$$

Specializing to a plasma of ions i and electrons e, we find that the sum

$$-\sum_s\sum_r \nu_{sr}\left(\overline{n}_s - \overline{n}_r\right) = (\nu_{ei} - \nu_{ie})(\overline{n}_i - \overline{n}_e) = 0, \tag{10.20}$$

since it must true that $\nu_{ei} = \nu_{ie}$.

Next, we check the conservation of momentum (10.6),

$$\sum_s \int d^3\mathbf{v} m_s\mathbf{v}\mathcal{C}(f_s) = -\sum_s\sum_r \nu_{sr}\left(\int d^3\mathbf{v} m_s\mathbf{v} f_s - \int d^3\mathbf{v} m_s\mathbf{v}\overline{F}_r\right) \tag{10.21}$$

Again the first integral is just $\int d^3\mathbf{v} m_s \mathbf{v} f_s = m_s \overline{n}_s \overline{\mathbf{U}}_s$. Using the integral,

$$\frac{1}{\sqrt{\pi}} \int v_i \frac{dv_i}{v_{tr}} e^{-(v_i - \overline{U}_{ri})^2/v_{tr}^2} = \overline{U}_{ri}, \tag{10.22}$$

the second integral gives $\int d^3\mathbf{v} m_s \mathbf{v} \overline{F}_r = m_s \overline{n}_r \overline{U}_r$. Our result is then

$$\sum_s \int d^3\mathbf{v} m_s \mathbf{v} \mathcal{C}(f_s) = -\sum_s \sum_r \nu_{sr} \left(m_s \overline{n}_s \overline{\mathbf{U}}_s - m_s \overline{n}_r \overline{U}_r \right). \tag{10.23}$$

Taking an ion-electron plasma and using the fact that $\nu_{ei} = \nu_{ie}$, the sum becomes

$$-\sum_s \sum_r \nu_{sr} \left(m_s \overline{n}_s \overline{\mathbf{U}}_s - m_s \overline{n}_r \overline{U}_r \right) = \nu_{ie} [(m_i - m_e) \overline{n}_e \overline{U}_e + (m_e - m_i) \overline{n}_i \overline{U}_i]. \tag{10.24}$$

Unless the two species have the same particle mass, the collision operator (10.25) does not conserve total momentum over interspecies collisions. In fact, this form of the operator conserves velocity, not momentum. We need to adjust the form of the operator to find a form that does conserve total momentum.

The literature suggests a slightly different form

Hence, we find that, in order for this operator to be conservative, the following properties must be true:

1. $\nu_{sr} = \nu_{rs}$

10.2.2 BGK Collision Operator

It is stated in the literature (for example [Snyder et al., 1997]) that mass, momentum and density are conserved by a simple BGK operator of the form

$$\mathcal{C}(f_s) = -\sum_r \nu_{sr}(f_s - \overline{F}_{sr}) \tag{10.25}$$

where we have defined

$$\overline{F}_{sr} = \frac{\overline{n}_s m_s^{3/2}}{(2\pi\overline{T}_s)^{3/2}} \exp\left[-\frac{m_s(\mathbf{v}-\overline{\mathbf{U}}_r)^2}{2\overline{T}_s}\right], \tag{10.26}$$

$$\overline{n}_s = \int d^3\mathbf{v} f_s, \tag{10.27}$$

$$\overline{n}_r\overline{\mathbf{U}}_r = \int d^3\mathbf{v}\mathbf{v} f_r, \tag{10.28}$$

and

$$\overline{n}_s\left(\frac{3\overline{T}_s}{2} + \frac{m_s\overline{U}_s^2}{2}\right) = \int d^3\mathbf{v}\frac{m_s v^2}{2} f_s. \tag{10.29}$$

I do not see how this form conserves total momentum.

We can go ahead and integrate the conservation of momentum equation (10.6) over velocity space,

$$\sum_s \int d^3\mathbf{v} m_s \mathbf{v}\mathcal{C}(f_s) = -\sum_s\sum_r \nu_{sr}\left(\int d^3\mathbf{v} m_s\mathbf{v} f_s - \int d^3\mathbf{v} m_s\mathbf{v}\overline{F}_{sr}\right) \tag{10.30}$$

The first integral is just $\int d^3\mathbf{v} m_s\mathbf{v} f_s = m_s\overline{n}_s\overline{\mathbf{U}}_s$. Using the integral

$$\frac{1}{\sqrt{\pi}}\int v_i \frac{dv_i}{v_{ts}} e^{-(v_i-\overline{U}_{ri})^2/v_{ts}^2} = \overline{U}_{ri}, \tag{10.31}$$

the second integral gives $\int d^3\mathbf{v} m_s\mathbf{v}\overline{F}_{sr} = m_s\overline{n}_s\overline{U}_r$. Our result is then

$$\sum_s \int d^3\mathbf{v} m_s\mathbf{v}\mathcal{C}(f_s) = -\sum_s\sum_r \nu_{sr}\left(m_s\overline{n}_s\overline{\mathbf{U}}_s - m_s\overline{n}_s\overline{U}_r\right). \tag{10.32}$$

Specializing to an ion-electron plasma and using the property that $n_e\nu_{ei} = n_i\nu_{ie}$, this sum simplifies to

$$-\sum_s\sum_r \nu_{sr}\left(m_s\overline{n}_s\overline{\mathbf{U}}_s - m_s\overline{n}_s\overline{U}_r\right) = \overline{n}_e\nu_{ei}(m_i - m_e)(\overline{U}_e - \overline{U}_i) \tag{10.33}$$

This is the expression for the total change of momentum, and in general it is not zero. Perhaps I am missing some approximation whereby this sum is almost always nearly zero for some conditions.

In the original paper [Gross and Krook, 1956], the form of the operator is slightly different, using instead

$$\overline{F}_{sr} = \frac{\overline{n}_s m_s^{3/2}}{(2\pi\overline{T}_{rs})^{3/2}} \exp\left[-\frac{m_s(\mathbf{v}-\overline{\mathbf{U}}_{rs})^2}{2\overline{T}_{rs}}\right], \tag{10.34}$$

where the flow velocity $\overline{\mathbf{U}}_{rs}$ and temperature T_{rs} are not yet specified for interspecies collisions. The final form of the total momentum conservation, if this operator is used, becomes

$$= m_i(\overline{U}_{ei} - \overline{U}_i) + m_e(\overline{U}_{ie} - \overline{U}_e) \tag{10.35}$$

where the quantities $\overline{U}_{ei}$ and $\overline{U}_{ie}$ are chosen such that this expression equals zero. I do not know if there is a more simple way to implement this to achieve momentum conservation between species.

10.2.3 Like-Particle Only Krook Collision Operator

Let us consider the gyrokinetic equation using a Krook collision operator and including only like-particles collisions. Thus, the collision operator has the form

$$\mathcal{C}(f_s) = -\nu_{ss}(f_s - \overline{F}_s) \tag{10.36}$$

where we have defined

$$\overline{F}_s = \frac{\overline{n}_s m_s^{3/2}}{(2\pi T_s)^{3/2}} \exp\left[-\frac{m_s(\mathbf{v} - \overline{\mathbf{U}}_s)^2}{2T_s}\right], \tag{10.37}$$

$$\overline{n}_s = \int d^3\mathbf{v} f_s, \tag{10.38}$$

$$\overline{n}_s \overline{\mathbf{U}}_s = \int d^3\mathbf{v}\mathbf{v} f_s, \tag{10.39}$$

The form presented here conserves density and momentum for like-particle collisions.

Linearized, Gyroaveraged Krook Collision Operator

Given the form of the distribution function

$$f_s = F_{0s}\left(1 - \frac{q_s\phi}{T_s}\right) + h_s, \tag{10.40}$$

we substitute into (10.38) and (10.39) to find

$$\overline{n}_s = n_{0s}\left(1 - \frac{q_s\phi}{T_s}\right) + \int d^3\mathbf{v} h_s \tag{10.41}$$

and

$$\overline{\mathbf{U}}_s = \frac{1}{\overline{n}_s}\int d^3\mathbf{v}\mathbf{v} h_s. \tag{10.42}$$

[NOTE: Should these integrals over velocity space of h_s be done at constant position $\mathbf{r}$? In this case, I may need to add another Bessel function, but maybe I've already done this right.]

We can now linearize the collision operator by expanding the exponential in $\overline{F}_s$

$$\exp\left[-\frac{(\mathbf{v} - \overline{\mathbf{U}}_s)^2}{v_{ts}^2}\right] \simeq \exp\left[-\frac{v^2}{v_{ts}^2}\right]\left(1 + \frac{2\mathbf{v}\cdot\overline{\mathbf{U}}_s}{v_{ts}^2}\right) \tag{10.43}$$

where we have used the definition of thermal velocity $v_{ts}^2 = 2T_s/m_s$. Our linearization has used the approximations

$$\int d^3\mathbf{v} h_s \ll n_{0s} \tag{10.44}$$

and

$$\int d^3\mathbf{v}\mathbf{v} h_s \ll \mathbf{v}. \tag{10.45}$$

The first inequality is the same as the gyrokinetic approximation; the second, although not true over the entire range of $\mathbf{v}$, will be true for the case that values of $v \sim v_{ts}$. Defining the quantities,

$$\delta n_s = \int d^3\mathbf{v} h_s \tag{10.46}$$

and

$$\delta \mathbf{U}_s = \frac{1}{n_{0s}} \int d^3\mathbf{v}\mathbf{v} h_s, \tag{10.47}$$

the linearized collision operator reduces to

$$\mathcal{C}(f_s) = -\nu_{ss} \left(h_s - \frac{2\mathbf{v} \cdot \delta\mathbf{U}_s}{v_{ts}^2} F_{0s} - \frac{\delta n_s}{n_{0s}} F_{0s} \right). \tag{10.48}$$

Next we need to gyroaverage the linearized collision operator. Writing the gyrokinetic distribution function as a Fourier series,

$$h_s(\mathbf{R}_s, v, v_\perp) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} h_{\mathbf{k}s}(v, v_\perp) \tag{10.49}$$

The integral over velocity space at constant position $\mathbf{r}$ of h_s is given by

$$\int d^3\mathbf{v}\Big|_{\mathbf{r}} h_s = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \int 2\pi v_\perp dv_\perp dv_\parallel J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v, v_\perp). \tag{10.50}$$

Now taking the gyroaverage of this quantity gives

$$\left\langle \int d^3\mathbf{v}\Big|_{\mathbf{r}} h_s \right\rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \int 2\pi v'_\perp dv'_\perp dv'_\parallel J_0\left(\frac{k_\perp v'_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v', v'_\perp). \tag{10.51}$$

Likewise,

$$\int d^3\mathbf{v}\Big|_{\mathbf{r}} \mathbf{v} h_s = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \left[\int 2\pi v_\perp dv_\perp v_\parallel dv_\parallel J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v, v_\perp)\hat{\mathbf{z}} + i \int 2\pi v_\perp^2 dv_\perp dv_\parallel J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v, v_\perp)\hat{\mathbf{x}} \right] \tag{10.52}$$

and

$$\begin{aligned} \left\langle \mathbf{v} \cdot \int d^3\mathbf{v}\Big|_{\mathbf{r}} \mathbf{v} h_s \right\rangle_{\mathbf{R}_s} &= \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_s}\right) v_\parallel \int 2\pi v'_\perp dv'_\perp v'_\parallel dv'_\parallel J_0\left(\frac{k_\perp v'_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v', v'_\perp) \right. \\ &- \left. J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right) v_\perp \int 2\pi v'^2_\perp dv'_\perp dv'_\parallel J_1\left(\frac{k_\perp v'_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v', v'_\perp) \right] \end{aligned} \tag{10.53}$$

where we have taken, without loss of generality, $\mathbf{k} = k_\parallel \hat{\mathbf{z}} + k_\perp \hat{\mathbf{x}}$.

We can define the quantities

$$\delta n_{\mathbf{k}s} = \int 2\pi v'_\perp dv'_\perp dv'_\parallel J_0\left(\frac{k_\perp v'_\perp}{\Omega_s}\right) h_{\mathbf{k}s}(v', v'_\perp), \tag{10.54}$$

$$\delta U_{\|\mathbf{k}s} = \frac{1}{n_{0s}} \int 2\pi v'_\perp dv'_\perp v'_\| dv'_\| J_0 \left(\frac{k_\perp v'_\perp}{\Omega_s} \right) h_{\mathbf{k}s}(v', v'_\perp), \tag{10.55}$$

$$\delta U_{\perp\mathbf{k}s} = \frac{1}{n_{0s}} \int 2\pi v'^2_\perp dv'_\perp dv'_\| J_1 \left(\frac{k_\perp v'_\perp}{\Omega_s} \right) h_{\mathbf{k}s}(v', v'_\perp), \tag{10.56}$$

and write the linearized, gyroaveraged Krook collision operator as

$$\langle \mathcal{C}(f_s) \rangle_{\mathbf{R}_s} = -\nu_{ss} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \left(h_{\mathbf{k}s} - \frac{2v_\| J_0 \delta U_{\|\mathbf{k}s}}{v_{ts}^2} F_{0s} + \frac{2v_\perp J_1 \delta U_{\perp\mathbf{k}s}}{v_{ts}^2} F_{0s} - \frac{J_0 \delta n_{\mathbf{k}s}}{n_{0s}} F_{0s} \right). \tag{10.57}$$

Solving for the Distribution Function

Fourier transforming the gyrokinetic equation using $\exp[i(\mathbf{k}\cdot\mathbf{R}_s - \omega t)]$, we can solve for the distribution function

$$h_{\mathbf{k}s} = \frac{\frac{q_s F_{0s}}{T_s} \omega \langle \chi \rangle_{\mathbf{k}\mathbf{R}_s} + i\nu_{ss}\mathcal{C}^*}{\omega + i\nu_{ss} - k_\| v_\|} \tag{10.58}$$

where we have defined

$$\mathcal{C}^*(v, v_\perp, h_{\mathbf{k}s}) = \frac{2v_\| \delta U_{\|\mathbf{k}s} J_0 - 2v_\perp \delta U_{\perp\mathbf{k}s} J_1}{v_{ts}^2} F_{0s} + \frac{\delta n_{\mathbf{k}s} J_0}{n_{0s}} F_{0s} \tag{10.59}$$

Note that we can write this alternatively as

$$\mathcal{C}^*(v, v_\perp, h_{\mathbf{k}s}) = \ 2J_0 \frac{v_\|}{v_{ts}} \frac{\delta U_{\|\mathbf{k}s}}{v_{ts}} F_{0s} - k_\perp \rho_s \frac{J_1}{\gamma_s} \frac{2v_\perp^2}{v_{ts}^2} \frac{\delta U_{\perp\mathbf{k}s}}{v_{ts}} F_{0s} + \frac{\delta n_{\mathbf{k}s} J_0}{n_{0s}} F_{0s} \tag{10.60}$$

where $\gamma_s = k_\perp v_\perp / \Omega_s$. In solving for $h_{\mathbf{k}s}$, we treat the local perturbations of density $\delta n_{\mathbf{k}s}$, parallel fluid velocity $\delta U_{\|\mathbf{k}s}$, and perpendicular fluid velocity $\delta U_{\perp\mathbf{k}s}$ as independet of $h_{\mathbf{k}s}$; this is consistent with the approximations (10.44) and (10.45) used in the linearization.

Using the gyroaveraged gyrokinetic potential

$$\langle \chi \rangle_{\mathbf{k}\mathbf{R}_s} = J_0(\frac{k_\perp v_\perp}{\Omega})(\phi_{\mathbf{k}} - \frac{v_\| A_{\|\mathbf{k}}}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} \frac{2v_\perp^2}{v_{ts}^2} \left(\frac{T_s}{q_s} \frac{\delta B_{\|\mathbf{k}}}{B_0} \right), \tag{10.61}$$

we find the expression for the distribution function is

$$h_{\mathbf{k}s} = \frac{q_s F_{0s}}{T_s} \left\{ J_0 \frac{\omega A_{\|\mathbf{k}}}{k_\| c} + \left(\frac{\omega}{\omega + i\nu_{ss} - k_\| v_\|} \right) \left[J_0 \left(\phi_{\mathbf{k}} - \frac{\omega A_{\|\mathbf{k}}}{k_\| c} \right) + \frac{J_1}{\gamma_s} \frac{2v_\perp^2}{v_{ts}^2} \left(\frac{T_s}{q_s} \frac{\delta B_{\|\mathbf{k}}}{B_0} \right) \right] \right. \tag{10.62}$$

$$\left. - \left(\frac{i\nu_{ss}}{\omega + i\nu_{ss} - k_\| v_\|} \right) J_0 \frac{\omega A_{\|\mathbf{k}}}{k_\| c} \right\} + \frac{i\nu_{ss}\mathcal{C}^*}{\omega + i\nu_{ss} - k_\| v_\|} \tag{10.63}$$

where the argument of the Bessel functions is $\gamma_s = \frac{k_\perp v_\perp}{\Omega_s}$.

Solving for the Perturbed Quantities

To solve for the local perturbations of density $\delta n_{\mathbf{k}s}$, parallel fluid velocity $\delta U_{\|\mathbf{k}s}$, and perpendicular fluid velocity $\delta U_{\perp\mathbf{k}s}$, we can drop the collisional term $\mathcal{C}^*$ in the solution for the distribution function and perform the integrations. Establishing the following definitions,

$$\xi_s = \frac{\omega}{k_\| v_{th_s}}, \tag{10.64}$$

$$\zeta_s = \frac{i\nu_{ss}}{k_\| v_{th_s}}, \tag{10.65}$$

$$\psi_s = \frac{\omega + i\nu_{ss}}{k_\| v_{th_s}}, \tag{10.66}$$

and

$$Z_s = Z(\psi_s) \tag{10.67}$$

we obtain the following relations,

$$\frac{\delta n_{\mathbf{k}s}}{n_{0s}} = \frac{q_s}{T_s}\left[\Gamma_{0s}(1+\zeta_s Z_s)\frac{\omega A_{\|\mathbf{k}}}{k_\| c} - \Gamma_{0s}\xi_s Z_s\left(\phi_{\mathbf{k}} - \frac{\omega A_{\|\mathbf{k}}}{k_\| c}\right) - \Gamma_{1s}\xi_s Z_s \frac{T_s}{q_s}\frac{\delta B_{\|\mathbf{k}}}{B_0}\right], \tag{10.68}$$

$$\frac{\delta U_{\|\mathbf{k}s}}{v_{ts}} = \frac{q_s}{T_s}\left[\Gamma_{0s}\zeta_s(1+\psi_s Z_s)\frac{\omega A_{\|\mathbf{k}}}{k_\| c} - \Gamma_{0s}\xi_s(1+\psi_s Z_s)\left(\phi_{\mathbf{k}} - \frac{\omega A_{\|\mathbf{k}}}{k_\| c}\right) - \Gamma_{1s}\xi_s(1+\psi_s Z_s)\frac{T_s}{q_s}\frac{\delta B_{\|\mathbf{k}}}{B_0}\right], \tag{10.69}$$

and

$$\frac{\delta U_{\perp\mathbf{k}s}}{v_{ts}} = \frac{q_s}{T_s}\frac{k_\perp \rho_s}{2}\left[\Gamma_{1s}(1+\zeta_s Z_s)\frac{\omega A_{\|\mathbf{k}}}{k_\| c} - \Gamma_{1s}\xi_s Z_s\left(\phi_{\mathbf{k}} - \frac{\omega A_{\|\mathbf{k}}}{k_\| c}\right) - \Gamma_{2s}\xi_s Z_s\frac{T_s}{q_s}\frac{\delta B_{\|\mathbf{k}}}{B_0}\right]. \tag{10.70}$$

Final Dispersion Relation

The form of the dispersion relation including the Krook collision operator is analagous to the collisionless dispersion relation

$$\begin{pmatrix} A & B & C \\ A-B & \alpha & C+E \\ C & -E & D \end{pmatrix}\begin{pmatrix} \hat{\phi} - \frac{\omega\hat{A}_\|}{k_\| c} \\ \frac{\omega\hat{A}_\|}{k_\| c} \\ \frac{T_i}{q_i}\frac{\delta\hat{B}_\|}{B} \end{pmatrix} = 0 \tag{10.71}$$

where the definitions of the symbols above are:

$$A = \sum_s \frac{T_i}{T_s}\left[(1+\Gamma_{0s}\xi_s Z_s) - (\Gamma_{0s}^2 - \Gamma_{1s}^2\alpha_s)\zeta_s\xi_s Z_s^2 - 2\Gamma_{0s}^2\zeta_s\xi_s(1+\psi_s Z_s)^2\right] \tag{10.72}$$

$$B = \sum_s \frac{T_i}{T_s}\left[(1-\Gamma_{0s} - \Gamma_{0s}\zeta_s Z_s) + (\Gamma_{0s}^2 - \Gamma_{1s}^2\alpha_s)\zeta_s Z_s(1+\zeta_s Z_s) + 2\Gamma_{0s}^2\zeta_s^2(1+\psi_s Z_s)^2\right] \tag{10.73}$$

$$C = \sum_s \frac{q_s}{q_i}\left[\Gamma_{1s}\xi_s Z(\xi_s) - (\Gamma_{0s}\Gamma_{1s} - \Gamma_{1s}\Gamma_{2s}\alpha_s)\zeta_s\xi_s Z_s^2 - 2\Gamma_{0s}\Gamma_{1s}\zeta_s\xi_s(1+\psi_s Z_s)^2\right] \tag{10.74}$$

$$D = -\frac{2}{\beta_i} + \sum_s \frac{T_s}{T_i} \left[\Gamma_{2s} \xi_s Z(\xi_s) - (\Gamma_{1s}^2 - \Gamma_{2s}^2 \alpha_s) \zeta_s \xi_s Z_s^2 - 2\Gamma_{1s}^2 \zeta_s \xi_s (1 + \psi_s Z_s)^2 \right] \tag{10.75}$$

$$E = \sum_s \frac{q_s}{q_i} \left[\Gamma_{1s}(1 + \zeta_s Z_s) - (\Gamma_{0s}\Gamma_{1s} - \Gamma_{1s}\Gamma_{2s}\alpha_s)\zeta_s Z_s (1 + \zeta_s Z_s) - 2\Gamma_{0s}\Gamma_{1s}\zeta_s^2 (1 + \psi_s Z_s)^2 \right] \tag{10.76}$$

$$\alpha = \alpha_i/\overline{\omega}^2 - \sum_s \frac{T_i}{T_s} \left[\Gamma_{0s} \frac{\zeta_s}{\xi_s}(1 + \psi_s Z_s) - (\Gamma_{0s}^2 - \Gamma_{1s}^2 \alpha_s)\frac{\zeta_s}{\xi_s}(1 + \zeta_s Z_s)(1 + \psi_s Z_s) - 2\Gamma_{0s}^2 \frac{\zeta_s}{\xi_s} \zeta_s \psi_s (1 + \psi_s Z_s)^2 \right] \tag{10.77}$$

$$\alpha_s = \frac{k_\perp^2 \rho_s^2}{2} \tag{10.78}$$

Solving for this dispersion relation, we get

$$\left(A\alpha - AB + B^2\right)\left(-AD + C^2\right) - \left(AE + BC\right)^2 = 0 \tag{10.79}$$

10.3 GS2's Implementation of Collisions

Let us be specific about the implementation of collisions and hypercollisionality in GS2. This is well described in a note by Greg Hammett dated June 26, 2003; this note also includes insightful discussion that is not included here. I will summarize his results here and update them to include the classical diffusion operator that arises from taking pitch angle derivative with at constant guiding center $\mathbf{R}$ rather than constant position $\mathbf{r}$.

The electron collision operator includes both electron-electron and electron-ion collisions with the equilibrium Maxwellian distribution,

$$\langle \mathcal{C}_e(h_e) \rangle_{\mathbf{R}_e} = \langle \mathcal{C}_{ee}(h_e, F_{0e}) + \mathcal{C}_{ei}(h_e, F_{0i}) \rangle_{\mathbf{R}_e} \tag{10.80}$$

while the ion collision operator includes only ion-ion collisions with the equilibrium Maxwellian distribution,

$$\langle \mathcal{C}_i(h_i) \rangle_{\mathbf{R}_i} = \langle \mathcal{C}_{ii}(h_i, F_{0i}) \rangle_{\mathbf{R}_i}. \tag{10.81}$$

`Question: What happens to the adiabatic piece of the distribution function in the collision operator?`

10.3.1 Electron Collision Operator

The electron collision operator is given by

$$\langle \mathcal{C}_e(h_e) \rangle_{\mathbf{R}_e} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_e} \nu_e(v) \frac{1}{2} \left\{ \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}e}}{\partial \xi} \right] - \frac{v^2}{v_{te}^2}(1+\xi^2) \frac{k_\perp^2 \rho_e^2}{2} h_{\mathbf{k}e} \right\} \tag{10.82}$$

where the coefficient, dependent only on the magnitude of velocity, is given by

$$\nu_e(v) = \nu_{ei} \left(\frac{v_{te}}{v} \right)^3 \left[Z_i^2 + H_{ss}\left(\frac{v}{v_{te}} \right) \right] \tag{10.83}$$

(NOTE here that in the GS2code, the factor of 1/2 in (19.3) is actually absorbed into the constant $\nu_e(v)$ and so actually appears only in (19.4)) with the like-particle collision coefficient is given by

$$H_{ss}(x) = \left(1 - \frac{1}{2x^2} \right) \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} + \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \tag{10.84}$$

and with

$$\nu_{ei} = \frac{4\pi e^4 n_e \lambda}{m_e^{1/2} (2T_e)^{3/2}}. \tag{10.85}$$

Here λ denotes the Coulomb logarithm $\lambda = \ln(\Lambda_{ee}) \simeq \ln(\Lambda_{ei})$, the pitch angle coordinate is defined by $\xi \equiv v_\parallel / v$, and the thermal velocity is defined by $v_{te}^2 = 2T_e/m_e$. The first term in (19.3) is the standard pitch-angle collision operator and the second term is a classical-diffusion correction due to the fact that the derivatives $\partial/\partial\xi$ are evaluated at constant guiding center $\mathbf{R}_s$ rather than at constant position $\mathbf{r}$. At the moment the second term is included when `cfac=1.0` in the collisions namelist; this is the default behavior (a change from earlier versions of the code). This correction term is motivated and derived in Schekochihin et al. (2006).

Figure 19.1 plots the variation of the functions $H_{ss}(v/v_{ts})$ and $\nu_e(v/v_{te})$ with energy, or magnitude of velocity.

10.3.2 Ion Collision Operator

Analogous to the electron collision operator, but neglecting ion-electron collisions, the ion collision operator is given by

$$\langle \mathcal{C}_i(h_i) \rangle_{\mathbf{R}_i} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_i} \nu_i(v) \frac{1}{2} \left\{ \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}i}}{\partial \xi} \right] - \frac{v^2}{v_{ti}^2}(1+\xi^2) \frac{k_\perp^2 \rho_i^2}{2} h_{\mathbf{k}i} \right\} \tag{10.86}$$

where

$$\nu_i(v) = \nu_{ii} \left(\frac{v_{ti}}{v} \right)^3 H_{ss} \left(\frac{v}{v_{ti}} \right) \tag{10.87}$$

and

$$\nu_{ii} = \frac{4\pi Z_i^4 e^4 n_i \lambda}{m_i^{1/2} (2T_i)^{3/2}}. \tag{10.88}$$

Figure 19.1 plots the variation of the function $\nu_i(v/v_{ti})$ with energy, or magnitude of velocity.

10.3.3 Collisional Coefficients

In the `GS2` input file, the collisional coefficients $\texttt{vnewk}_s$ are used as the values of ν_{ei} and ν_{ii} in the electron and ion collision operators above. If we denote these user supplied values as $\texttt{vnewk}_e = \hat{\nu}_{ce}$ and $\texttt{vnewk}_i = \hat{\nu}_{ci}$, then the normalization is given by

$$\hat{\nu}_{cs} = \frac{\nu_{cs} a_0}{v_{t0}}. \tag{10.89}$$

10.3.4 Collisional Heating

To estimate the collisional heating, we take the gyrokinetic equation,

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left[\langle \chi \rangle_{\mathbf{R}_s}, h_s \right] - \langle \mathcal{C}(h_s) \rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} F_{0s}, \tag{10.90}$$

multiply by $T_s h_s / F_{0s}$ and integrate over all space and velocity to give the entropy-balance equation. (In fact, to be more precise, in complex space we must actually take $\int_{\mathbf{R}} \int_{\mathbf{v}} T_s h_s^* \mathrm{GK}/F_{0s} + \int_{\mathbf{R}} \int_{\mathbf{v}} T_s h_s \mathrm{GK}^*/F_{0s}$, where GK denotes the gyrokinetic equation; but we will be somewhat loose with notation here.) The second and third terms on the left-hand side give nothing when integrated over all space for periodic boundary conditions, leaving the result

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} h_s - \frac{d}{dt} \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s \mathcal{C}(h_s) \rangle_{\mathbf{R}_s} = 0 \tag{10.91}$$

Let us now take a closer look at the heating from the collisional term. Writing the gyrokinetic distribution function as a Fourier series

$$h_s(\mathbf{R}_s, \mathbf{v}, t) = \sum_{\mathbf{k}} h_{\mathbf{k}s}(\mathbf{k}, \mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{10.92}$$

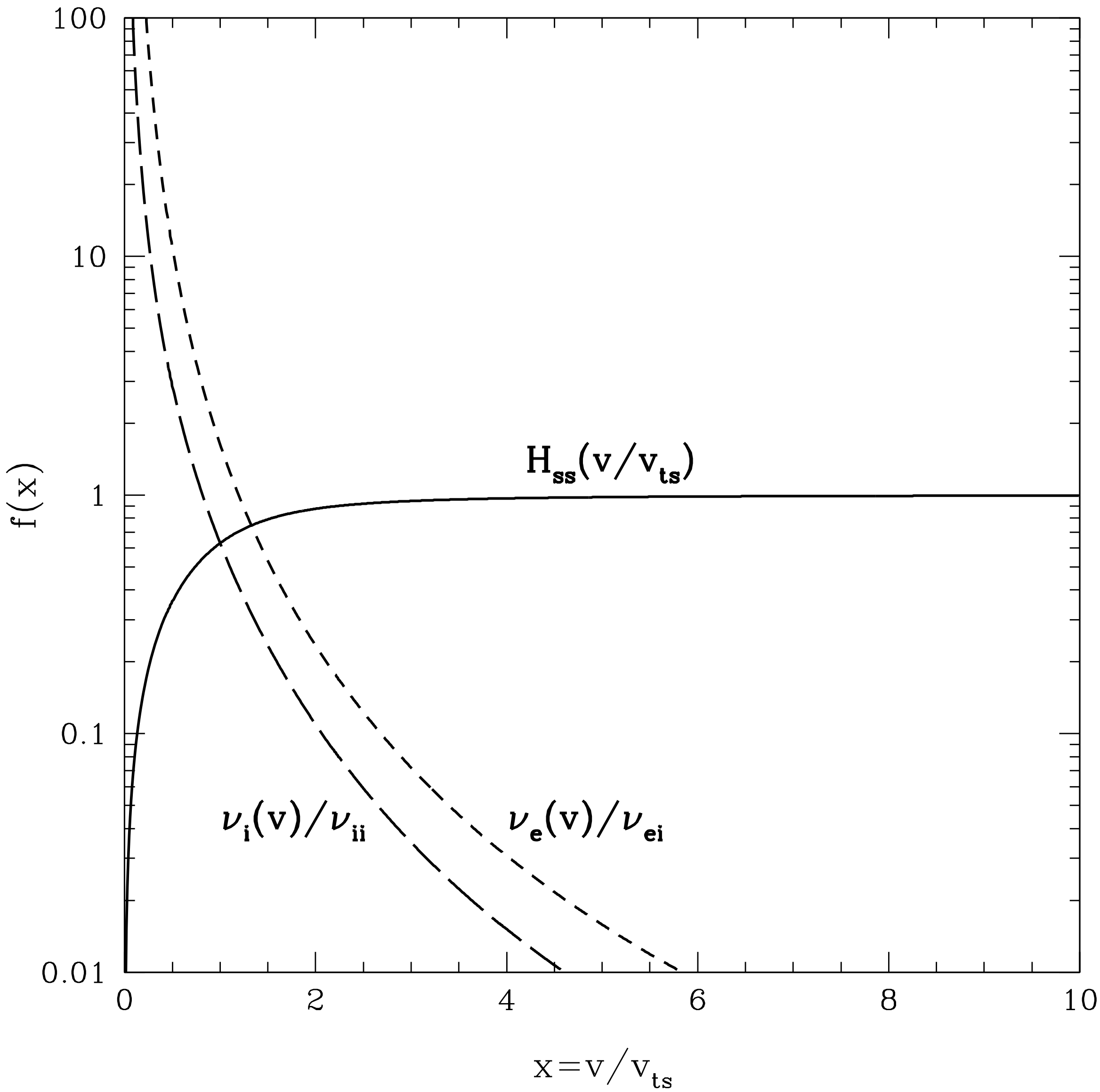

Figure 10.1: Plot of the functions $H_{ss}(v/v_{ts})$, $\nu_e(v/v_{te})/\nu_{ei}$, and $\nu_i(v/v_{ti})/\nu_{ii}$ to demonstrate how these coefficients vary with energy (magnitude of velocity).

and using a generalization of (19.3) and (19.7), we find

$$\begin{aligned}
&\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}}\left\langle h_s\mathcal{C}(h_s)\right\rangle_{\mathbf{R}_s} \qquad (10.93)\\
&= \sum_{\mathbf{k}}\sum_{\mathbf{k}'}\int\frac{d^3\mathbf{R}_s}{V}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_s}T_s\int d^3\mathbf{v}\frac{\nu_s(v)}{2F_{0s}}h_{\mathbf{k}'s}\left\{\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right]-\frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\}
\end{aligned}$$

Using the property

$$\int d^3\mathbf{R}_s e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_s} = \delta(\mathbf{k}+\mathbf{k}'), \tag{10.94}$$

the reality condition that

$$h_{-\mathbf{k}s} = h^*_{\mathbf{k}s}, \tag{10.95}$$

and transforming to velocity v, pitch angle $\xi = v_\parallel/v$, and gyrophase angle θ coordinates

$$\int d^3\mathbf{v} = \int_0^\infty v^2 dv \int_{-1}^1 d\xi \int_0^{2\pi} d\theta, \tag{10.96}$$

this simplifies to

$$\begin{aligned}
&\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \left\langle h_s \mathcal{C}(h_s) \right\rangle_{\mathbf{R}_s} \\
&= \sum_{\mathbf{k}} \pi T_s \int_0^\infty v^2 dv \frac{\nu_s(v)}{F_{0s}(v)} \left\{ \int_{-1}^1 d\xi h^*_{\mathbf{k}s} \frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right] - \frac{v^2}{v_{ts}^2}\frac{k_\perp^2\rho_s^2}{2}\int_{-1}^1 d\xi(1+\xi^2)|h_{\mathbf{k}s}|^2 \right\}.
\end{aligned} \tag{10.97}$$

We may now perform an integration by parts in pitch angle on the pitch angle scattering term to obtain the final, sign-definite result

$$\begin{aligned}
&\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \left\langle h_s \mathcal{C}(h_s) \right\rangle_{\mathbf{R}_s} \\
&= -\sum_{\mathbf{k}} \pi T_s \int_0^\infty v^2 dv \frac{\nu_s(v)}{F_{0s}(v)} \left\{ \int_{-1}^1 d\xi(1-\xi^2)\left|\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right|^2 + \frac{v^2}{v_{ts}^2}\frac{k_\perp^2\rho_s^2}{2}\int_{-1}^1 d\xi(1+\xi^2)\left|h_{\mathbf{k}s}\right|^2 \right\}.
\end{aligned} \tag{10.98}$$

10.3.5 Normalization of Collisional Heating

We begin with (19.19) and multiply by $\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}$ to find

$$\begin{aligned}
&\int \frac{d^3\mathbf{R}_s/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{T_s}{T_0}\frac{F_{0s}}{F_{00}} \left\langle \left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\frac{\mathcal{C}a_0}{v_{t0}}\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\right\rangle_{\mathbf{R}_s} \\
&= -\sum_{\mathbf{k}} \pi \frac{T_s}{T_0}\int_0^\infty \frac{v^2 dv}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}}\left(\frac{\nu_s a_0}{v_{t0}}\right)\left\{\int_{-1}^1 d\xi(1-\xi^2)\left|\frac{\partial}{\partial\xi}\left(\frac{h_{\mathbf{k}s}}{F_{0s}}\frac{a_0}{\rho_0}\right)\right|^2\right. \\
&+ \left.\frac{v^2}{v_{ts}^2}\frac{(k_\perp\rho_0)^2}{2}\left(\frac{\rho_s}{\rho_0}\right)^2\int_{-1}^1 d\xi(1+\xi^2)\left|\frac{h_{\mathbf{k}s}}{F_{0s}}\frac{a_0}{\rho_0}\right|^2\right\}.
\end{aligned} \tag{10.99}$$

Note here that the variable ξ is already dimensionless

$$\hat{\xi} = \xi = \frac{v_\parallel}{v} \tag{10.100}$$

and the collision frequency $\nu_s(v)$ is normalized by

$$\hat{\nu}_s = \frac{\nu_s a_0}{v_{t0}}. \tag{10.101}$$

We also write the

$$\frac{\rho_s}{\rho_0} = \frac{v_{ts}}{\Omega_s}\frac{\Omega_0}{v_{t0}} = \left(\frac{2T_s}{m_s}\right)^{1/2}\frac{m_s c}{q_s B_0}\frac{q_0 B_0}{m_0 c}\left(\frac{m_0}{2T_0}\right)^{1/2} = \left(\frac{T_s}{T_0}\right)^{1/2}\left(\frac{m_s}{m_0}\right)^{1/2}\frac{q_0}{q_s} = \frac{(\hat{T}_s\hat{m}_s)^{1/2}}{\hat{q}_s} \tag{10.102}$$

Writing all of this in normalized, dimensionless variables yields

$$\begin{aligned} &\int \frac{d^3\hat{\mathbf{R}}_s}{\hat{V}}\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left\langle \hat{h}_s\hat{\mathcal{C}}\hat{h}_s\right\rangle \qquad (10.103)\\ = \quad & -\sum_{\mathbf{k}}\frac{1}{\pi^{1/2}}\hat{n}_s\hat{T}_s\int_0^\infty \hat{v}_s^2 d\hat{v}_s e^{-\hat{v}_s^2}\hat{\nu}_s\left\{\int_{-1}^1 d\hat{\xi}(1-\hat{\xi}^2)\left|\frac{\partial\hat{h}_{\mathbf{k}s}}{\partial\hat{\xi}}\right|^2 + \hat{v}_s^2\frac{\hat{k}_\perp^2}{2}\frac{\hat{T}_s\hat{m}_s}{\hat{q}_s^2}\int_{-1}^1 d\hat{\xi}(1+\hat{\xi}^2)\left|\hat{h}_{\mathbf{k}s}\right|^2\right\}. \end{aligned}$$

10.4 Hypercollisionality

The hypercollisionality operator for a species s is defined by

$$\langle \mathcal{C}_{ns}(h_s) \rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s} \nu_{Hs}(k_\perp \rho_i)^n \frac{1}{2} \left\{ \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}s}}{\partial \xi} \right] - \frac{v^2}{v_{ts}^2}(1+\xi^2) \frac{k_\perp^2 \rho_s^2}{2} h_{\mathbf{k}s} \right\} \tag{10.104}$$

where ν_{Hs} is a constant coefficient, independent of velocity.

10.4.1 Goal of Hypercollisionality

The aim of any hyperdamping term is to model the transfer of energy from the smallest resolved scales in the box to smaller, unresolved scales. Without such a mechanism, energy builds up at the smallest scales because there are no smaller scale modes with which to couple nonlinearly. Hence, a bottleneck in the energy spectrum results.

One task to be completed is to estimate the nonlinear energy transfer rate (as a function of amplitude) and determine the minimum necessary hyperdamping to remove that energy. Although, at the moment the magnitude of linear hypercollisional damping is not well understood (in relation to the coefficient supplied in the input file), in principle we can at least connect the required effective hyperdamping to the nonlinear energy transfer rate. Due to *critical balance*, the nonlinear transfer rate should simply be of order the linear wave frequency, $k_\perp v_\perp \sim k_\parallel v_A$. Hence, the requirement for hyperdamping is to achieve $\gamma/\omega > 1$ at the end of the cascade, meaning that the energy will be damped out in roughly one wave period.

A more elegant technique, rather than to just choose some constant coefficient for the hypercollisional damping in any given run, is to allow the coefficient for the hypercollisional damping to vary as a function of simulation quantities (such as $|\chi|^2$), so that the effective damping supplies just the right amount of energy transfer for a given amplitude at the smallest resolves scales in the box. Hence, rather than specifying two parameters, the driving amplitude and hyperdamping amplitude, we simply specify the driving amplitude and allow the hyperdamping to vary in such a way as to always provide at least the minimum required damping rate.

10.4.2 Hypercollisionality in the Reduced MHD Limit

To find the what form a hypercollisional term takes in the Reduced MHD Limit will shed light on the effect of hypercollisionality.

We begin with the gyronkinetic equation written in terms of g_s,

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} + \frac{q_s}{T_s} v_\parallel F_{0s} \frac{\partial \tilde{\phi}}{\partial z} + \frac{c}{B}\left[\tilde{\phi} - A, h_s\right] - \langle \mathcal{C}(h_s) \rangle - \langle \mathcal{C}_n(h_s) \rangle = -\frac{q_s}{T_s} F_{0s} \frac{\partial A}{\partial t} \tag{10.105}$$

where we have used following definitions

$$g_s \equiv h_s - \frac{q_s \langle \phi \rangle}{T_s} F_{0s} + \frac{q_s \langle \mathbf{v}_\perp \cdot \mathbf{A}_\perp \rangle}{c T_s} F_{0s}, \tag{10.106}$$

$$\tilde{\phi} \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s})\hat{\phi} + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}} \frac{m_s v_\perp^2}{q_s} \frac{\delta \hat{B}_\parallel}{B_0}, \tag{10.107}$$

and

$$A \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s})\frac{v_\parallel A_\parallel}{c} \tag{10.108}$$

To derive the vorticity equation, the first step is to multiply the gyrokinetic equation by q_s, ring average at constant position $\mathbf{r}$, integrate the equation over velocity, and sum over species. We will consider each term in turn. Using the quasineutrality condition in terms of g_s, integrating over velocity for the potential terms, and solving for the integral of g_s gives us a form for the first term

$$\frac{\partial}{\partial t}\sum_s \int_{\mathbf{v}} q_s \langle g_s\rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(1-\Gamma_{0s})\frac{\partial\hat{\phi}}{\partial t} - \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\Gamma_{1s}\frac{\partial}{\partial t}\left(\frac{T_s}{q_s}\frac{\delta\hat{B}_\parallel}{B_0}\right). \tag{10.109}$$

The second term, after using the Parallel Ampere's Law in terms of g_s, becomes

$$\frac{\partial}{\partial z}\sum_s \int_{\mathbf{v}} q_s v_\parallel \langle g_s\rangle_{\mathbf{r}} = -\frac{c}{4\pi}\frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel. \tag{10.110}$$

The third term integrates to zero because it is odd in $v_\parallel$.

The fourth term, the nonlinear term, we neglect because it does not affect the form of the hypercollisional term.

The fifth term is the physical collisional term. Since we are generally interested in collisionless problems for which $\omega > \nu_c$, we neglect this term as well.

The term on the right-hand side is also odd in $v_\parallel$ and so contributes nothing after integration over velocity.

The sixth term on the left-hand side is the hypercollisional term. The ring average at constant position $\mathbf{r}$ of the hypercollisional operator becomes

$$\langle\langle \mathcal{C}_{ns}(h_s)\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} J_0(\alpha_s)\nu_{Hs}(k_\perp\rho_i)^n \frac{1}{2}\left\{\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right] - \frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\} \tag{10.111}$$

where $\alpha_s = k_\perp v_\perp/\Omega_s$ Summing over species and integrating over velocity gives

$$\sum_s \int_{\mathbf{v}} q_s \langle\langle \mathcal{C}_{ns}(h_s)\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} \tag{10.112}$$

$$= \sum_s \int_{\mathbf{v}} q_s \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} J_0(\alpha_s)\nu_{Hs}(k_\perp\rho_i)^n \frac{1}{2}\left\{\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right] - \frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\} \tag{10.113}$$

10.4.3 Nonlinear Cascade of Energy and Hypercollisionality

10.4.4 Eliot's Questions

1. What is the $\nu_H \int h_s \nabla^4 h_s$ heating? How does it depend on the parameters? What happens if $\nu_{He} \neq \nu_{Hi}$?
2. Can we put in a simple momentum conserving Krook collision operator to see what happens?
3. How does collision operator look in the entropy balance? What would be the heating for regular/hypercollisionality?
4. Why is $P_{He} \gg P_{Hi}$?

10.4.5 A Few Additional Thoughts on Hypercollisionality

Is hypercollisionality really the correct approach to achieve hyperdamping at the end of a cascade?

1. As demonstrated by Figure 10.2, the effective total damping as the hypercollisionality coefficient ν_{Hi} is increased actually decreases substantially before finally increasing, only reaching a value of $\gamma/\omega > 1$ for a ratio of $\nu_{Hi}/\nu_i \sim 10^7$. The initial decrease is expected since, as the collisions increase, the linear damping rate decreases.
2. Is it not true that, as the collisionality is increased, damping should decrease as you approach the fluid limit in which there is no damping?
3. In terms of the mean free path λ_{mfp}, the collision frequency is given by $\nu_c \sim v_t/\lambda_{mfp}$ and the parallel viscosity is given by $\nu_{\parallel visc} \sim \lambda_{mfp} v_t$. Thus, as the mean free path decreases, the collision frequency increases and the parallel viscosity decreases. Therefore, a more collisional plasma will have lower viscosity and therefore less damping of waves, not more.
4. The viscosity is related to the ion-ion collisions and the resistivity to the electron-ion collisions. But do increasing collision rates actually mean that you will achieve damping of waves in the MHD limit? Using a momentum-conserving collision operator, wouldn't the infinitely collisional limit be a fluid without any viscous damping?
5. Is the damping at very large ν_{Hi} seen in Figure 10.2 a result of perpendicular viscosity (since the Larmour radius is important here), or is it an effect of the collision operator not conserving momentum?
6. Is the increase of the mode frequency ω as ν_{Hi} increases, as seen in Figure 10.3, a sign that the dynamics are being corrupted, most likely because the collision operator is not conserving momentum?

10.5 Thoughts on Collisions and Velocity Space

These thoughts arise from discussions with Bill.

1. I think of Landau damping as the opposite of the bump-on-tail instability. When any part of the distribution function has $\partial f/\partial v_\parallel > 0$, the instability kicks in driving waves that dissipate the free energy in the bump of the distribution at $\omega = k_\parallel v_\parallel$. Landau damping is the same process operating in reverse: Landau damping extracts wave energy, forming structure, or a bump, in the distribution function at $v_\parallel = \omega/k_\parallel$. If that bump increases enough that $\partial f/\partial v_\parallel\big|_{v_\parallel=\omega/k_\parallel} > 0$, then the instability will drive more unstable waves, opposing the Landau damping and leading to a saturation of the damping process, probably with the $\partial f/\partial v_\parallel\big|_{v_\parallel=\omega/k_\parallel} = 0$.
2. Bill pointed out that the term responsible for driving the bump-on-tail instability, $\delta E_\parallel \partial f/\partial v_\parallel$, is ordered out of the equations in the gyrokinetic ordering. So what process, then, can saturate the Landau damping?
3. For this reason, we must have collisions ordered $\nu > \mathcal{O}(\epsilon\omega)$ so that collisions can serve to smooth out the growing bump on the tail and saturate the damping. Without these collisions, the problem is ill-defined; we see this in poor performance of `GS2` if collisions are turned off. Collisions must be present to smooth out growing structure in the distribution function, increasing the entropy by driving the distribution function towards a Maxwellian, and thus capturing the dynamics of the collisionless damping correctly.

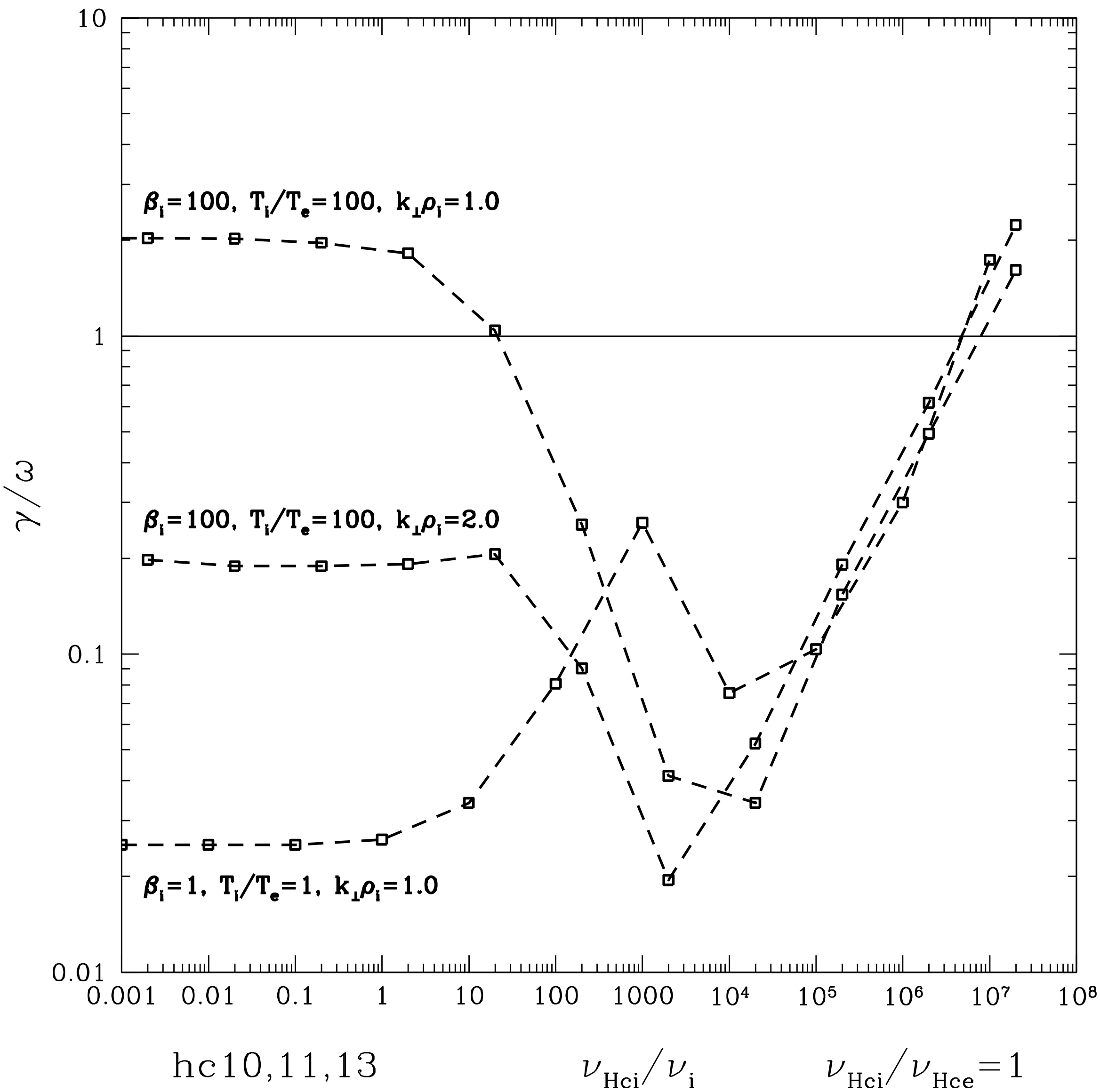

Figure 10.2: Plot of the total damping rate (relative to the wave frequency) as the hypercollisionality coefficient is increased. NOTE: In this figure, the hypercollisional damping coefficient is normalized by the largest $k_\perp \rho_i$ in the box; since these are linear simulations, effectively $k_\perp \rho_i = 1$ for the determination of the hypercollisional coefficient $\nu_{Hi}(k_\perp \rho_i)^4$.

4. We would like to plot $f(v_\parallel)$ as a function of time to see the rise of structure in velocity space; we expect the wavenumber of the velocity space structure to be inversely related to the damping rate (similar behavior to the shape of a Lorentzian in frequency sweep runs).

5. Bill suggested that the collisions cause an inverse cascade in velocity wavenumber space. Weak damping implies driving in velocity space at high wavenumber, strong damping at low wavenumber. Collisions take energy at high

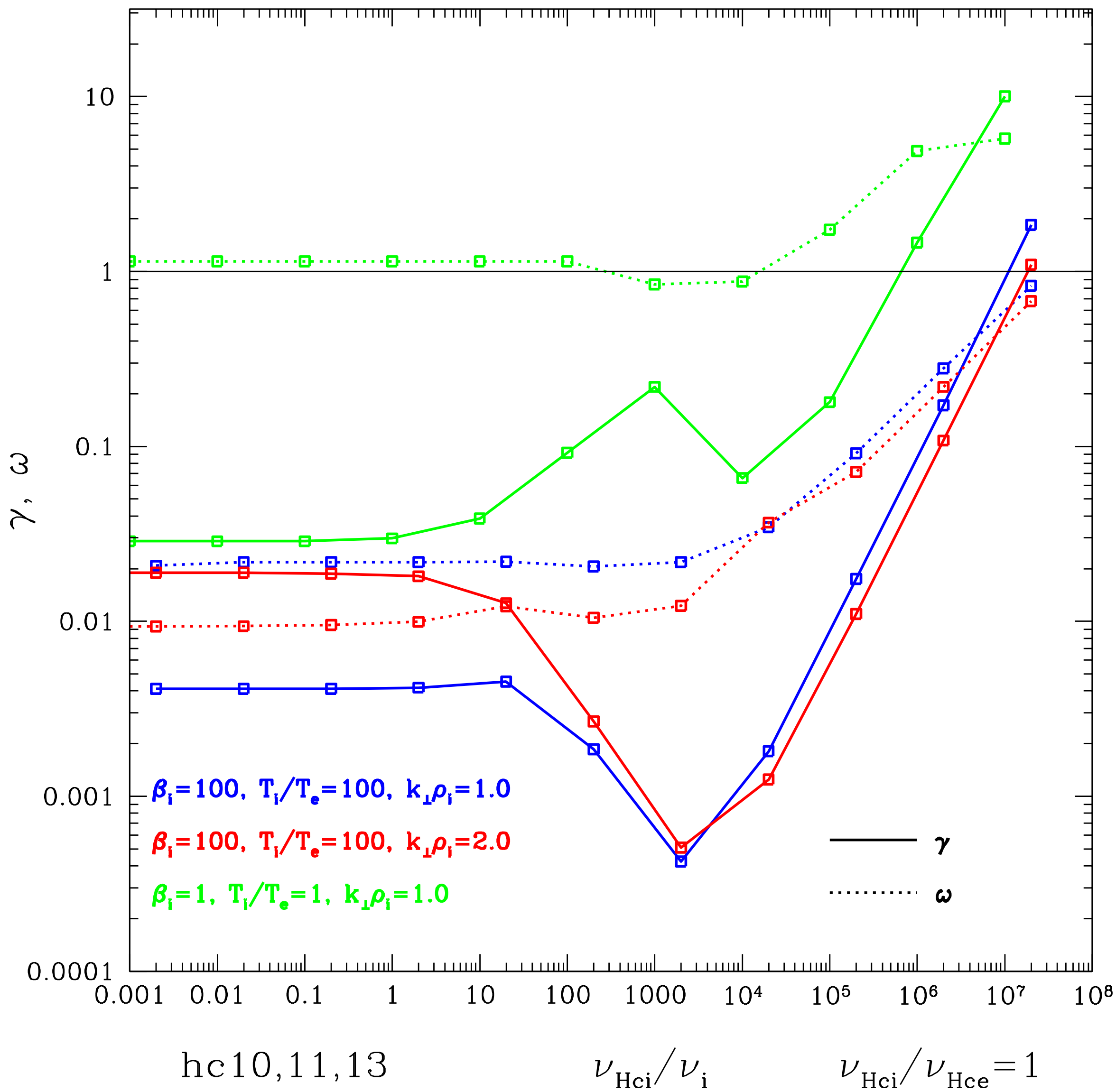

Figure 10.3: Plot of the mode frequency ω and the total damping rate γ as the hypercollisionality coefficient is increased; this are the same runs as in Figure 10.2.

wavenumber and inverse cascade it to low wavenumber, smoothing out the velocity space structure driven by Landau damping.

6. If we have a run with strong ion damping dominating the heating, we should be able to do two things that should not destroy the damping rate:

   (a) We can turn up the electron collisionality. Although it will should then underestimate damping from electrons correctly, that damping is negligible and so it will not matter much.

(b) We can turn down the ion collisionality. Since the ions are strongly damping, the bump in velocity space is large, so less collisionality may still work. Or maybe not. See next item.

7. For a given converged linear run with collisionless damping, the actual rate of entropy production by the collisional term must be the same, regardless of the collision frequency or the velocity space resolution (I believe this is correct). If that is so, what happens when we reduce the collision frequency. With fewer collisions, the amplitude of bumps in velocity space raises to higher amplitudes, thus meaning that $\partial h/\partial v_{\|}$ becomes larger. It should increase until the product of the collisional frequency with the pitch angle derivatives reaches the same value as before. But, if there is not enough resolution in velocity space, then numerical diffusion can limit the amplitude of $\partial h/\partial v_{\|}$ and this product can never achieve the same value as before; the behavior becomes unpredicatable, as numerical diffusion does not behave as a real physical process, and you are underresolved in velocity space.

Chapter 11

Highly Collisional Limit of Gyrokinetics

11.1 Strong Electron Collision Limit

Consider the linearized electron gyrokinetic equation

$$\frac{\partial h_e}{\partial t} + v_\parallel \frac{\partial h_e}{\partial z} - \langle \mathcal{C}(h_e) \rangle_{\mathbf{R}_e} = \frac{q_e F_{0e}}{T_e} \frac{\partial \langle \chi \rangle_{\mathbf{R}_e}}{\partial t} \tag{11.1}$$

where

$$\langle \chi \rangle_{\mathbf{R}_e} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_e} \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)\left(\hat{\phi}_{\mathbf{k}} - \frac{v_\parallel \hat{A}_{\parallel\mathbf{k}}}{c}\right) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_e}\right)}{\frac{k_\perp v_\perp}{\Omega_e}} \frac{2v_\perp^2}{v_{te}^2} \frac{T_e}{q_e} \frac{\delta \hat{B}_{\parallel\mathbf{k}}}{B_0} \right] \tag{11.2}$$

and

$$\langle \mathcal{C}(h_e) \rangle_{\mathbf{R}_e} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_e} \frac{\nu_{ee}(v)}{2} \left\{ \frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}e}}{\partial \xi}\right] - \frac{k_\perp^2 \rho_e^2}{2} \frac{v^2}{v_{te}^2}(1+\xi^2) h_{\mathbf{k}e} \right\}. \tag{11.3}$$

The pitch angle is defined by $\xi \equiv v_\parallel / v$ and the coefficient ν_{ee} is given by the formula

$$\nu_{ee}(v) = \frac{\sqrt{2}\pi n_e q_e^4 \ln \Lambda_{ee}}{m_e^{1/2} T_e^{3/2}} \left(\frac{v_{te}}{v}\right)^3 \left[\left(1 - \frac{1}{2}\frac{v_{te}^2}{v^2}\right) \frac{2}{\sqrt{\pi}} \int_0^{v/v_{te}} dx e^{-x^2} + \frac{1}{\sqrt{\pi}} \frac{v_{te}}{v} e^{-v^2/v_{te}^2} \right] \tag{11.4}$$

Note that here we are considering only electron-electron collisions and are not including electron-ion collisions. As shorthand, we denote $J_{ns} = J_n(\gamma_s)$ where $\gamma_s = \frac{k_\perp v_\perp}{\Omega_s}$. Also, we take $\nu_e = \nu_{ee}(v)$ to be simply a constant of our choice, independent of v. The Fourier-transformed electron gyrokinetic equation then becomes

$$\omega h_{\mathbf{k}e} - k_\parallel v_\parallel h_{\mathbf{k}e} - \frac{i\nu_e}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}e}}{\partial \xi}\right] + \frac{i\nu_e}{2}\frac{k_\perp^2\rho_e^2}{2}\frac{v^2}{v_{te}^2}(1+\xi^2)h_{\mathbf{k}e} = \omega \frac{q_e F_{0e}}{T_e}\left[J_{0e}\phi_{\mathbf{k}} - J_{0e}\frac{v_\parallel A_{\parallel\mathbf{k}}}{c} + \frac{J_{1e}}{\gamma_e}\frac{2v_\perp^2}{v_{te}^2}\frac{T_e}{q_e}\frac{\delta B_{\parallel\mathbf{k}}}{B_0}\right] \tag{11.5}$$

We are interested in the limit that $k_\perp \rho_e \sim \epsilon$. In this large-wavelength limit (compared to the electron Larmor radius ρ_e), we can take the small argument limits of the Bessel functions,

$$\lim_{z\to 1} J_0(z) = 1, \quad \lim_{z\to 1} \frac{J_1(z)}{z} = \frac{1}{2}.$$

The following subsidiary ordering, in addition to the usual gyrokinetic ordering, is imposed:

$$\begin{array}{ccc} k_\perp \rho_e & \sim & \epsilon \\ \frac{\omega}{k_\parallel v_{te}} & \sim & \epsilon^{1/2} \\ \frac{\nu_e}{\omega} & \sim & \epsilon^{-1} \end{array} \tag{11.6}$$

The balance of terms, as we shall see, occurs in this ordering for

$$\frac{\nu_e}{\omega} \sim \left(\frac{k_\parallel v_{te}}{\omega} \right)^2 \tag{11.7}$$

The order of each term in the gyrokinetic equation thus becomes

$$\underset{1}{\omega h_{\mathbf{k}e}} - \underset{\epsilon^{-1/2}}{k_\parallel v_\parallel h_{\mathbf{k}e}} - \underset{\epsilon^{-1}}{\frac{i\nu_e}{2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}e}}{\partial \xi} \right]} + \underset{\epsilon}{\frac{i\nu_e}{2} \frac{k_\perp^2 \rho_e^2}{2} \frac{v^2}{v_{te}^2} (1+\xi^2) h_{\mathbf{k}e}} = \omega \frac{q_e F_{0e}}{T_e} \left[\underset{1}{\phi_{\mathbf{k}}} - \underset{\epsilon^{-1/2}}{\frac{v_\parallel A_{\parallel \mathbf{k}}}{c}} + \underset{1}{\frac{v_\perp^2}{v_{te}^2} \frac{T_e}{q_e} \frac{\delta B_{\parallel \mathbf{k}}}{B_0}} \right] \tag{11.8}$$

Expanding the electron gyrokinetic distribution function $h_{\mathbf{k}e}$ in terms powers of $\epsilon^{1/2}$,

$$h_{\mathbf{k}e} = h_{\mathbf{k}e}^{(0)} + \epsilon^{1/2} h_{\mathbf{k}e}^{(1)} + \epsilon h_{\mathbf{k}e}^{(2)} + \cdots \tag{11.9}$$

we can solve for the distribution function order by order

11.1.1 Order ϵ^{-1}

At this order, only the pitch-angle scattering term appears,

$$\frac{i\nu_e}{2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}e}^{(0)}}{\partial \xi} \right] = 0. \tag{11.10}$$

Simple integration yields the solution

$$h_{\mathbf{k}e}^{(0)} = \int d\xi \frac{A}{1-\xi^2}, \tag{11.11}$$

which is infinite unless the constant of integration $A = 0$. Thus, at this order we discover that $h_{\mathbf{k}e}^{(0)}$ has no pitch angle dependence,

$$h_{\mathbf{k}e}^{(0)} = h_{\mathbf{k}e}^{(0)}(v) \tag{11.12}$$

11.1.2 Order $\epsilon^{-1/2}$

At this order, we find

$$k_\parallel v_\parallel h_{\mathbf{k}e}^{(0)} + \frac{i\nu_e}{2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}e}^{(1)}}{\partial \xi} \right] = \frac{q_e F_{0e}}{T_e} k_\parallel v_\parallel \frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c} \tag{11.13}$$

We want to solve for $h^{(1)}_{\mathbf{k}e}$, so we obtain

$$\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h^{(1)}_{\mathbf{k}e}}{\partial \xi}\right] = -\frac{i2k_\parallel v}{\nu_e}\left(\frac{q_e F_{0e}}{T_e}\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c} - h^{(0)}_{\mathbf{k}e}\right)\xi \tag{11.14}$$

Defining a variable

$$\overline{A} = -\frac{i2k_\parallel v}{\nu_e}\left[\frac{q_e F_{0e}}{T_e}\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c} - h^{(0)}_{\mathbf{k}e}\right], \tag{11.15}$$

we can solve for $h^{(1)}_{\mathbf{k}e}$ in two ways: by direct integration of the equation

$$\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h^{(1)}_{\mathbf{k}e}}{\partial \xi}\right] = \overline{A}\xi \tag{11.16}$$

or by noticing that the solutions of this equation are the Legendre Polynomials. The solution is $h^{(1)}_{\mathbf{k}e} = -\overline{A}\xi/2$, or

$$h^{(1)}_{\mathbf{k}e} = \frac{ik_\parallel v_\parallel}{\nu_e}\left[\frac{q_e F_{0e}}{T_e}\left(\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c}\right) - h^{(0)}_{\mathbf{k}e}\right]. \tag{11.17}$$

11.1.3 Order 1

At this order, we find

$$\omega h^{(0)}_{\mathbf{k}e} - k_\parallel v_\parallel h^{(1)}_{\mathbf{k}e} - \frac{i\nu_e}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h^{(2)}_{\mathbf{k}e}}{\partial \xi}\right] = \omega\frac{q_e F_{0e}}{T_e}\left[\phi_{\mathbf{k}} + \frac{v_\perp^2}{v_{te}^2}\left(\frac{T_e}{q_e}\frac{\delta B_{\parallel \mathbf{k}}}{B_0}\right)\right] \tag{11.18}$$

We integrate this equation over $\int_{-1}^{1} d\xi$ to annihilate the $h^{(2)}_{\mathbf{k}e}$ term, leaving us, after substituting for $h^{(1)}_{\mathbf{k}e}$ from the Order $\epsilon^{-1/2}$ solution, with

$$\omega h^{(0)}_{\mathbf{k}e} - \frac{ik_\parallel^2 v^2}{3\nu_e}\left[\frac{q_e F_{0e}}{T_e}\left(\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c}\right) - h^{(0)}_{\mathbf{k}e}\right] = \omega\frac{q_e F_{0e}}{T_e}\left[\phi_{\mathbf{k}} + \frac{2}{3}\frac{v^2}{v_{te}^2}\left(\frac{T_e}{q_e}\frac{\delta B_{\parallel \mathbf{k}}}{B_0}\right)\right] \tag{11.19}$$

Factoring, we find

$$\omega\left\{h^{(0)}_{\mathbf{k}e} - \frac{q_e F_{0e}}{T_e}\left[\phi_{\mathbf{k}} + \frac{2}{3}\frac{v^2}{v_{te}^2}\left(\frac{T_e}{q_e}\frac{\delta B_{\parallel \mathbf{k}}}{B_0}\right)\right]\right\} - \frac{ik_\parallel^2 v^2}{3\nu_e}\left[\frac{q_e F_{0e}}{T_e}\left(\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c}\right) - h^{(0)}_{\mathbf{k}e}\right] = 0 \tag{11.20}$$

Solving for $h^{(0)}_{\mathbf{k}e}$, we obtain the final result

$$h^{(0)}_{\mathbf{k}e} = F_{0e}\left\{\frac{1}{1+\frac{ik_\parallel^2 v^2}{3\nu_e\omega}}\left[\left(\frac{q_e\phi_{\mathbf{k}}}{T_e}\right) + \frac{2}{3}\frac{v^2}{v_{te}^2}\left(\frac{\delta B_{\parallel \mathbf{k}}}{B_0}\right)\right] + \frac{\frac{ik_\parallel^2 v^2}{3\nu_e\omega}}{1+\frac{ik_\parallel^2 v^2}{3\nu_e\omega}}\left(\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c}\right)\right\} \tag{11.21}$$

11.1.4 Summary

We define a new set of dimensionless basis functions

$$\tilde{E}_{\parallel} \equiv \frac{q_e}{T_e}\left(\phi_{\mathbf{k}} - \frac{\omega A_{\parallel\mathbf{k}}}{k_{\parallel}c}\right) \tag{11.22}$$

$$\tilde{B}_{\perp} \equiv \frac{q_e}{T_e}\frac{\omega A_{\parallel\mathbf{k}}}{k_{\parallel}c} \tag{11.23}$$

$$\tilde{B}_{\parallel} \equiv \frac{\delta B_{\parallel\mathbf{k}}}{B_0}. \tag{11.24}$$

We also express the solution in terms of normalized velocity

$$x \equiv \frac{v}{v_{te}} \tag{11.25}$$

and pitch angle

$$\xi \equiv \frac{v_{\parallel}}{v}. \tag{11.26}$$

We normalize the electron collisional coefficient using

$$\hat{\nu}_e \equiv \frac{\nu_e}{k_{\parallel}v_{te}}. \tag{11.27}$$

and define the dimensionless parameter y which measures the strength of the collisionality

$$y \equiv \frac{-i3\nu_e\omega}{k_{\parallel}^2 v_{te}^2}. \tag{11.28}$$

The solution is given by

$$h_{\mathbf{k}e} = \left(1 - \frac{i\xi x}{\hat{\nu}_e}\right) h_{\mathbf{k}e}^{(0)}(x) + \frac{i\xi x}{\hat{\nu}_e} F_{0e}\tilde{B}_{\perp} \tag{11.29}$$

with

$$h_{\mathbf{k}e}^{(0)}(x) = \frac{1}{1+\frac{x^2}{y}} F_{0e}\left(\tilde{E}_{\parallel} + \frac{2}{3}x^2\tilde{B}_{\parallel}\right) + F_{0e}\tilde{B}_{\perp} \tag{11.30}$$

11.1.5 Simple Limit Comparing ν_e/ω to $k_{\parallel}^2 v_{te}^2/\omega^2$

From (11.20) we can discern two simple limits: semi-collisionality and high collisionality.

In the semi-collisional limit of $\nu_e/\omega \ll k_{\parallel}^2 v_{te}^2/\omega^2$, the solution is simply

$$h_{\mathbf{k}e}^{(0)} = \frac{q_e F_{0e}}{T_e}\left(\frac{\omega A_{\parallel\mathbf{k}}}{k_{\parallel}c}\right). \tag{11.31}$$

The perturbed distribution function is given by

$$\delta f_e = -\frac{q_e F_{0e}}{T_e}\phi_{\mathbf{k}} + h^{(0)}_{\mathbf{k}e} + h^{(1)}_{\mathbf{k}e}, \tag{11.32}$$

so we find in this limit

$$\delta f_e = -\frac{q_e F_{0e}}{T_e}\left[\phi_{\mathbf{k}} - \left(\frac{\omega A_{\|\mathbf{k}}}{k_\| c}\right)\right] \tag{11.33}$$

Hence, using the definitions (11.22)–(11.24), this becomes

$$\frac{\delta f_e}{F_{0e}} = -\tilde{E}_\|. \tag{11.34}$$

The distribution function is driven by $E_\|$.

In the highly collisional limit of $\nu_e/\omega \gg k_\|^2 v_{te}^2/\omega^2$, the solution becomes

$$h^{(0)}_{\mathbf{k}e} = \frac{q_e F_{0e}}{T_e}\left[\phi_{\mathbf{k}} + \frac{2}{3}\frac{v^2}{v_{te}^2}\left(\frac{T_e}{q_e}\frac{\delta B_{\|\mathbf{k}}}{B_0}\right)\right]. \tag{11.35}$$

Hence, the perturbed distribution function becomes

$$\frac{\delta f_e}{F_{0e}} = -\frac{ik_\| v_\|}{\nu_e}\tilde{E}_\| + \left(1 - \frac{ik_\| v_\|}{\nu_e}\right)\frac{2}{3}\frac{v^2}{v_{te}^2}\tilde{B}_\|. \tag{11.36}$$

The perturbed distribution function damps $E_\|$, effectively providing a resistance to the parallel current. Hence, we expect strong damping in this limit of high collisionality.

Note, however, that the necessary collision rate to be in the highly collisional regime is very high. For kinetic Alfvén waves, the frequency is given by

$$\overline{\omega}^2 = \frac{2\alpha_i}{\beta_i + \frac{2\tau}{1+\tau}} \tag{11.37}$$

where the bar denotes normalization to $k_\| v_A$; so we have

$$\overline{\nu}_e \gg \frac{\mu\beta_i}{\overline{\omega}\tau} = \frac{\mu\beta_i\sqrt{\beta_i + \frac{2\tau}{1+\tau}}}{\sqrt{2}\alpha_i^{1/2}\tau} \tag{11.38}$$

For $\beta_i \sim 1$ and $\tau \sim 1$ and $k_\perp\rho_i \sim 10$, this means $\overline{\nu}_e \gg 260$, or $\overline{\nu}_e = \mathcal{O}(m_i/m_e)$.

11.2 Define the Collisional Dispersion Function

Here we define the Collisional Dispersion Function by

$$\chi_n(y) = \int_0^\infty \frac{dx}{\sqrt{\pi}} \frac{x^n e^{-x^2}}{1 + \frac{x^2}{y}} \tag{11.39}$$

for a complex argument y. This function has poles at

$$x = |y|^{1/2} \left(\mp \sin \frac{\theta}{2} \pm i \cos \frac{\theta}{2} \right), \tag{11.40}$$

where the complex argument is given by

$$y = y_r + iy_i = |y|e^{i\theta} \tag{11.41}$$

and

$$\theta = \tan^{-1} \frac{y_i}{y_r}. \tag{11.42}$$

These poles fall along the real axis, on the path of integration, at $x = \pm|y|^{1/2}$ when $y_i = 0$ and $y_r < 0$ (or $\theta = \pm\pi$). In the course of solving for the dispersion relation, we will need χ_n for values $n = 2, 4, 6$. Contour plots of the real and imaginary components of $\chi_n(y)$ over complex y-space for values $n = 2, 4, 6$ are presented in Figure 11.1.

11.2.1 Small Argument Limit of the Collisional Dispersion Function

For $|y| \ll x$ or $|y| \ll 1$,

$$\chi_n(y) \simeq y \int_0^\infty \frac{dx}{\sqrt{\pi}} x^{n-2} e^{-x^2}. \tag{11.43}$$

For values $n = 2, 4, 6$, this limit gives

$$\begin{array}{rcl} \chi_2(y) & \simeq & \frac{y}{2} \\ \chi_4(y) & \simeq & \frac{y}{4} \\ \chi_6(y) & \simeq & \frac{3y}{8} \end{array} . \tag{11.44}$$

11.2.2 Large Argument Limit of the Collisional Dispersion Function

For $|y| \gg x$ or $|y| \gg 1$,

$$\chi_n(y) \simeq \int_0^\infty \frac{dx}{\sqrt{\pi}} \left(x^n - x^{n+2} \frac{y_r}{|y|^2} + i x^{n+2} \frac{y_i}{|y|^2} \right) e^{-x^2}. \tag{11.45}$$

For values $n = 2, 4, 6$, this limit gives

$$\begin{array}{rcl} \chi_2(y) & \simeq & \frac{1}{4}\left(1 - \frac{3}{2y}\right) \\ \chi_4(y) & \simeq & \frac{3}{8}\left(1 - \frac{5}{2y}\right) \\ \chi_6(y) & \simeq & \frac{15}{16}\left(1 - \frac{7}{2y}\right) \end{array} . \tag{11.46}$$

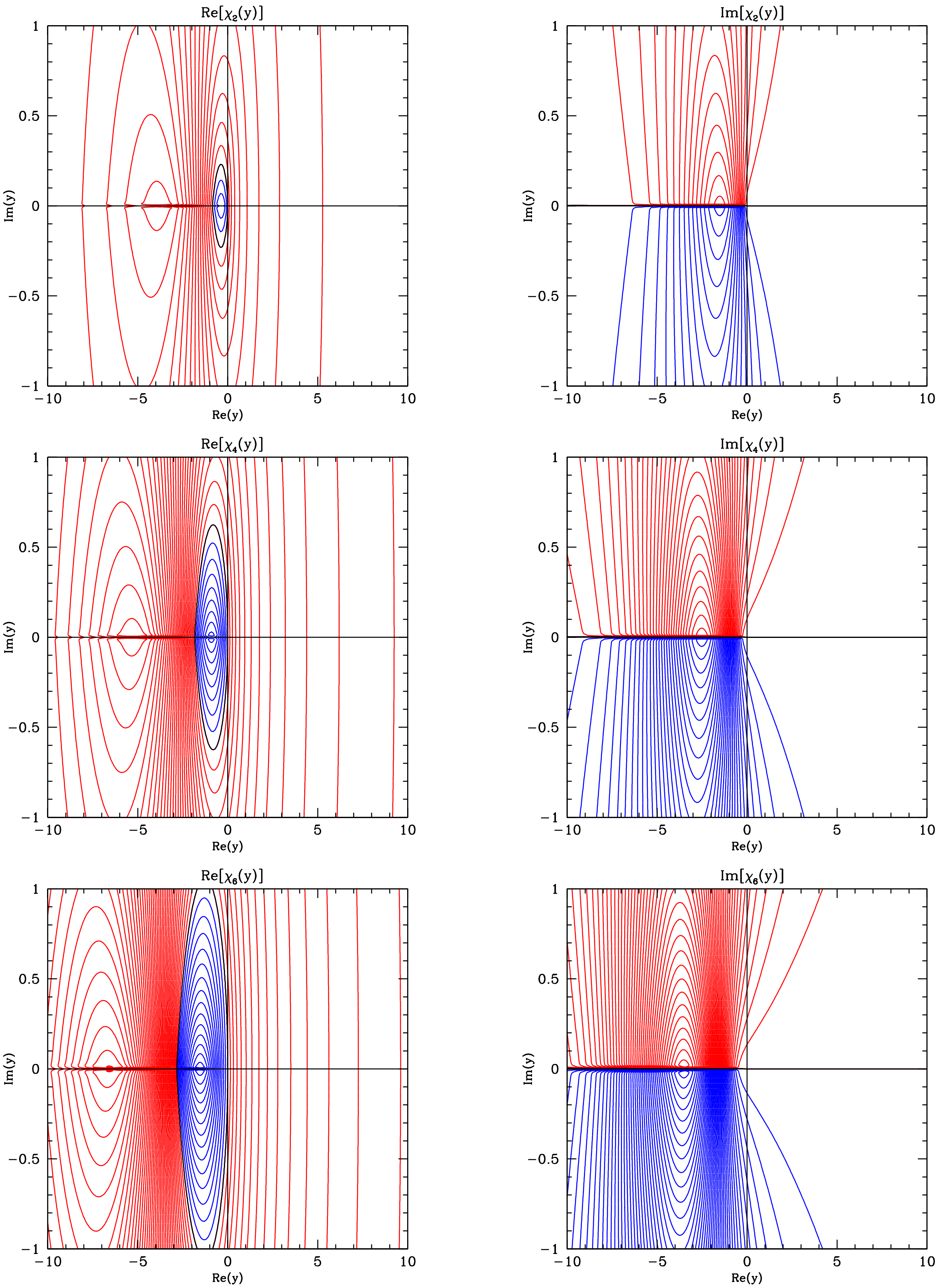

Figure 11.1: Contour plots of the real and imaginary components of $\chi_n(y)$ over complex y-space for values $n = 2, 4, 6$. Positive values have red contours, negative have blue contours, and zero is a black contour.

11.3 Dispersion Relation for Strong Electron Collision Limit

To determine the dispersion relation for the strong electron collision limit of gyrokinetics, we must substitute the solution for the distribution function given by (11.29) and (11.30) into Maxwells equations and determine the solubility condition.

We will solve for the dispersion relation in the kinetic Alfven wave limit given by the ordering

$$\begin{array}{ccc} \alpha_e & \sim & \epsilon \\ \alpha_i & \sim & \epsilon^{-1} \\ \frac{\omega}{k_\| v_{te}} & \sim & \epsilon^{1/2} \\ \frac{\nu_e}{\omega} & \sim & \epsilon^{-1} \end{array} . \tag{11.47}$$

In this limit,

$$J_{0i} = 0, \quad J_{0e} = 1, \quad \frac{J_{1i}}{\gamma_i} = 0, \quad \frac{J_{1e}}{\gamma_e} = \frac{1}{2}.$$

In this limit, the ion response is strictly Boltzmann, so $h_{\mathbf{k}i} = 0$.

Velocity integration is done over dimensionless variables $x = v/v_{te}$ and $\xi = v_\|/v$,

$$\int d^3\mathbf{v} = 2\pi v_{te}^3 \int_0^\infty x^2 dx \int_{-1}^1 d\xi. \tag{11.48}$$

11.3.1 Poisson's Equation, or the Quasineutrality Condition

The quasineutrality condition is given by

$$\sum_s \frac{q_s^2 n_s}{T_s}\phi - \sum_s q_s \int d^3\mathbf{v} \langle h_s \rangle_{\mathbf{r}} = 0. \tag{11.49}$$

Fourier transforming and noting that $h_{\mathbf{k}i} = 0$, this becomes

$$\left(\frac{q_i^2 n_i}{T_i} + \frac{q_e^2 n_e}{T_e} \right) \phi_{\mathbf{k}} - q_e \int d^3\mathbf{v} J_{0e} h_{\mathbf{k}e} = 0. \tag{11.50}$$

Noting that the terms proportional to ξ in (11.29) are odd in ξ and so contribute nothing, substituting $J_{0e} = 1$, and using $F_{0e} = \frac{n_e}{\pi^{3/2} v_{te}^3} \exp(-v^2/v_{te}^2)$, the integral becomes

$$q_e \int d^3\mathbf{v} h_{\mathbf{k}e} = \frac{2\pi q_e v_{te}^3 n_e}{\pi v_{te}^3} \int_0^\infty \frac{x^2 dx}{\sqrt{\pi}} \int_{-1}^1 d\xi \left[\frac{1}{1 + \frac{x^2}{y}} \left(\tilde{E}_\| + \frac{2}{3} x^2 \tilde{B}_\| \right) + \tilde{B}_\perp \right] e^{-x^2} \tag{11.51}$$

Noting $\int_{-1}^1 d\xi = 2$ and simplifying, we end up with

$$= 4 q_e n_e \left[\tilde{E}_\| \int_0^\infty \frac{dx}{\sqrt{\pi}} \frac{x^2 e^{-x^2}}{1 + \frac{x^2}{y}} + \frac{2}{3} \tilde{B}_\| \int_0^\infty \frac{dx}{\sqrt{\pi}} \frac{x^4 e^{-x^2}}{1 + \frac{x^2}{y}} + \tilde{B}_\perp \int_0^\infty \frac{dx}{\sqrt{\pi}} x^2 e^{-x^2} \right] \tag{11.52}$$

Using the definition of the Collisional Dispersion Function χ_n, given by (11.39), we find the quasineutrality condition becomes

$$q_e n_e \left(\frac{1}{\tau}+1\right)\left(\tilde{E}_\parallel + \tilde{B}_\perp\right) - q_e n_e \left[4\chi_2 \tilde{E}_\parallel + \frac{8}{3}\chi_4 \tilde{B}_\parallel + \tilde{B}_\perp\right] = 0 \tag{11.53}$$

where we have used the shorthand for the temperature ratio $\tau = T_i/T_e$. Simplifying and collecting terms, this results in

$$\left(1+\frac{1}{\tau}-4\chi_2\right)\tilde{E}_\parallel + \frac{1}{\tau}\tilde{B}_\perp - \frac{8}{3}\chi_4 \tilde{B}_\parallel = 0 \tag{11.54}$$

11.3.2 Parallel Ampere's Law

The parallel component of Ampere's Law is given by

$$-\nabla_\perp^2 A_\parallel = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} v_\parallel \langle h_s \rangle_{\mathbf{r}} \tag{11.55}$$

Fourier transforming, noting that $h_{\mathbf{k}i} = 0$, and simplifying, this becomes

$$\frac{ck_\perp^2}{4\pi} A_{\parallel\mathbf{k}} = q_e \int d^3\mathbf{v} v_\parallel J_{0e} h_{\mathbf{k}e} \tag{11.56}$$

Noting that the terms even in ξ in (11.29) are odd in ξ in this integral and so contribute nothing, substituting $J_{0e} = 1$, and using $F_{0e} = \frac{n_e}{\pi^{3/2} v_{te}^3} \exp(-v^2/v_{te}^2)$, the integral becomes

$$q_e \int d^3\mathbf{v} v_\parallel h_{\mathbf{k}e} = \frac{2\pi q_e v_{te}^4 n_e}{\pi v_{te}^3} \int_0^\infty \frac{x^2 dx}{\sqrt{\pi}} \int_{-1}^1 d\xi \left[\left(-i\frac{\xi^2 x^2}{\hat{\nu}_e}\right) h_{\mathbf{k}e}^{(0)}(x) + i\frac{\xi^2 x^2}{\hat{\nu}_e}\tilde{B}_\perp\right] e^{-x^2} \tag{11.57}$$

where the collision frequency has been normalized to the electron thermal velocity $\hat{\nu}_e = \nu_e/(k_\parallel v_A)$. Noting that the $\tilde{B}_\perp$ terms cancel, using $\int_{-1}^1 d\xi \xi^2 = 2/3$, and simplifying, the integral becomes

$$= \frac{4}{3} q_e n_e v_{te} \frac{i}{\hat{\nu}_e}\left[-\tilde{E}_\parallel \int_0^\infty \frac{dx}{\sqrt{\pi}} \frac{x^4 e^{-x^2}}{1+\frac{x^2}{y}} - \frac{2}{3}\tilde{B}_\parallel \int_0^\infty \frac{dx}{\sqrt{\pi}} \frac{x^6 e^{-x^2}}{1+\frac{x^2}{y}}\right] \tag{11.58}$$

Putting everything together, we find

$$\frac{T_e}{q_e}\frac{c^2 k_\perp^2 k_\parallel}{4\pi\omega}\frac{q_e}{T_e}\frac{\omega A_{\parallel\mathbf{k}}}{k_\parallel c} = \frac{4}{3} q_e n_e v_{te} \frac{i}{\hat{\nu}_e}\left[-\chi_4 \tilde{E}_\parallel - \frac{2}{3}\chi_6 \tilde{B}_\parallel\right] \tag{11.59}$$

Further simplification yields

$$-i\frac{T_e}{q_e}\frac{c^2 k_\perp^2 \nu_e}{4\pi q_e n_e v_{te}^2 \omega}\tilde{B}_\perp = -\frac{4}{3}\chi_4 \tilde{E}_\parallel - \frac{8}{9}\chi_6 \tilde{B}_\parallel \tag{11.60}$$

The constants on the left-hand side simplify to

$$-i\frac{T_e}{q_e}\frac{c^2 k_\perp^2 \nu_e}{4\pi q_e n_e v_{te}^2 \omega} = -i\frac{\nu_e}{\omega}\frac{\alpha_i}{\beta_i \mu} \tag{11.61}$$

where the mass ratio is given by $\mu = m_i/m_e$ and $\alpha_i = k_\perp^2 \rho_i^2/2$. Thus, our final result is

$$\frac{4}{3}\chi_4 \tilde{E}_\parallel - i\frac{\nu_e}{\omega}\frac{\alpha_i}{\mu\beta_i}\tilde{B}_\perp + \frac{8}{9}\chi_6 \tilde{B}_\parallel = 0. \tag{11.62}$$

11.3.3 Perpendicular Ampere's Law

The perpendicular component of Ampere's Law is given by

$$\nabla_\perp \delta B_\parallel = \sum_s \frac{4\pi}{c} q_s \int d^3\mathbf{v} \langle \hat{\mathbf{z}} \times \mathbf{v}_\perp h_s \rangle_{\mathbf{r}}. \tag{11.63}$$

Fourier transforming, noting that $h_{\mathbf{k}i} = 0$, and simplifying, this becomes

$$-\frac{c}{4\pi}\delta B_{\parallel\mathbf{k}} = \frac{q_e}{\Omega_e}\int d^3\mathbf{v} \frac{J_{1e}}{\frac{k_\perp v_\perp}{\Omega_e}} v_\perp^2 h_{\mathbf{k}e} \tag{11.64}$$

Noting that the terms proportional to ξ in (11.29) are odd in ξ and so contribute nothing, substituting $\frac{J_{1e}}{\gamma_e} = \frac{1}{2}$, replacing $v_\perp^2 = x^2(1-\xi^2)$, and using $F_{0e} = \frac{n_e}{\pi^{3/2} v_{te}^3}\exp(-v^2/v_{te}^2)$, the integral becomes

$$\frac{q_e}{\Omega_e}\int d^3\mathbf{v} \frac{J_{1e}}{\frac{k_\perp v_\perp}{\Omega_e}} v_\perp^2 h_{\mathbf{k}e} = \frac{2\pi q_e v_{te}^5 n_e}{2\pi \Omega_e v_{te}^3}\int_0^\infty \frac{x^2 dx}{\sqrt{\pi}}\int_{-1}^1 d\xi x^2(1-\xi^2)\left[\frac{1}{1+\frac{x^2}{y}}\left(\tilde{E}_\parallel + \frac{2}{3}x^2\tilde{B}_\parallel\right) + \tilde{B}_\perp\right]e^{-x^2} \tag{11.65}$$

Using $\int_{-1}^1 d\xi(1-\xi^2) = 4/3$, and simplifying, the integral becomes

$$= \frac{4q_e n_e v_{te}^2}{3\Omega_e}\left[\tilde{E}_\parallel \int_0^\infty \frac{dx}{\sqrt{\pi}}\frac{x^4 e^{-x^2}}{1+\frac{x^2}{y}} + \frac{2}{3}\tilde{B}_\parallel \int_0^\infty \frac{dx}{\sqrt{\pi}}\frac{x^6 e^{-x^2}}{1+\frac{x^2}{y}} + \tilde{B}_\perp \int_0^\infty \frac{dx}{\sqrt{\pi}} x^4 e^{-x^2}\right] \tag{11.66}$$

Putting everything together, we find

$$-\frac{c\Omega_e B_0}{4\pi q_e n_e v_{te}^2}\tilde{B}_\parallel = \frac{4}{3}\chi_4\tilde{E}_\parallel + \frac{8}{9}\chi_6\tilde{B}_\parallel + \frac{1}{2}\tilde{B}_\perp \tag{11.67}$$

The constants on the left-hand side simplify to

$$-\frac{c\Omega_e B_0}{4\pi q_e n_e v_{te}^2} = -\frac{\tau}{\beta_i}, \tag{11.68}$$

so we end up with

$$\frac{8}{3}\chi_4\tilde{E}_\parallel + \tilde{B}_\perp + \left(\frac{2\tau}{\beta_i} + \frac{16}{9}\chi_6\right)\tilde{B}_\parallel = 0 \tag{11.69}$$

11.3.4 Solving for the Dispersion Relation

Defining the quantities

$$A = 1 + \frac{1}{\tau} - 4\chi_2 \tag{11.70}$$

$$B = \frac{4}{3}\chi_4 \tag{11.71}$$

$$C = \frac{8}{9}\chi_6, \tag{11.72}$$

we obtain the matrix equation

$$\left(\begin{array}{ccc} A & \frac{1}{\tau} & -2B \\ B & -i\frac{\nu_e}{\omega}\frac{\alpha_i}{\mu\beta_i} & C \\ 2B & 1 & \frac{2\tau}{\beta_i} + 2C \end{array} \right) \left(\begin{array}{c} \tilde{E}_\parallel \\ \tilde{B}_\perp \\ \tilde{B}_\parallel \end{array} \right) = 0 \tag{11.73}$$

This yields the dispersion relation

$$\frac{i\nu_e}{\omega}\frac{2\alpha_i}{\mu\beta_i}\left(\frac{A\tau}{\beta_i} + 2B^2 + AC\right) + \frac{2B}{\beta_i} + 2B^2 + AC = 0 \tag{11.74}$$

11.4 Limits of the Strong Electron Collision Dispersion Relation

Analytical solutions of the dispersion relation can be derived in the limits $|y| \ll 1$ and $|y| \gg 1$. Here we describe the assumed ordering in detail and derive the solutions for both of these limits.

11.4.1 Semi-collisional Limit, $|y| \ll 1$

For the semi-collisional limit, we assume the following ordering of parameters:

$$\begin{array}{ccc} \mu & \sim & \epsilon^{-2} \\ \tau & \sim & 1 \\ \beta_i & \sim & 1 \\ \alpha_i & \sim & \epsilon^{-1} \\ \nu_e/\omega & \sim & \epsilon^{-1/2} \end{array} . \tag{11.75}$$

In this semi-collisional limit, we expect to reproduce the basic dynamics of the kinetic Alfvén wave, so we can use this formula to estimate the ordering of the frequency. The ordering imposed above gives the following order for secondary parameters:

$$\begin{array}{ccc} \overline{\omega} = \left(\frac{\alpha_i}{\beta_i/2 + 2\tau/(1+\tau)}\right)^{1/2} & \sim & \epsilon^{-1/2} \\ \frac{\omega}{k_\parallel v_{te}} = \frac{\overline{\omega}\tau^{1/2}}{\mu^{1/2}\beta_i^{1/2}} & \sim & \epsilon^{1/2} \\ \alpha_e = \frac{\alpha_i}{\mu\tau} & \sim & \epsilon \\ y = -3\frac{i\nu_e}{\omega}\frac{\overline{\omega}^2\tau}{\mu\beta_i} & \sim & \epsilon^{1/2} \end{array} . \tag{11.76}$$

In this limit, the coefficients in the dispersion relation simplify to

$$\begin{array}{rcc} A & \simeq & 1+1/\tau-2y \\ B & \simeq & y/3 \\ C & \simeq & y/3 \end{array} . \tag{11.77}$$

Hence, the order of the terms in the dispersion relation is given by

$$\frac{i\nu_e}{\omega}\frac{2\alpha_i}{\mu\beta_i}\left(\underset{(\epsilon^{1/2},\epsilon)}{\frac{A\tau}{\beta_i}}+\underset{\epsilon^{3/2}}{2B^2}+\underset{(\epsilon,\epsilon^{3/2})}{AC}\right)+\underset{\epsilon^{1/2}}{\frac{2B}{\beta_i}}+\underset{\epsilon}{2B^2}+\underset{(\epsilon^{1/2},\epsilon)}{AC}=0 \tag{11.78}$$

Keeping all terms to order ϵ except for the order ϵ part of the first term and simplifying produces the result

$$\alpha_i-\overline{\omega}^2\left(\frac{1}{A'}+\frac{\beta_i}{2}\right)+-\frac{i\nu_e}{\mu\omega}\left(\alpha_i\overline{\omega}^2+\frac{2\tau}{A'}\overline{\omega}^4\right)=0. \tag{11.79}$$

where we have used $A'=1+1/\tau$.

Assuming $|\gamma|\ll|\omega_r|$, we can expand about $\omega=\omega_r$, where $D_r(\omega_r)=0$, and solve for the damping rate using

$$\gamma=-\frac{D_i(\omega_r)}{\frac{\partial D_r(\omega_r)}{\partial\omega_r}}. \tag{11.80}$$

The resulting solution is

$$\overline{\omega}_r^2=\frac{2\alpha_i}{\beta_i+\frac{2\tau}{1+\tau}} \tag{11.81}$$

$$\overline{\gamma}=-\frac{\overline{\nu}_e\alpha_i}{\mu\beta_i}\frac{\left[\frac{\tau^2\beta_i}{1+\tau}+\frac{\beta_i}{2}\left(\frac{\tau}{1+\tau}+\frac{\beta_i}{2}\right)\right]}{\left(\frac{\tau}{1+\tau}+\frac{\beta_i}{2}\right)^2} \tag{11.82}$$

Comparison of this limit with the numerical solution to (11.74) is shown in Figure 11.2.

11.4.2 Highly Collisional Limit, $|y|\gg 1$

For the highly collisional limit, we assume the following ordering of parameters:

$$\begin{array}{rcc} \mu & \sim & \epsilon^{-2} \\ \tau & \sim & 1 \\ \beta_i & \sim & 1 \\ \alpha_i & \sim & \epsilon^{-1} \\ \nu_e/\omega & \sim & \epsilon^{-3/2} \end{array} . \tag{11.83}$$

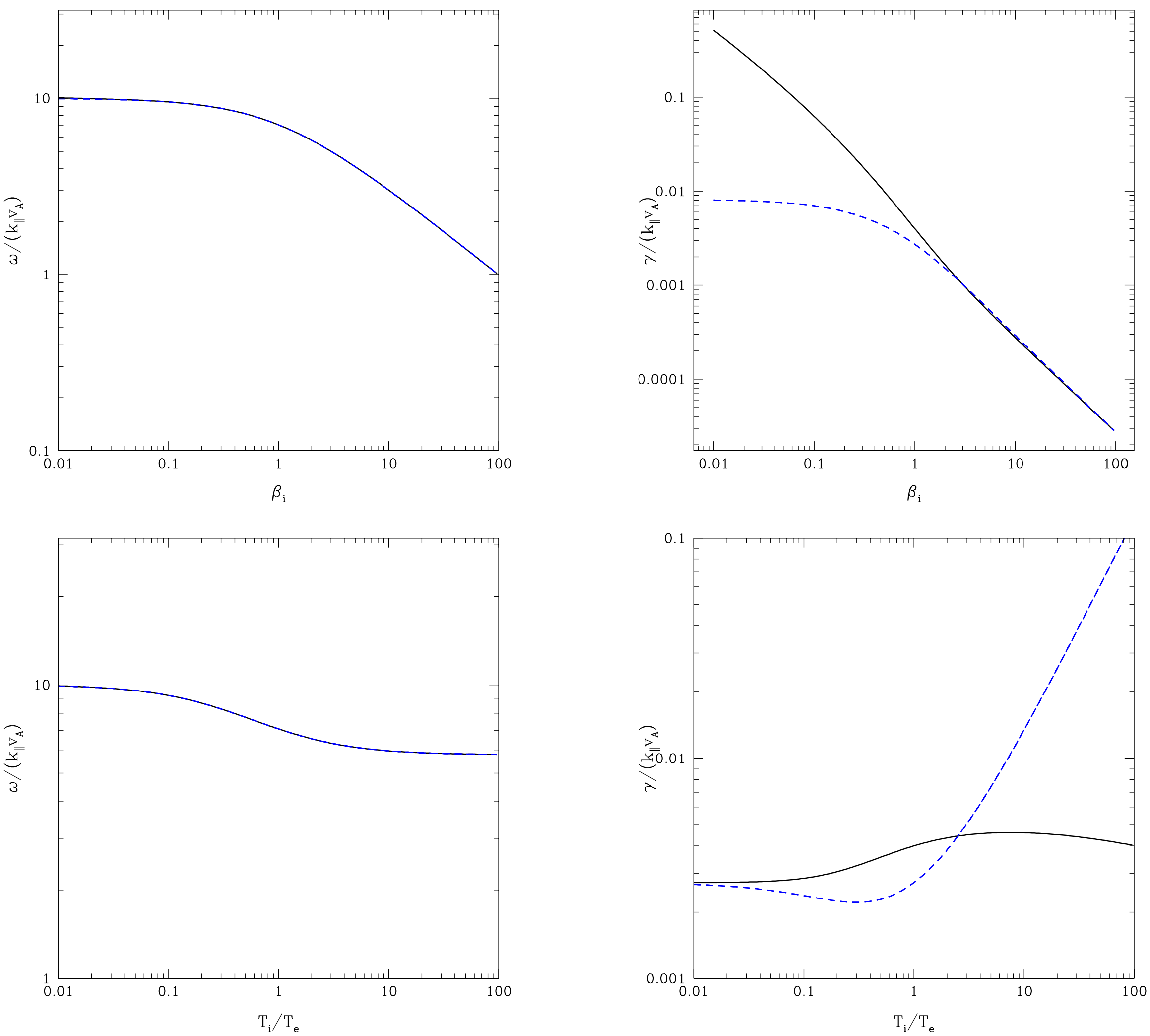

Figure 11.2: Semi-collisional, $|y| \ll 1$, limit of the highly collisional gyrokinetic dispersion relation. Values of parameters not plotted are $\beta_i = 1$, $T_i/T_e = 1$, and $k_\perp \rho_i = 10$.

Using once again the kinetic Alfvén wave solution to estimate the ordering of the frequency, the secondary parameters assume the following ordering:

$$\begin{array}{rcl} \overline{\omega} = \left(\frac{\alpha_i}{\beta_i/2+2\tau/(1+\tau)}\right)^{1/2} & \sim & \epsilon^{-1/2} \\ \frac{\omega}{k_\parallel v_{te}} = \frac{\overline{\omega}\tau^{1/2}}{\mu^{1/2}\beta_i^{1/2}} & \sim & \epsilon^{1/2} \\ \alpha_e = \frac{\alpha_i}{\mu\tau} & \sim & \epsilon \\ y = -3\frac{i\nu_e}{\omega}\frac{\overline{\omega}^2\tau}{\mu\beta_i} & \sim & \epsilon^{-1/2} \end{array} . \tag{11.84}$$

In this limit, the coefficients in the dispersion relation simplify to

$$\begin{array}{rcl} A & \simeq & \frac{1}{\tau}+\frac{3}{2y} \\ B & \simeq & \frac{1}{2}-\frac{5}{4y} \\ C & \simeq & \frac{5}{6}-\frac{35}{12y} \end{array} . \tag{11.85}$$

Hence, the order of the terms in the dispersion relation is given by

$$\frac{i\nu_e}{\omega}\frac{2\alpha_i}{\mu\beta_i}\left(\underset{(\epsilon^{-1/2},1)}{\frac{A\tau}{\beta_i}} + \underset{(\epsilon^{-1/2},1)}{2B^2} + \underset{(\epsilon^{-1/2},1)}{AC}\right) + \underset{(1,\epsilon^{1/2})}{\frac{2B}{\beta_i}} + \underset{(1,\epsilon^{1/2})}{2B^2} + \underset{(1,\epsilon^{1/2})}{AC} = 0 \tag{11.86}$$

Keeping all terms to order 1, we obtain

$$\frac{1}{2}\overline{\omega}^2 + \frac{i\overline{\nu}_e\alpha_i}{\mu\beta_i}\overline{\omega} - \frac{\alpha_i}{3\tau}\frac{\left(\frac{3\tau}{2\beta_i}-\frac{5}{4}-\frac{35}{12\tau}\right)}{\left(\frac{1}{\beta_i}+\frac{1}{2}+\frac{5}{6\tau}\right)} = 0 \tag{11.87}$$

The solution is

$$\overline{\omega} = -i\frac{\overline{\nu}_e\alpha_i}{\mu\beta_i} \pm \left[\frac{2\alpha_i}{3\tau}\frac{\left(\frac{3\tau}{2\beta_i}-\frac{5}{4}-\frac{35}{12\tau}\right)}{\left(\frac{1}{\beta_i}+\frac{1}{2}+\frac{5}{6\tau}\right)} - \left(\frac{\overline{\nu}_e\alpha_i}{\mu\beta_i}\right)^2\right]^{1/2} \tag{11.88}$$

Comparison of this limit with the numerical solution to (11.74) is shown in Figure 11.3.

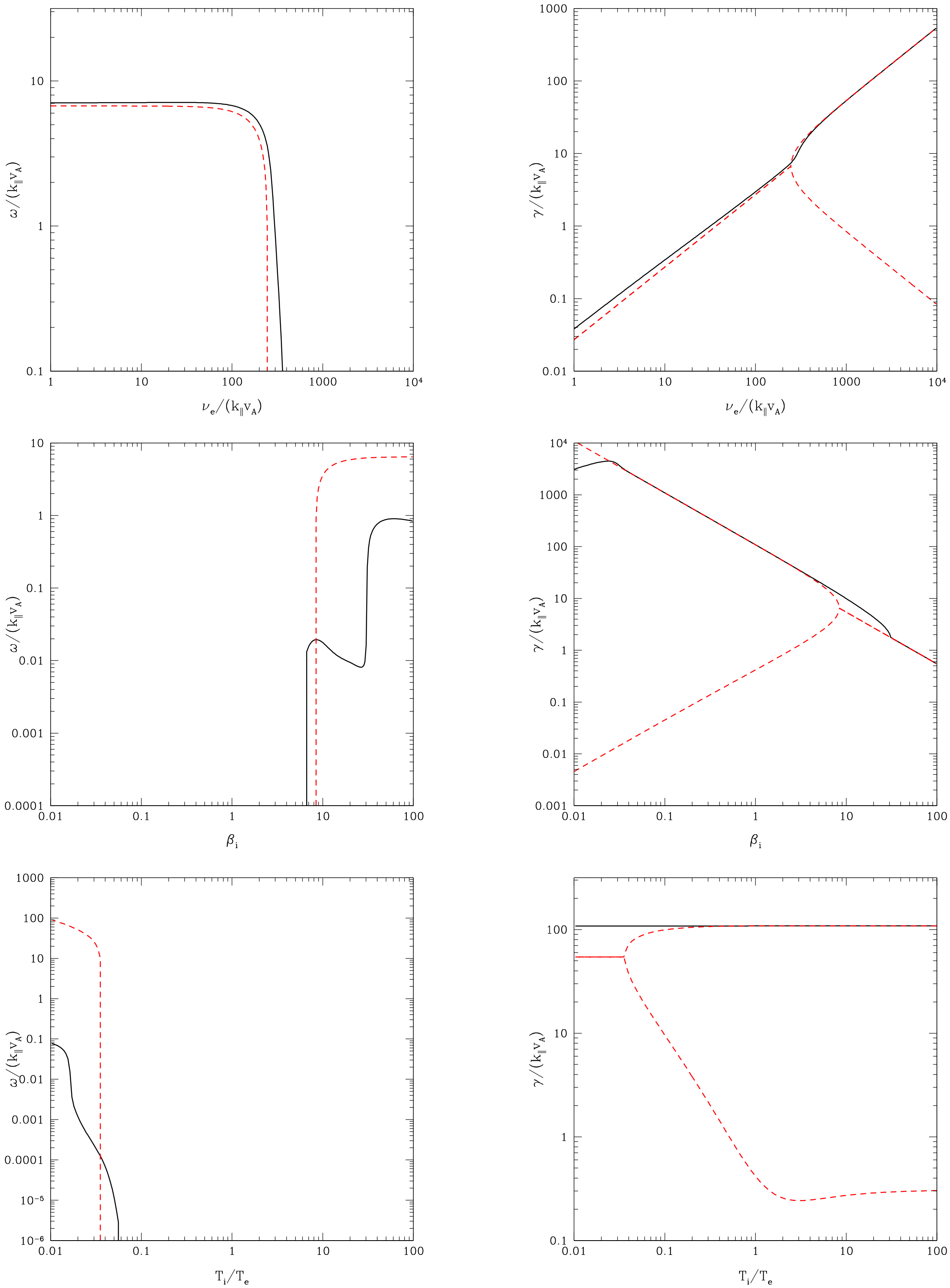

Figure 11.3: Highly collisional, $|y| \gg 1$, limit of the highly collisional gyrokinetic dispersion relation. Values of parameters not plotted are $\beta_i = 1$, $T_i/T_e = 1$, and $k_\perp \rho_i = 10$.

11.5 Strong Electron Collision Dispersion Relation for $\alpha_i \sim 1$

The previous dispersion relation assumed the kinetic Alfvén wave regime was appropriate, which requires $\alpha_i \gg 1$, severely restricting the applicability of the result. Here we combine the standard gyrokinetic distribution function for ions with the strong electron collisional distribution function for electrons to find a dispersion relation valid for finite α_i.

Note that although this dispersion relation is valid for a much greater range of $k_\perp \rho_i$, it still assumes collisions dominate the evolution of the electron distribution function, so it can never recover the correct collisionless damping by electrons, although it does recover the ion collisionless damping.

11.5.1 Solving for the Dispersion Relation

We define a slightly different set of dimensionless basis functions (normalized to ion quantities)

$$\tilde{E}_{\parallel i} \equiv \frac{q_i}{T_i}\left(\phi_{\mathbf{k}} - \frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c}\right) \tag{11.89}$$

$$\tilde{B}_{\perp i} \equiv \frac{q_i}{T_i}\frac{\omega A_{\parallel \mathbf{k}}}{k_\parallel c} \tag{11.90}$$

$$\tilde{B}_{\parallel} \equiv \frac{\delta B_{\parallel \mathbf{k}}}{B_0}. \tag{11.91}$$

Defining the quantities

$$A = 1 + \Gamma_{0i}\xi_i Z(\xi_i) \tag{11.92}$$

$$B = 1 - \Gamma_{0i} \tag{11.93}$$

$$C = \Gamma_{1i}\xi_i Z(\xi_i) \tag{11.94}$$

$$D = \Gamma_{2i}\xi_i Z(\xi_i) \tag{11.95}$$

$$E = \Gamma_{1i} \tag{11.96}$$

$$F = \tau - 4\tau\chi_2 \tag{11.97}$$

$$G = \frac{4}{3}\chi_4 \tag{11.98}$$

$$H = \frac{8}{9}\frac{\chi_6}{\tau} \tag{11.99}$$

$$Q = \frac{i\omega}{\nu_e}\frac{\beta_i \mu}{\overline{\omega}^2} \tag{11.100}$$

we obtain the matrix equation

$$\begin{pmatrix} A+F & B & C+2G \\ A-B+QG & \alpha_i/\overline{\omega}^2 & C+E-QH \\ C+2G & 1-E & D-\frac{2}{\beta_i}-2H \end{pmatrix} \begin{pmatrix} \tilde{E}_{\parallel i} \\ \tilde{B}_{\perp i} \\ \tilde{B}_{\parallel} \end{pmatrix} = 0 \tag{11.101}$$

This yields the dispersion relation

$$
\begin{aligned}
&\left[\frac{\alpha_i(A+F)}{\overline{\omega}^2} - B(A-B+QG)\right]\left[\left(D-\frac{2}{\beta_i}-2H\right)(A+F)-(C+2G)^2\right] \\
&-\left[(A+F)(C+E-QH)-(A-B+QG)(C+2G)\right]\left[(A+F)(1-E)-B(C+2G)\right]=0
\end{aligned}
\tag{11.102}
$$

Part II

AstroGK Code Description

Chapter 12

General Overview of the Code

12.1 Equations Advanced by `AstroGK`

The system for driven gyrokinetics is given by the gyrokinetic equation

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{\mathbf{z}} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} F_{0s} \tag{12.1}$$

and Maxwell's equations, which in the gyrokinetic limit become the quasineutrality condition

$$-\frac{1}{4\pi}\nabla_\perp^2(\phi+\phi_a) + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\phi = q_s \int d^3\mathbf{v}\langle h_s\rangle_{\mathbf{r}}, \tag{12.2}$$

the parallel component of Ampère's law,

$$-\frac{c}{4\pi}\nabla_\perp^2(A_\parallel + A_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle h_s\rangle_{\mathbf{r}}, \tag{12.3}$$

and the perpendicular component of Ampère's law,

$$\frac{c}{4\pi}\nabla_\perp(\delta B_\parallel + \delta B_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v}\langle(\hat{\mathbf{z}}\times\mathbf{v}_\perp)h_s\rangle_{\mathbf{r}}, \tag{12.4}$$

where ϕ_a, $A_{\parallel a}$, and $\delta B_{\parallel a}$ are antenna driving terms. The gyrokinetic potential is

$$\chi = \phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}. \tag{12.5}$$

Note that in the undriven case, the first term of (12.2) is dropped; I am not certain if it is consistent to retain it when attempting to drive the gyrokinetic system electrostatically. To drive electrostatically, it may be necessary to add the driving term to the right-hand side of the gyrokinetic equation as a source of the form $q_s F_{0s}/T_s \partial\langle\phi_a\rangle_{\mathbf{R}_s}/\partial t$.

The gyroaveraged gyrokinetic potential can be written, following the procedure in Chapter (4), as

$$\langle\chi\rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} \left[J_0(\alpha_s)\phi_{\mathbf{k}} - J_0(\alpha_s)\frac{v_\parallel}{c}A_{\parallel\mathbf{k}} + \frac{T_{0s}}{q_s}\frac{2v_\perp^2}{v_{ts}^2}\frac{J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel\mathbf{k}}}{B_0}\right] e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{12.6}$$

We can each Fourier mode in the slightly less cumbersome notation

$$\langle\chi\rangle_{\mathbf{R}_s\mathbf{k}} = \langle\phi\rangle_{\mathbf{R}_s\mathbf{k}} - \frac{v_\parallel}{c}\langle A_\parallel\rangle_{\mathbf{R}_s\mathbf{k}} + \frac{T_{0s}}{q_s}\frac{2v_\perp^2}{v_{ts}^2}\frac{\langle\delta B_\parallel\rangle_{\mathbf{R}_s\mathbf{k}}}{B_0} \tag{12.7}$$

using the definitions

$$\langle\phi\rangle_{\mathbf{R}_s\mathbf{k}} \equiv J_0(\alpha_s)\phi_{\mathbf{k}} \tag{12.8}$$

$$\langle A_\parallel\rangle_{\mathbf{R}_s\mathbf{k}} = J_0(\alpha_s)A_{\parallel\mathbf{k}} \tag{12.9}$$

$$\langle\delta B_\parallel\rangle_{\mathbf{R}_s\mathbf{k}} = \frac{J_1(\alpha_s)}{\alpha_s}\delta B_{\parallel\mathbf{k}} \tag{12.10}$$

.

Now, we define two terms to simplify notation

$$\langle\tilde{\phi}\rangle \equiv \langle\phi\rangle_{\mathbf{R}_s\mathbf{k}} + \frac{T_{0s}}{q_s}\frac{2v_\perp^2}{v_{ts}^2}\frac{\langle\delta B_\parallel\rangle_{\mathbf{R}_s\mathbf{k}}}{B_0} \tag{12.11}$$

and

$$\langle\tilde{A}\rangle \equiv \frac{v_\parallel}{c}\langle A_\parallel\rangle_{\mathbf{R}_s\mathbf{k}} \tag{12.12}$$

so that we have

$$\langle\chi\rangle_{\mathbf{R}_s\mathbf{k}} = \langle\tilde{\phi}\rangle - \langle\tilde{A}\rangle \tag{12.13}$$

In `AstroGK`, the distribution function used is g_s, and is related to the distribution function in the gyrokinetic equation above by

$$g_{s\mathbf{k}} = h_{s\mathbf{k}} - \frac{q_s\langle\phi\rangle_{\mathbf{R}_s\mathbf{k}}}{T_{0s}}F_{0s} - \frac{2v_\perp^2}{v_{ts}^2}\frac{\langle\delta B_\parallel\rangle_{\mathbf{R}_s\mathbf{k}}}{B_0}F_{0s}, \tag{12.14}$$

or in the more simplified notation

$$g_{s\mathbf{k}} = h_{s\mathbf{k}} - \frac{q_s\langle\tilde{\phi}\rangle}{T_{0s}}F_{0s}. \tag{12.15}$$

Thus, the gyrokinetic equation becomes

$$\frac{\partial g_{s\mathbf{k}}}{\partial t} + v_\parallel\frac{\partial g_{s\mathbf{k}}}{\partial z} + v_\parallel\frac{\partial\langle\tilde{\phi}\rangle}{\partial z} + \frac{c}{B_0}\left[\langle\tilde{\phi}\rangle - \langle\tilde{A}\rangle, h_s\right] - \left\langle\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s} = -\frac{q_s}{T_{0s}}\frac{\partial\langle\tilde{A}\rangle}{\partial t}F_{0s} \tag{12.16}$$

In terms of $g_{s\mathbf{k}}$, the Maxwell's equations become

$$-\frac{1}{4\pi}\nabla_\perp^2(\phi + \phi_a) + \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(\phi + \langle\langle\phi\rangle\rangle_s) = q_s\int d^3\mathbf{v}\langle g_{s\mathbf{k}}\rangle_{\mathbf{r}}, \tag{12.17}$$

the parallel component of Ampère's law,

$$-\frac{c}{4\pi}\nabla_\perp^2(A_\parallel + A_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle g_{s\mathbf{k}} \rangle_\mathbf{r}, \tag{12.18}$$

and the perpendicular component of Ampère's law,

$$\frac{c}{4\pi}\nabla_\perp(\delta B_\parallel + \delta B_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} \langle (\hat{\mathbf{z}} \times \mathbf{v}_\perp) h_s \rangle_\mathbf{r}, \tag{12.19}$$

where ϕ_a, $A_{\parallel a}$, and $\delta B_{\parallel a}$ are antenna driving terms. The gyrokinetic potential is

$$\chi = \phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}. \tag{12.20}$$

12.1.1 OLD

Dropping the species subscript s and equilibrium subscripts 0, using $\mathcal{C}(h)$ to denote the collision operator, and substituting for $\langle\chi\rangle$, we get

$$\frac{\partial h}{\partial t} + v_\parallel \frac{\partial h}{\partial z} + \frac{c}{B}\left[\tilde{\phi} - A, h\right] - \langle \mathcal{C}(h) \rangle = \frac{q}{T}\frac{\partial(\tilde{\phi} - A)}{\partial t} F \tag{12.21}$$

Next, we define an alternative version of the gyrokinetic collision operator,

$$g \equiv h - \frac{q\tilde{\phi}}{T} F \tag{12.22}$$

Thus, the relation between g and h is

$$g \equiv h - \frac{q\langle\phi\rangle}{T} F - \frac{m v_\perp^2}{T}\frac{\langle \delta B_\parallel \rangle}{B_0} F \tag{12.23}$$

Writing the gyrokinetic equation in terms of g gives

$$\frac{\partial g}{\partial t} + v_\parallel \frac{\partial g}{\partial z} + \frac{q}{T} v_\parallel F \frac{\partial \tilde{\phi}}{\partial z} + \frac{c}{B}\left[\tilde{\phi} - A, h\right] - \langle \mathcal{C}(h) \rangle = -\frac{q}{T} F \frac{\partial A}{\partial t} \tag{12.24}$$

12.2 Variables

12.2.1 General Variables

The dimensions of the variables in the code are:

Space	x, y, and z
Energy	$E = m_s v^2/2$
Pitch Angle	$\lambda = v_\perp^2/(v^2 B_0)$
Time	t

The variables in the code are:

Distribution Function	$g(x, y, z, E, \lambda, t)$
Scalar Potential	$\phi(x, y, z, t)$
Parallel Vector Potential	$A_{\parallel}(x, y, z, t)$
Parallel Magnetic Field Perturbation	$\delta B_{\parallel}(x, y, z, t)$

The treatment of each of the dimensions of these variables are:

x	Fourier Spectral
y	Fourier Spectral
z	Compact Finite Difference
E	Spectral Integration by quadrature
λ	Finite difference and spectral integration
t	Linear terms are implicit, nonlinear terms are 3rd order Adams-Bashforth

12.3 General Flow of the Code

1. Initialization
2. Main Timestep Loop (`advance_implicit` in `fields_implicit.f90`)
   (a) Antenna Amplitudes
   (b) $\mathbf{E} \times \mathbf{B}$ shear
   (c) Advance Distribution Function (Initial)
      i. Nonlinear Terms (3rd order Adams-Bashforth)
      ii. Invert Matrix
      iii. Collisions
   (d) Add antenna driving term to fields
   (e) Advance Fields (Implicit)
   (f) Advance Distribution Function (Final—same as above)
   (g) Save for restart if necessary
   (h) Loop Diagnostics
      i. Heating
      ii. Density and Velocity fluctuations
      iii. External Current
      iv. Movie
      v. Field magnitudes by mode
      vi. Fluxes
      vii. Polar spectra (Raw and log-averaged)
      viii. Antenna frequency sweep
      ix. Nonlinear heat fluxes
      x. Linear output
   (i) Check timestep Courant condition
   (j) Check for stop
3. Final Diagnostics
   (a) Output Distribution Function (Linear)
   (b) Final Fields
   (c) $E_{\|}$
   (d) Final moments
   (e) Save for restart
   (f) Finish NetCDF
   (g) $k_{\|}$-field line following spectra
4. End

12.4 Main Algorithm

The code is designed under a basis of the Beam-Warming algorithm which discretizes the finite difference in z and t. Other component of the code is, in some sense, attached to the main frame in a consistent way. In this section, we will describe the basic flow of the linear algorithm on the basis of Beam-Warming scheme. This corresponds to a part of the steps 2 (c), (e), (f) in the flow of the code. Other details are described separately in the following chapters.

In `AstroGK` the GK eqn and the coupled field eqns (we call this system coupled GK eqns) are written in the semi-Fourier space k_x-k_y-z and real space in E and λ. Since the linear terms in the coupled GK eqns do not include any differentiation except for z and t, the following procedure may be carried out independently for k_x, k_y, λ and E, which enables an efficient parallelization.

Here we consider an electrostatic case for simplicity since the electromagnetic case is merely an extension to a largeer matrix and fields. We discretize the GK eqn at the middle of the z and t grid points, namely at $i+1/2$ for z and $n+1/2$ for t. For example, $\partial g/\partial t$ term yields

$$\left(\frac{\partial g}{\partial t}\right)_{i+1/2}^{n+1/2} \sim \frac{g_{i+1/2}^{n+1} - g_{i+1/2}^{n}}{\Delta t} \sim \frac{g_{i+1}^{n+1} + g_{i}^{n+1} - g_{i+1}^{n} - g_{i}^{n}}{2\Delta t}, \tag{12.25}$$

where we have approximated $g_{i+1/2}$ by the arithmetic mean of neighboring grid points. In the same way, we may write

$$\left(\frac{\partial \phi}{\partial z}\right)_{i+1/2}^{n+1/2} \sim \frac{\phi_{i+1}^{n+1/2} - \phi_{i}^{n+1/2}}{\Delta z} \sim \frac{\phi_{i+1}^{n+1} + \phi_{i+1}^{n} - \phi_{i}^{n+1} - \phi_{i}^{n}}{2\Delta z}. \tag{12.26}$$

Thus we may symbolically write the GK eqn (12.16) as

$$C_1 g_i^n + C_2 g_{i+1}^n + D_1 g_i^{n+1} + D_2 g_{i+1}^{n+1} = F_1 \phi_i^n + F_2 \phi_{i+1}^n + G_1 \phi_i^{n+1} + G_2 \phi_{i+1}^{n+1} + \text{other terms}, \tag{12.27}$$

where 'other terms' include nonlinear and various source terms. In fact nonlinear term is treated by an independent scheme (3rd order Adams-Bashforth scheme) and added on the right hand side as one of the source terms as is explained later. This implicit scheme is described by Beam-Warming and it is a second order scheme both in space and time. Notice that the scheme is only applied to linear terms and by this we become free from the Courant condition due to the convection along background field direction.

In order for the implicit scheme to work, we have to obtain the field at the future time step. For this we use a response matrix scheme developed by Kotschenreuther [Kotschenreuther et al., 1995]. The scheme first starts from splitting the distribution function into homogeneous and inhomogeneous parts:

$$g^{n+1} = g_{\text{inh}}^{n+1} + g_{\text{h}}^{n+1}, \tag{12.28}$$

where g_{inh} is the solution of GK eqn with setting $\phi^{n+1} = 0$ and g_{h} is that with $g^n = \phi^n = 0$. The g_{inh}^{n+1} is readily obtained as soon as all quantities at time step n is known [Step 2(c)]. Here we assumed that the edge value is know from the boundary conditions. It is subtle for periodic conditions, but we may make two sweeps with specified values at the edge as is explained later.

The next step is to write g_{h}^{n+1} in terms of ϕ^{n+1} using the response matrix. In the initialization, the GK eqn is solved for each j with

$$\begin{aligned}\phi_i^{n+1} &= \delta_{ij} \quad (i \in N_z), \\ \phi^n &= 0, \quad g^n = 0,\end{aligned} \tag{12.29}$$

where δ_{ij} denotes Kornecker's delta and we write the solution $\delta g_i/\delta\phi_j$. For this step we have to solve GK eqn $N_z \times N_s$ times for each $j \in N_z$ and each species. Since the general ϕ^{n+1} may be written by the sum of these Kronecker's delta and both the response of the distribution function and the quasi-neutrality condition are linear, we may write

$$g_{\mathrm{h},i}^{n+1} = \frac{\delta g_i}{\delta \phi_j}\phi_j^{n+1} \tag{12.30}$$

in the symbolic form of the quasi-neutrality condition (12.2):

$$Q_\phi \phi^{n+1} = \mathcal{Q}_d(g^{n+1}). \tag{12.31}$$

This will yield

$$Q_\phi \phi_i^{n+1} - \mathcal{Q}_d\left(\frac{\delta g_i}{\delta \phi_j}\right)\phi_j^{n+1} = \mathcal{Q}_d(g_{\mathrm{inh},i}^{n+1}), \tag{12.32}$$

where the left hand side is merely a matrix of size $N_z \times N_z$ (only for one dimension along the background field line) which may be inverted at each time step [Step 2(e)]. If there is a source term in the quasi-neutrality condition, it is added on the right hand side.

Finally, the GK eqn is solved again with proper future fields and we obtain the distribution function at the future time step [Step 2(f)].

12.5 Velocity space grids

We basically use Legendre polynomials for both energy and pitch angle grid points, so we first review how the Legendre zeros and weights are obtained here, and then describe about the details for each grids in separate subsections.

The zeros and weights of Legendre polynomials are obtained from the routine `gauleg` in Numerical Recipes [Press et al., 1992]. It obtains the zeros by the Newton scheme from the definition of the Legendre polynomial

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \tag{12.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) = \frac{1}{x^2-1}[xP_n(x) - P_{n-1}(x)]. \tag{12.34}$$

Weights are obtained from the formula

$$w_i = \frac{2}{(1-z_i^2)\left[P_n(z_i)'\right]^2} \qquad (i = 1, 2, \dots, N), \tag{12.35}$$

where z_i are the zeros of $P_N(x)$.

12.5.1 Energy grids

This is the documentation of the module `egrid`.

We define a function

$$
\begin{aligned}
x &= \frac{2}{\sqrt{\pi}} \int_0^E dE' \, e^{-E'} \sqrt{E'} \\
&= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, E\right) = \frac{2e^{-E}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \left(E^{k+3/2} \prod_{j=0}^{k} \frac{1}{j+3/2} \right)
\end{aligned} \tag{12.36}
$$

and convert energy integral with Jacobian $\sqrt{E}$ and weight e^{-E} corresponding to Maxwellian into the one for x, where Γ denotes the incomplete gamma function and the summation is taken $k \leq 100$ in the code. This function yields a monotonic transform from $E \in [0, \infty)$ to $x \in [0, 1)$.

The maximum value of the energy `ecut` corresponds to the value `x0` in x:

$$
\mathtt{x0} = \frac{2}{\sqrt{\pi}} \int_0^{\mathtt{ecut}} dE' \, e^{-E'} \sqrt{E'}. \tag{12.37}
$$

Legendre zeros and weights of ($\mathtt{negrid} - 1$)-th order polynomial are used in $x \in (0, \mathtt{x0})$.

The last weight representing the integral for higher energy than `ecut` is obtained by

$$
w_{\mathtt{negrid}} = 1 - \mathtt{x0} \tag{12.38}
$$

as is suggested by Candy and Waltz [Candy and Waltz, 2003].

Therefore, the energy grids are the $\mathtt{negrid} - 1$ zeros of the Legendre polynomial $P_{\mathtt{negrid}-1}[x(E)]$, $\{E_i(z_i) | \, i = 1, \ldots, \mathtt{negrid} - 1\}$, which are obtained from `gauleg`, plus `ecut`. Legendre zeros are concentrated to both edges in x, but in most reasonable cases, it is not in E-space. In other words, you should choose `ecut` and `negrid` in such a way that the grid doesn't concentrate on the edge in E. `gs2` gives a warning message if this is not fulfilled.

The advantage of this scheme is that we can take the weights of the energy integral unity by putting the square root in the variable transformation (12.36). Thus, the exponential accuracy of the quadrature is kept.

12.5.2 Pitch angle grids

This algorithm appears in the module `le_grids`.

By defining

$$
\check{\xi} = \sqrt{1 - \check{\lambda}}, \tag{12.39}
$$

the pitch angle integration for the untrapped particle is converted into

$$
\int_0^1 d\check{\lambda} \, \frac{1}{\sqrt{1 - \check{\lambda}}} = 2 \int_0^1 d\check{\xi}. \tag{12.40}
$$

Legendre zeros and weights are used again to evaluate the integral in $\check{\xi}$. The factor multiplied to the quadrature weights are, thus, 2. In the same way as before, numerical values of Legendre zeros and weights are obtained by `gauleg` in the code.

12.5.3 Michael's Advanced Velocity Space Grids

This implementation uses `nesub` and `nesuper` as variables for the energy grid (with `negrid`=`nesub`+`nesuper`), where `nesub` is the number of points up to the maximum velocity `vcut` (instead of `ecut` in the old implementation). To use this improved implementation, choose `vgrid`=T.

You may choose to only specify `negrid`, and the code will choose default values using

$$\texttt{nesuper} = (\texttt{negrid}/16) + 1$$

up to a maximum value of either 3 or 5 (CHECK), and `nesub`=`negrid`−`nesuper`.

12.6 Tomo's Variable Description

scalars

name	type	default value	description
`nx`	int	0	number of grid points in x real space
`ny`	int	0	number of grid points in y real space
`nakx`	int	`2*((nx-1)/3)+1`	number of valid modes in x
`naky`	int	`(ny-1)/3+1`	half number of valid modes in y
`ntheta`	int	24	number of grid points in z
`ntgrid`	int	`ntheta`/2	half number of grid points in z
`negrid`	int	16	total number of energy grid
`ngauss`	int	8	half number of λ grid points
`ng2`	int	`ngauss` $*\,2$	
`nlambda`	int	`ng2`	number of grid points in $\lambda = \mu/E$
`x0`	real	10.0	box length in x by multiple of 2π
`y0`	real	10.0	box length in y by multiple of 2π
`z0`	real	1.0	box length in z by multiple of 2π
`ecut`	real	6.0	cutoff of energy grid
`igomega`	int	0	`ig` to output in 2d

1d arrays

name	type	dim	description
`akx`	real	`nakx`	$\hat{k}_x$ wavenumbers in x (reversed in the middle)
`aky`	real	`naky`	$\hat{k}_y$ wavenumbers in y
`al`	real	`nlambda`	pitch-angle grid $\lambda = \mu/E$
`wl`	real	`nlambda`	pitch-angle weights ($= \int_0^1 \frac{1}{\sqrt{1-\lambda}} \cdot d\lambda$)
`vperp2`	real	`glo`	$\check{\lambda}\check{E} = \check{v}_\perp^2$
`aj0`	real	`glo`	Bessel function $J_0(a)$
`aj1`	real	`glo`	Bessel function $J_1(a)/a$

where $a = k_\perp v_\perp/\Omega$.

2d arrays

name	type	dim	description
`e`	real	`negrid` $\times$ `nspec`	energy grid
`w`	real	`negrid` $\times$ `nspec`	energy weights ($= \frac{1}{2\sqrt{\pi}} \int_0^{\texttt{ecut}} \sqrt{E} e^{-E} \cdot\, dE$)
`anon`	real	`negrid` $\times$ `nspec`	equals unity unless `slowing_down_species`
`vpa`	real	$2 \times$ `glo`	$\check{v}_\parallel$
`vpar`	real	$2 \times$ `glo`	$\frac{Z}{\sqrt{mT}} \frac{\Delta t}{\Delta z} \check{v}_\parallel = \frac{Z}{T} \frac{\Delta t}{\Delta z} \hat{v}_\parallel$
`gamtot`	real	`nakx` $\times$ `naky`	$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} [1 - \Gamma_0(\alpha_s)]$
`gamtot1`	real	`nakx` $\times$ `naky`	$\sum_s q_s n_{0s} \Gamma_1(\alpha_s)$
`gamtot2`	real	`nakx` $\times$ `naky`	$\frac{1}{2} \sum_s q_s T_{0s} \Gamma_2(\alpha_s)$

where $ntg = -$`ntgrid` : `ntgrid`.

3d arrays

name	type	dim	description
`phi`	complex	$ntg \times$ `nakx` $\times$ `naky`	electrostatic field
`apar`	complex	$ntg \times$ `nakx` $\times$ `naky`	$A_\parallel$
`bpar`	complex	$ntg \times$ `nakx` $\times$ `naky`	$\delta B_{\parallel \boldsymbol{k}}$
`g`	complex	$ntg \times 2 \times$ `glo`	distribution function

Chapter 13

AstroGK Algorithm

13.1 Simulation Equations

In the gyrokinetic limit, the evolution of a magnetized plasma with a uniform mean magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ is governed by the Gyrokinetic-Maxwell equations; see Howes et al. (2006) [?] and Schekochihin et al. (2007) [?] for derivations of these equations expressly intended for the study of astrophysical plasmas. The gyrokinetic equation evolves the ring-averaged, non-adiabatic part of the perturbed distribution function $h_s(X, Y, z, v_\parallel, v_\perp, t)$ for plasma species s,

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left\{\langle\chi\rangle_{\mathbf{R}_s}, h_s\right\} = -\frac{q_s F_{0s}}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} + \left(\frac{\partial h_s}{\partial t}\right)_c \tag{13.1}$$

where X and Y are the guiding center positions in the plane perpendicular to the equilibirum magnetic $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ giving a full guiding-center position $\mathbf{R}_s = (X, Y, z)$, the equilibrium distribution function is $F_{0s}(v)$, the gyrokinetic potential $\chi(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \mathbf{v}\cdot\mathbf{A}(\mathbf{r}, t)/c$, and the collision operator is represented by $(\ldots)_c$. Maxwell's equations in the gyrokinetic limit reduce to the quasineutrality condition

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\phi = \sum_s q_s \int d^3\mathbf{v}\langle h_s\rangle_{\mathbf{r}}, \tag{13.2}$$

the parallel component of Ampere's Law,

$$-\frac{c}{4\pi}\nabla_\perp^2(A_\parallel + A_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle h_s\rangle_{\mathbf{r}}, \tag{13.3}$$

and perpendicular component of Ampere's Law,

$$\frac{c}{4\pi}\nabla_\perp \delta B_\parallel = \sum_s q_s \int d^3\mathbf{v}\langle(\hat{\mathbf{z}}\times\mathbf{v}_\perp)h_s\rangle_{\mathbf{r}}. \tag{13.4}$$

The electromagnetic fields are completely described by the scalar potential $\phi(\mathbf{r}, t) = \phi(x, y, z, t)$, the parallel vector potential $A_\parallel(\mathbf{r}, t)$, and the parallel component of the perturbed magnetic field $\delta B_\parallel(\mathbf{r}, t)$; the term $A_{\parallel,a}$ is the driving term from an external antenna current $\mathbf{J}_a = -\frac{c}{4\pi}\nabla_\perp^2 A_{\parallel a}$. The equilibrium distribution function is a Maxwellian, uniform in space,

$$F_{0s}(v,t) = \frac{n_{0s}}{\pi^{3/2} v_{ts}^3} \exp\left(-\frac{v^2}{v_{ts}^2}\right). \tag{13.5}$$

The nonlinear term in (13.1) is given in Poisson bracket notation, defined by

$$\{\langle\chi\rangle_{\mathbf{R}_s}, h_s\} = \hat{\mathbf{z}} \cdot \left(\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial \mathbf{R}_s} \times \frac{h_s}{\partial \mathbf{R}_s}\right). \tag{13.6}$$

In the Gyrokinetic-Maxwell equations, the ring average of a function $a(\mathbf{r},t)$ at constant guiding center $\mathbf{R}_s$ is given by

$$\langle a(\mathbf{r},t)\rangle_{\mathbf{R}_s} = \frac{1}{2\pi}\int_0^{2\pi} d\phi a(\mathbf{R}_s - \mathbf{v}_\perp \times \hat{\mathbf{z}}/\Omega_s, t) \tag{13.7}$$

and the ring average of a function $a(\mathbf{R}_s,\mathbf{v},t)$ at constant position $\mathbf{r}$ is given by

$$\langle a(\mathbf{R}_s,\mathbf{v},t)\rangle_{\mathbf{r}} = \frac{1}{2\pi}\int_0^{2\pi} d\phi a(\mathbf{r} + \mathbf{v}_\perp \times \hat{\mathbf{z}}/\Omega_s, \mathbf{v}, t). \tag{13.8}$$

The cyclotron frequency is given by Ω_s.

For computational reasons (why? or is it more historical reasons now? This seems to have to do with isolating the Alfvén wave from compressible fluctuations), we transform to a complementary distribution function g_s given by

$$g_s = h_s - \frac{q_s F_{0s}}{T_{0s}}\left\langle \phi - \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s} \tag{13.9}$$

Performing this change leads to the following transformed set of the Gyrokinetic-Maxwell equations:

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} = -v_\parallel \frac{q_s F_{0s}}{T_{0s}} \frac{\partial\left\langle \phi - \frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}}{\partial z} + \frac{q_s F_{0s}}{T_{0s}}\frac{\partial\langle\frac{v_\parallel A_\parallel}{c}\rangle_{\mathbf{R}_s}}{\partial t} + \frac{c}{B_0}\{\langle\chi\rangle_{\mathbf{R}_s}, h_s\} + \left(\frac{\partial h_s}{\partial t}\right)_c \tag{13.10}$$

$$\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\phi - \sum_s \frac{q_s^2}{T_{0s}}\int d^3\mathbf{v}\left\langle\left\langle \phi - \frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} F_{0s} = \sum_s q_s\int d^3\mathbf{v}\langle g_s\rangle_{\mathbf{r}}, \tag{13.11}$$

$$-\frac{c}{4\pi}\nabla_\perp^2(A_\parallel + A_{\parallel a}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel \langle g_s\rangle_{\mathbf{r}}, \tag{13.12}$$

$$\frac{c}{4\pi}\nabla_\perp \delta B_\parallel - \sum_s \frac{q_s^2}{T_{0s}}\int d^3\mathbf{v}\left\langle (\hat{\mathbf{z}}\times\mathbf{v}_\perp)\left\langle \phi - \frac{\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s}\right\rangle_{\mathbf{r}} F_{0s} = \sum_s q_s\int d^3\mathbf{v}\langle(\hat{\mathbf{z}}\times\mathbf{v}_\perp)g_s\rangle_{\mathbf{r}}. \tag{13.13}$$

where $\langle\langle\ldots\rangle\rangle_s$ denotes two ring-averages taking in sucessively.

Next, we express the electromagnetic potentials as complex Fourier series in position $\mathbf{r}$ over the perpendicular plane of the form $\phi(\mathbf{r},t) = \sum_{\mathbf{k}_\perp}\phi_{\mathbf{k}}(t)\exp[i\mathbf{k}_\perp\cdot\mathbf{r}]$, and the distribution function as a complex Fourier series in guiding center

$\mathbf{R}_s$ over the perpendicular plane of the form $g_s(\mathbf{R}_s, v_\parallel, v_\perp, t) = \sum_{\mathbf{k}_\perp} g_{s\mathbf{k}}(v_\parallel, v_\perp, t)\exp[i\mathbf{k}_\perp \cdot \mathbf{R}_s]$. The ring-averages take a particularly simple form for the Fourier components; for example, the ring average in the definition of g_s becomes

$$\frac{q_s}{T_{0s}}\left\langle \phi - \frac{\mathbf{v}_\perp \cdot \mathbf{A}_\perp}{c}\right\rangle_{\mathbf{R}_s} F_{0s} = \sum_{\mathbf{k}_\perp} e^{i\mathbf{k}_\perp\cdot\mathbf{R}_s}\left[J_0(a_s)\frac{q_s}{T_{0s}}\phi_{\mathbf{k}} + \frac{2v_\perp^2}{v_{ts}^2}\frac{J_1(a_s)}{a_s}\frac{\delta B_{\parallel\mathbf{k}}}{B_0}\right]F_{0s}, \tag{13.14}$$

where J_0 and J_1 are the zeroth- and first-order Bessel functions, $a_s = k_\perp v_\perp/\Omega_s$, and the thermal velocity is defined by $v_{ts} = \sqrt{2T_{0s}/m_s}$.

The Fourier tranformed Gyrokinetic-Maxwell equations then become

$$\frac{\partial g_{s\mathbf{k}}}{\partial t} + v_\parallel \frac{\partial g_{s\mathbf{k}}}{\partial z} = -v_\parallel J_0(a_s)F_{0s}\frac{\partial}{\partial z}\left(\frac{q_s\phi_{\mathbf{k}}}{T_{0s}}\right) - v_\parallel \frac{2v_\perp^2}{v_{ts}^2}\frac{J_1(a_s)}{a_s}\frac{\partial}{\partial z}\left(\frac{\delta B_{\parallel\mathbf{k}}}{B_0}\right) + \sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\frac{c}{B_0}\left\{\langle\chi\rangle_{\mathbf{R}_s\mathbf{k}'}, h_{s\mathbf{k}''}\right\}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'') + \left(\frac{\partial h_s}{\partial t}\right)_{c\mathbf{k}} \tag{13.15}$$

$$\sum_s q_s n_{0s}(1-\Gamma_{0s})\frac{q_s\phi_{\mathbf{k}}}{T_{0s}} + \sum_s q_s n_{0s}\Gamma_{1s}\frac{\delta B_{\parallel\mathbf{k}}}{B_0} = \sum_s q_s \int d^3\mathbf{v} J_{0s} g_{s\mathbf{k}} \tag{13.16}$$

$$-\frac{ck_\perp^2}{4\pi}(A_{\parallel\mathbf{k}} + A_{\parallel a\mathbf{k}}) = \sum_s q_s \int d^3\mathbf{v} v_\parallel J_{0s} g_{s\mathbf{k}}, \tag{13.17}$$

$$\frac{ck_\perp B_0}{4\pi}\frac{\delta B_{\parallel\mathbf{k}}}{B_0} - \sum_s q_s n_{0s} v_{ts}\left(\frac{k_\perp\rho_s}{2}\right)\Gamma_{1s}\frac{q_s\phi_{\mathbf{k}}}{T_{0s}} + \sum_s q_s n_{0s} v_{ts}\left(\frac{k_\perp\rho_s}{2}\right)\Gamma_{2s}\frac{\delta B_{\parallel\mathbf{k}}}{B_0} = \sum_s q_s v_{ts}\left(\frac{k_\perp\rho_s}{2}\right)\int d^3\mathbf{v}\frac{2v_\perp^2}{v_{ts}^2}\frac{J_1(a_s)}{a_s} g_{s\mathbf{k}} \tag{13.18}$$

Normalizing these equations, we find (dropping the hats)

$$\frac{\partial g_{s\mathbf{k}}}{\partial t} + v_{\parallel s}\frac{T_{0s}}{m_s}\frac{\partial g_{s\mathbf{k}}}{\partial z} = -v_{\parallel s}\frac{q_s}{m_s}J_0(a_s)\frac{\partial\phi_{\mathbf{k}}}{\partial z} - v_{\parallel s}2v_{\perp s}^2\frac{J_1(a_s)}{a_s}\frac{\partial\delta B_{\parallel\mathbf{k}}}{\partial z} + \sum_{\mathbf{k}',\mathbf{k}''}\frac{a_0}{\rho_0}\left\{\langle\chi\rangle_{\mathbf{R}_s\mathbf{k}'}, h_{s\mathbf{k}''}\right\}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'') + \left(\frac{\partial h_s}{\partial t}\right)_{c\mathbf{k}} \tag{13.19}$$

$$\phi_{\mathbf{k}}\sum_s \frac{q_s^2}{T_{0s}}(1-\Gamma_{0s}) + \delta B_{\parallel\mathbf{k}}\sum_s \Gamma_{1s} = \sum_s q_s\int d^3\mathbf{v}_s J_{0s} g_{s\mathbf{k}} \tag{13.20}$$

$$-\frac{k_\perp^2}{2\beta_i}\left(A_{\parallel\mathbf{k}} + A_{\parallel a\mathbf{k}}\right) = \sum_s q_s\frac{T_{0s}}{m_s}\int d^3\mathbf{v}_s v_{\parallel s} J_{0s} g_{s\mathbf{k}}, \tag{13.21}$$

$$\delta B_{\parallel\mathbf{k}}\left(\frac{2}{\beta_i} + \sum_s T_{0s}\Gamma_{2s}\right) - \phi_{\mathbf{k}}\sum_s q_s\Gamma_{1s} = \sum_s T_{0s}\int d^3\mathbf{v}_s 2v_{\perp s}^2\frac{J_1(a_s)}{a_s} g_{s\mathbf{k}} \tag{13.22}$$

13.2 Simplified Algorithm

In general, we can discuss a simplified set of equations that has all of the same properties as the Gyrokinetic-Maxwell Equations. Thus, we consider the following of two equations governing the evolution of the distribution function $f(y,z,v,t)$ and the field $\phi(y,z,t)$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = a(v)\frac{\partial \phi}{\partial z} + b(v)\frac{\partial \phi}{\partial t} + \mathcal{N}(v, \partial\phi/\partial y, \partial f/\partial y) + \mathcal{C}(v,f) \tag{13.23}$$

$$\phi = \int_{-\infty}^{\infty} dv \quad c(v) f \tag{13.24}$$

The distribution function $f(y,z,v,t)$ is periodic in the two spatial dimensions (y,z), and the nonlinear term $\mathcal{N}$ is a function of derivatives of f and ϕ in the y-direction. If we substitute (13.24) into (13.23), it is clear that this is an integro-differential equation for $f(y,z,v,t)$.

We can express the distribution function and field in terms of Fourier series in y—for example, $\phi(y,z,t) = \sum \phi_k(z,t)e^{iky}$, where the sum is over all possible wavenumbers k. Thus, the equations for the Fourier coefficients of the distribution function $f_k(z,v,t)$ and the field $\phi_k(z,t)$ become

$$\frac{\partial f_k}{\partial t} + v\frac{\partial f_k}{\partial z} = a(v)\frac{\partial \phi_k}{\partial z} + b(v)\frac{\partial \phi_k}{\partial t} + \mathcal{N}_k(v, \phi_1, \ldots, \phi_{N_y}, f_1, \ldots, f_{N_y}) + \mathcal{C}_k(v, f_k) \tag{13.25}$$

$$\phi_k = \int_{-\infty}^{\infty} dv \quad c(v) f_k \tag{13.26}$$

Some important properties of these equations are:

1. For all terms but the collision term $\mathcal{C}$, the velocity space coordinate appears merely as a parameter
2. For all terms but the nonlinear term $\mathcal{N}$, the evolution of a Fourier mode k is independent of the other Fourier coefficients $k' \neq k$
3. All terms are linear save for the nonlinear term $\mathcal{N}$.

13.2.1 Implicit Advance of Linear Terms

Now, let us describe how the linear terms may be advanced nonlinearly. This implicit method for the advance of an integro-differential equation was first used by Kotschenreuther [Kotschenreuther et al., 1995]; we will apply this method to our system with the nonlinear term $\mathcal{N}$ and collision term $\mathcal{C}$ removed

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = a(v)\frac{\partial \phi}{\partial z} + b(v)\frac{\partial \phi}{\partial t} \tag{13.27}$$

$$\phi = \int dv \quad c(v) f. \tag{13.28}$$

Since we have dropped the nonlinear term, the Fourier modes are independent so we drop the subscript k from the Fourier coefficients f_k and ϕ_k. The differencing is done according to a Beam-Warming algorithm [**?**], where the time and space derivatives are given by

$$\frac{\partial f}{\partial t} = \frac{1}{2}\left(\frac{f^{n+1}_{i,j} - f^{n}_{i,j}}{\Delta t} + \frac{f^{n+1}_{i+1,j} - f^{n}_{i+1,j}}{\Delta t}\right) \tag{13.29}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{f^{n}_{i+1,j} - f^{n}_{i,j}}{\Delta z} + \frac{f^{n+1}_{i+1,j} - f^{n+1}_{i,j}}{\Delta z}\right) \tag{13.30}$$

where the discrete variables are given by $f^n_{i,j} = f(i\Delta z, j\Delta v, n\Delta t)$ with $i = 1, \ldots, N_z$ and $j = 1, \ldots, N_v$; the field is similarly given by $\phi^n_i = f(i\Delta z, n\Delta t)$. We can write the differencing of (13.27) symbolically as

$$D_j f^n_{i,j} + E_j f^n_{i+1,j} + F_j f^{n+1}_{i,j} + G_j f^{n+1}_{i+1,j} = H_j \phi^n_i + I_j \phi^n_{i+1} + J_j \phi^{n+1}_i + K_j \phi^{n+1}_{i+1} \tag{13.31}$$

and of (13.28) as

$$\phi^{n+1}_i = \sum_{j'} c_{j'} f^{n+1}_{i,j'} \tag{13.32}$$

Let us now consider the method by which we may step forward implicitly from the known values at the last timestep f^n and ϕ^n to the unknown values at the next timestep f^{n+1} and ϕ^{n+1}. Also, we apply a known boundary condition $f^n_{i=1,j} = 0$.

13.2.2 Brute-Force Implicit Solve

We can implement a brute-force implicit solution by substituting (13.32) into (13.31) to obtain the linear integro-differential equation

$$D_j f^n_{i,j} + E_j f^n_{i+1,j} + F_j f^{n+1}_{i,j} + G_j f^{n+1}_{i+1,j} = H_j \sum_{j'} c_{j'} f^n_{i,j'} + I_j \sum_{j'} c_{j'} f^n_{i+1,j'} + J_j \sum_{j'} c_{j'} f^{n+1}_{i,j'} + K_j \sum_{j'} c_{j'} f^{n+1}_{i+1,j'} \tag{13.33}$$

I can rewrite this into a form that makes the form of the matrix more apparent

$$G_j f^{n+1}_{i+1,j} - K_j \sum_{j'} c_{j'} f^{n+1}_{i+1,j'} + F_j f^{n+1}_{i,j} - J_j \sum_{j'} c_{j'} f^{n+1}_{i,j'} = -D_j f^n_{i,j} - E_j f^n_{i+1,j} + H_j \sum_{j'} c_{j'} f^n_{i,j'} + I_j \sum_{j'} c_{j'} f^n_{i+1,j'} \tag{13.34}$$

Given the values at the last timestep f^n and the boundary conditions $f_{i=1,j}$, we end up with $N_v(N_z - 1)$ linear difference equations. If the vector of unknown values is written with the j index increasing more rapidly, the form of the matrix is block bidiagonal as shown below:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ B & A & 0 & 0 \\ 0 & B & A & 0 \\ 0 & 0 & B & A \end{pmatrix} \begin{pmatrix} \tilde{f}^{n+1}_2 \\ \tilde{f}^{n+1}_3 \\ \tilde{f}^{n+1}_4 \\ \tilde{f}^{n+1}_5 \end{pmatrix} = \begin{pmatrix} \tilde{S}_2 + \tilde{C} \\ \tilde{S}_3 \\ \tilde{S}_4 \\ \tilde{S}_5 \end{pmatrix} \tag{13.35}$$

where the matrix blocks A and B both have dimensions $N_v \times N_v$ with elements given by

$$A_{kl} = G_k \delta_{kl} - K_k c_l \tag{13.36}$$

and

$$B_{kl} = F_k \delta_{kl} - J_k c_l \tag{13.37}$$

for $k = 1, \dots, N_v$ and $l = 1, \dots, N_v$. The source terms are due to the values at the last timestep f^n and are given by $S_{i,j} = -D_j f^n_{i-1,j} - E_j f^n_{i,j} + H_j \sum_{j'} c_{j'} f^n_{i-1,j'} + I_j \sum_{j'} c_{j'} f^n_{i,j'}$ for $i = 2, \dots, N_z$ and $j = 1, \dots, N_v$. The boundary conditions at $i = 1$ are imposed by the terms $C_j = -F_j f^{n+1}_{1,j} + J_j \sum_{j'} c_{j'} f^{n+1}_{1,j'}$ for $j = 1, \dots, N_v$.

In principle, solution of this equation is simple and the problem is solved. In practice, however, this matrix incorporates information from all N_z meshpoints along the field line with all of the meshpoints from velocity space (pitch angle N_λ and energy N_E) and species N_s. For a large run with $N_z = 128$, $N_\lambda = 64$, $N_E = 32$, and $N_s = 2$, this yields a matrix of dimensions $524,288 \times 524,288$. But we can use the linear properties of the equation to break this single large matrix solution into a large number of small matrix solutions. Of course, sparse matrix solvers may be able to do this just as efficiently, but for the moment we will focus on the presently implemented approach.

From the point of view of parallelization for scaling to thousands of processors, this approach is not optimal because it requires information from the all N_z meshpoints, and all meshpoints from velocity space (N_λ N_E) and species N_s. Because velocity space and species information appears in the collisionless gyrokinetic equation only as parameters, a more efficient parallelization scheme can be introduced which takes advantage of this characteristic.

13.2.3 Kotschenreuther's Green's Function Approach

Because equations (13.27)–(13.28) are linear, the solution to the equations may consist of any linear combination of solutions to parts of the equation. Thus, we may split the solution at timestep $n+1$ into an inhomogenous piece that depends only on the known distribution function and field values at timestep n and a homogenous piece that depends only on the field quantities at timestep $n+1$. These field quantities may then be solved as separate step using a Green's function approach.

First, we split the distribution function into an inhomogeneous piece $f^{n+1}_{(0)i,j}$ and a homogeneous piece $f^{n+1}_{(1)i,j}$,

$$f^{n+1}_{i,j} = f^{n+1}_{(0)i,j} + f^{n+1}_{(1)i,j}. \tag{13.38}$$

These two pieces of the solution solve the equations

$$D_j f^n_{i,j} + E_j f^n_{i+1,j} + F_j f^{n+1}_{(0)i,j} + G_j f^{n+1}_{(0)i+1,j} = H_j \phi^n_i + I_j \phi^n_{i+1} \tag{13.39}$$

$$F_j f^{n+1}_{(1)i,j} + G_j f^{n+1}_{(1)i+1,j} = J_j \phi^{n+1}_i + K_j \phi^{n+1}_{i+1} \tag{13.40}$$

Given the boundary condition $f_{i=1,j}$ and the distribution function and field values at the last timestep f^n and ϕ^n, (13.39) can be solved immediately for the inhomogeneous solution $f^{n+1}_{(0)i,j}$. The matrix equation of (13.39) takes the form

$$\begin{pmatrix} G_j & 0 & 0 & 0 \\ F_j & G_j & 0 & 0 \\ 0 & F_j & G_j & 0 \\ 0 & 0 & F_j & G_j \end{pmatrix} \begin{pmatrix} f^{n+1}_{(0)2,j} \\ f^{n+1}_{(0)3,j} \\ f^{n+1}_{(0)4,j} \\ f^{n+1}_{(0)5,j} \end{pmatrix} = \begin{pmatrix} S^n_{2,j} + C_{(0)j} \\ S^n_{3,j} \\ S^n_{4,j} \\ S^n_{5,j} \end{pmatrix} \tag{13.41}$$

where the matrix elements F_j and G_j depend only on the velocity coordinate j and the square matrix is of size $N_z - 1$. The source is given by

$$S^n_{i+1,j} = H_j \phi^n_i + I_j \phi^n_{i+1} - D_j f^n_{(0)i,j} - E_j f^n_{(0)i+1,j} \tag{13.42}$$

and the boundary condition, also dependent only on the velocity coordinate, is given by

$$C_{(0)j} = -F_j f^n_{1,j} \tag{13.43}$$

To solve for the homogeneous solution, however, we must know the values of the field at the next timestep ϕ^{n+1}. To solve for these required values, we use a Green's function approach.

First let us write the field (13.32) as a sum of the moments due to the inhomogeneous and homogeneous parts of the distribution function,

$$\phi^{n+1}_i = m_{(0)i} + m_{(1)i} \tag{13.44}$$

where the moment of the inhomogeneous part of the distribution function

$$m_{(0)i} = \sum_{j'} c_{j'} f^{n+1}_{(0)i,j'} \tag{13.45}$$

is known since we have solved for $f^{n+1}_{(0)i,j}$ and the moment of the homogeneous part is

$$m_{(1)i} = \sum_{j'} c_{j'} f^{n+1}_{(1)i,j'}. \tag{13.46}$$

Effectively, we need to determine the response in the moment of the homogeneous part of the distribution function due to the field at the next timestep ϕ^{n+1}. Because this response is linear, any linear combination of solutions is also a solution. Thus, we take a Green's function approach and find the response in the moment due to a trial function given by

$$\phi^{n+1}_i = \delta_{ik} \tag{13.47}$$

for $k = 2, \ldots, N_z$. Note the the value of ϕ^{n+1}_1 is determined by (13.32) using the boundary condition $f^{n+1}_{i=1,j}$. The matrix form of (13.40) is

$$\begin{pmatrix} G_j & 0 & 0 & 0 \\ F_j & G_j & 0 & 0 \\ 0 & F_j & G_j & 0 \\ 0 & 0 & F_j & G_j \end{pmatrix} \begin{pmatrix} f^{n+1}_{(1)2,j} \\ f^{n+1}_{(1)3,j} \\ f^{n+1}_{(1)4,j} \\ f^{n+1}_{(1)5,j} \end{pmatrix} = \begin{pmatrix} K_j & 0 & 0 & 0 \\ J_j & K_j & 0 & 0 \\ 0 & J_j & K_j & 0 \\ 0 & 0 & J_j & K_j \end{pmatrix} \begin{pmatrix} \phi^{n+1}_2 \\ \phi^{n+1}_3 \\ \phi^{n+1}_4 \\ \phi^{n+1}_5 \end{pmatrix} + \begin{pmatrix} C_{(1)j} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{13.48}$$

where the matrix elements F_j, G_j, J_j, and K_j depend only on the velocity coordinate j and the square matrices are of size $N_z - 1$. The boundary condition, also dependent only on the velocity coordinate, is given by

$$C_{(1)j} = J_j \sum_{j'} c_{j'} f^{n+1}_{1,j'} - F_j f^n_{1,j} \tag{13.49}$$

Calculation of the response matrix employs the following procedure:

1. Given the boundary condition $f^{n+1}_{i=1,j}$, we solve for the homogeneous solution $f^{n+1}_{(1)i,j}$ from (13.40) given a sequence of $N_z - 1$ trial functions chosen to be $\phi^{n+1}_i = \delta_{ik}$ for $k = 2, \dots, N_z$. Note that we must do this for each of the N_v velocity space coordinates j.

2. Summing over the N_v values of velocity space gives the $N_z - 1$ values for the moment of the homogeneous plasma response $m_{(1)i}$ for $i = 2, \dots, N_z$ due to each trial function.

3. All of these responses to the trial functions are combined into an $(N_z - 1) \times (N_z - 1)$ response matrix $\mathcal{M}$ such that the homogeneous moment $m_{(1)i}$ due to any field ϕ^{n+1} is given by

$$m_{(1)i} = \sum_k \mathcal{M}_{ik} \phi^{n+1}_k \tag{13.50}$$

Having calculated the response matrix, the full field (13.32) becomes

$$\phi^{n+1}_i = m_{(0)i}(f^n) + m_{(1)i}(\phi^{n+1}) \tag{13.51}$$

for $i = 2, \dots, N_z$. This produces a dense, square matrix of size $N_z - 1$ that must be inverted to solve implicitly for the field at the new timestep ϕ^{n+1}. Because the response to a field at a given meshpoint z_i is largely concentrated at points nearby z_i, the largest components fall roughly along the diagonal of this dense matrix. Therefore, the matrix is well-conditioned and may be solved easily using a simple Gauss elimination scheme without pivoting.

With the field at the new timestep ϕ^{n+1} solved, (13.40) is trivially solved for the homogeneous contribution to the distribution function at the next timestep,$f^{n+1}_{(1)}$. Thus we have the full solution of (13.31) for the distribution function at the next timestep $f^{n+1} = f^{n+1}_{(0)} + f^{n+1}_{(1)}$.

13.2.4 Comparison of the Brute-Force to the Kotschenreuther Approach

The "direct" solution of the integro-diffferential (13.33) requires the inversion of a block bidiagonal $(N_z-1)N_v \times (N_z-1)N_v$ matrix. The solution of the $(N_z - 1)N_v$ unknown values to give the distribution function at the next timestep f^{n+1} requires use of all the velocity-space values. This leads to an algorithm that has poor scalability.

However, the $(N_z - 1)N_v \times (N_z - 1)N_v$ matrix that must be solved by this technique is constant; therefore it could be solved in an initialization step. The matrix multiplication would probably involve some set of different velocity space values and so may not be ideally scalable. To get the next timestep solution, one needs to multiply the source and boundary condition vector in (13.35) by the inverted matrix. This matrix multiplication requires information from different velocity space coordinates.

The Kotschenreuther solution, on the other hand, goes through the following procedure:

1. The inhomogeneous is trivially solved given the boundary condition $f^{n+1}_{i=1,j}$ and requires only values at the current point in velocity space.

2. The solution of (13.51) requires the inversion of a dense $(N_z - 1) \times (N_z - 1)$ matrix a single time to get the field values, independent of velocity space, at the next timestep ϕ^{n+1}. Note that, to solve for the inhomogeneous moment, $m_{(0)i}(f^n)$, we do need to sum over all N_v velocity space coordinates at spatial position z_i. This *will* require some communication, and I don't currently know if this is handled in a clever way. Once $m_{(0)i}(f^n)$ is calculated, however, the solution of (13.51) does not require the information from other velocity-space coordinates.

3. Calculating the homogeneous solution is trivial with knowledge of ϕ^{n+1} and requires no information from other velocity space coordinates.

The beauty of the Kotschenreuther solution is that it takes advantage of the field's independence of velocity space to eliminate the need for values of the distribution function at other velocity space coordinates.

13.2.5 Nonlinear Term

The nonlinear term is handled using a 3rd-order Adams-Bashforth scheme such that the discretization of

$$\frac{df}{dt} = \mathcal{N} \tag{13.52}$$

is given by

$$\frac{f^{n+1} - f^n}{\Delta t} = \frac{23}{12}\mathcal{N}^n - \frac{4}{3}\mathcal{N}^{n-1} + \frac{5}{12}\mathcal{N}^{n-2} \tag{13.53}$$

Therefore, to write this in terms of the discretized $\partial f/\partial t$, we obtain

$$\frac{1}{2}\left(\frac{f^{n+1}_{k,i,j} - f^n_{k,i,j}}{\Delta t} + \frac{f^{n+1}_{k,i+1,j} - f^n_{k,i+1,j}}{\Delta t}\right) = \text{Linear terms} + \frac{23}{12}\mathcal{N}^n_k - \frac{4}{3}\mathcal{N}^{n-1}_k + \frac{5}{12}\mathcal{N}^{n-2}_k \tag{13.54}$$

13.2.6 Collision Term

The collision term is handled by operator splitting. The nonlinear, collisionless solution is calculated using the Kotschenreuther algorithm including the nonlinear term to obtain f^{n+1}_{*k}. Then the collisional solution is obtained by an implicit solve of

$$f^{n+1}_{k,i} - \Delta t \mathcal{C}_k(f^{n+1}_{k,i}) = f^{n+1}_{*k,i} \tag{13.55}$$

The actual collision operator is a pitch-angle scattering operator of the form

$$\mathcal{C}_k(f^{n+1}_{k,i}) = \nu(E)\frac{1}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f^{n+1}_{k,i}}{\partial \xi}\right] \tag{13.56}$$

where $E = v^2 = (v_\parallel^2 + v_\perp^2$, $\lambda = v_\perp^2/B$, and $\xi = v_\parallel/v = \sqrt{E - \lambda B}/v$.

13.3 Parallelization Scheme

The decomposition of the gyrokinetic distribution function is accomplishing using a flexible parallelization scheme that allows for different layouts. The gyrokinetic distribution function is

$$h(k_x, k_y, z, \lambda, E, s), \tag{13.57}$$

comprising a five-dimensional distribution function for each species s.

13.3.1 Questions

1. Is there a way to implement the direct solution by precomputing the inverse of the block bidiagonal $(N_z - 1)N_v \times (N_z - 1)N_v$ matrix? If the matrix multiplication to solve for all the $(N_z - 1)N_v$ unknown elements of f^{n+1} can be written in a way that minimizes interprocessor communication, this could be a viable strategy.
2. Can we not use a Green's function approach for each of the terms in the inhomogeneous solution as well? Doing this and putting it all together with homogeneous solution is probably somehow equivalent to the direct matrix solution.

Chapter 14

Antenna

14.1 Driving Antenna

The main parameters for the antenna are `amplitude` and frequency `w_antenna`. Setting up the antenna for a single driving wavenumber **k** and for traveling waves (the default is `travel=.true.`), then given `amplitude`$= A$ and `w_antenna`$= \omega_0$, we have

$$\texttt{a\_ant} = \frac{A}{2}(1+i)e^{-i\omega_0 t} \tag{14.1}$$

$$\texttt{b\_ant} = 0. \tag{14.2}$$

NOTE: The actual equation in the code is $a_{n+1} = a_n e^{-i\omega_0 \Delta t}$, which is seen to simply be $a = Ae^{-i\omega_0 t}$ where we take $a_n = Ae^{-i\omega_0 t_n}$, so for $t_{n+1} = t_n + \Delta t$

$$a_{n+1} = Ae^{-i\omega_0 t_{n+1}} = Ae^{-i\omega_0 t_n} e^{-i\omega_0 \Delta t} = a_n e^{-i\omega_0 \Delta t}. \tag{14.3}$$

Now, the driving potential for real driving frequency is given by

$$A_{\|a} = \frac{\texttt{a\_ant} + \texttt{b\_ant}}{\sqrt{2}} e^{ik_\| z} \tag{14.4}$$

and so for `a_ant` and `b_ant` above we find

$$A_{\|a} = \frac{A}{2}\frac{(1+i)}{\sqrt{2}} e^{i(k_\| z - \omega_0 t)} \tag{14.5}$$

The routine `get_volume_average` puts in a factor of 1/2 for all $k_y \neq 0$ modes, so the variable `apar2` (output for `write_flux_line=.true.`) becomes

$$\texttt{apar2} = \frac{|A_{\|a}|^2}{2} = \frac{A^2}{8}. \tag{14.6}$$

Hence, in the magnitude of $A_\|$ we find a factor of $\sqrt{8}$ difference between the analytical theory of Section (6.4.2) and the output file from `GS2`.

14.1.1 Amplitudes for Driving Strong Turbulence

According to the hypothesis of critical balance, the nonlinear energy cascade rate, which can be equated with the perpendicular eddy turnaround rate $\gamma_{nl} = k_\perp v_\perp$, balances with the linear wave frequency $\omega = k_\parallel v_A$ to give

$$k_\perp v_\perp \sim \omega \tag{14.7}$$

The reduced MHD limit of gyrokinetics, the perpendicular fluid velocity is given by

$$\mathbf{u}_\perp = \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla\phi \tag{14.8}$$

and the perpendicular magnetic field perturbation is given by

$$\mathbf{B}_\perp = -\hat{\mathbf{z}} \times \nabla A_\parallel. \tag{14.9}$$

To find the amplitude of $A_\parallel$ that corresponds to the critical balance of the Goldreich-Sridhar strong turbulent cascade, we note that in MHD $v_\perp \sim B_\perp/(4\pi n_i m_i)^{1/2}$, so we get

$$\mathbf{u}_\perp \sim \frac{\omega}{k_\perp} \sim \frac{k_\perp A_\parallel}{\sqrt{4\pi n_i m_i}}. \tag{14.10}$$

Hence, the resulting amplitude is

$$A_\parallel \sim \frac{\omega}{k_\perp^2}\sqrt{4\pi n_i m_i}. \tag{14.11}$$

Normalizing this relation to dimensionless code units, this becomes

$$\frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0} \sim 2\left(\frac{\omega}{k_\parallel v_A}\right)\frac{(k_\parallel \rho_0)}{(k_\perp \rho_0)^2}\left(\frac{\sqrt{4\pi n_i m_i}}{B_0}\right)\left(\frac{q_0 B_0}{m_0 c}\right)\left(\frac{m_0}{2T_0}\right)\frac{v_A v_{t0}\rho_0^2}{\rho_0} \tag{14.12}$$

which becomes

$$\hat{A}_\parallel \sim 2\frac{\overline{\omega}\hat{k}_\parallel}{\hat{k}_\perp^2} \tag{14.13}$$

This is the steady-state amplitude of $A_\parallel$ necessary to be in critical balance.

We now need to connect this steady state amplitude to the driving amplitude A_0 in the `GS2` input file. If the loss of energy in the driving mode is given purely by the linear damping, a Laplace-Fourier solution to the linear system tells us that the saturation amplitude for $A_\parallel$ in the long time limit is given by

$$|A_\parallel(t \to \infty)| = \frac{A_{\parallel 0}}{2D'(\overline{\omega}_0)} \tag{14.14}$$

where A_0 and ω_0 are the amplitude and frequency of the driving antenna and the dispersion relation $D'(\overline{\omega}_0)$ is given by

$$D'(\overline{\omega}) = \frac{\overline{\omega}^2}{\alpha_i A}\left[B(A-B) + \frac{(AE+BC)^2}{2A/\beta_i - AD + C^2}\right] - 1. \tag{14.15}$$

However, this neglects the drain of energy from the driving mode (or modes) through nonlinear transfer of energy to other wave modes. We can roughly include this transfer of energy by choosing a total energy loss rate for a given mode $\gamma_{tot} = \gamma + \gamma_{nl}$ where γ is the linear collisionless damping rate and γ_{nl} is the nonlinear rate of energy transfer to other modes. A Fourier solution for the saturation amplitude in steady state is given by

$$|A_{\|}(t \to \infty)|^2 = \frac{A_{\|0}^2}{(\overline{\omega} - \overline{\omega}_0)^2 + \overline{\gamma_{tot}}^2} \tag{14.16}$$

For almost any system in which we have interest, at the wavenumbers where we drive $\gamma_{nl} \gg \gamma$, so we can neglect the linear damping. Since the critical balance suggests the nonlinear transfer rate $\gamma_{nl} \sim \omega$, we find

$$|A_{\|}(t \to \infty)|^2 \simeq \frac{A_{\|0}^2}{(\overline{\omega} - \overline{\omega}_0)^2 + \overline{\omega}^2}. \tag{14.17}$$

If we drive fairly near the resonant frequency such that $|\overline{\omega} - \overline{\omega}_0| \leq |\overline{\omega}|$, we can neglect the driving term; we arrive at last at the result for the saturated amplitude in a strongly turbulence nonlinear simulation

$$|A_{\|}(t \to \infty)| \simeq \frac{A_{\|0}}{\overline{\omega}}. \tag{14.18}$$

Note, however, that in reality if the decorrelation of the driving frequnecy is of the same order as the driving frequency, then $|\overline{\omega} - \overline{\omega}_0| \simeq |\overline{\omega}|$, so a slightly more accurate solution may be $|A_{\|}(t \to \infty)| \simeq A_{\|0}/(\sqrt{2}\overline{\omega})$.

When the driver is not decorrellated, if travelling waves are specified, the effective amplitude of each mode is actually $A_{\|0}/2$ (see Section (14.1) above for details); if $|\Im(\omega_0)| > 0$, in other words if the driving antenna has a non-zero decorellation time, then the amplitude is $A_{\|0}$. Additionally, taking into account that we stir with N_{stir} driving modes, the total energy is increased by N_{stir}, and thus the saturation amplitude is increased by $\sqrt{N_{stir}}$. This modifies the formula above to

$$|A_{\|}(t \to \infty)| \simeq \frac{A_{\|0}\sqrt{N_{stir}}}{\overline{\omega}}. \tag{14.19}$$

All that remains now is to connect the driving amplitude in the `GS2` input file to the desired final saturated amplitude. Finally, then, the solution for the necessary driving amplitude to be in critical balance for strong turbulence is given by

$$A_{\|0} \sim \frac{2\overline{\omega}^2 \hat{k}_{\|}}{\sqrt{N_{stir}}\hat{k}_{\perp}^2} \tag{14.20}$$

14.1.2 Amplitudes for Driving Strong Turbulence (NEW)

From the cascade model, we have $\omega = \omega_{nl} = C_2 k_\perp v_\perp = C_2 k_\perp \alpha \delta B_\perp / \sqrt{4\pi n_i m_i}$. Since $\delta B_\perp = k_\perp A_\parallel$, we end up with

$$A_\parallel = \frac{\omega\sqrt{4\pi n_i m_i}}{k_\perp^2 C_2 \alpha(k_\perp)} \tag{14.21}$$

Normalizing this equation, we get

$$\frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0} = \left(\frac{\omega}{k_\parallel v_A}\right)\frac{(k_\parallel a_0)}{(k_\perp \rho_0)^2}\left(\frac{q_0 B_0}{m_0 c}\right)\left(\frac{m_0}{2T_0}\right)\frac{2\rho_0 v_{t0}}{C_2\alpha(k_\perp)} \tag{14.22}$$

giving

$$\hat{A}_\parallel = \frac{2\overline{\omega}\hat{k}_\parallel}{\hat{k}_\perp^2 C_2 \alpha(k_\perp)} \tag{14.23}$$

This is the steady state amplitude of $\hat{A}_\parallel$ necessary to satisfy critical balance for the normalized wavenumbers $\hat{k}_\parallel$ and $\hat{k}_\perp$.

Now, we must determine the antenna driving amplitude $\hat{A}_{\parallel 0}$necessary to achieve this steady state amplitude $\hat{A}_\parallel$. The Fourier solution for the saturation amplitude in steady state is given by

$$|\hat{A}_\parallel(t\to\infty)|^2 = \frac{\hat{A}_{\parallel 0}^2}{(\overline{\omega}-\overline{\omega}_0)^2 + \overline{\gamma_{tot}}^2} \tag{14.24}$$

where the damping term includes not only the linear damping but also the nonlinear energy transfer, $\overline{\gamma}_{tot} = \overline{gamma} + \overline{\omega}_{nl}$. For the driving wavenumber, the dissipation is generally weak, $\overline{gamma} \ll \overline{\omega}_{nl}$; as well, in critical balance we have $\overline{\omega} = \overline{\omega}_{nl}$, so this becomes

$$|\hat{A}_\parallel(t\to\infty)|^2 = \frac{\hat{A}_{\parallel 0}^2}{(\overline{\omega}_{nl}-\overline{\omega}_0)^2 + \overline{\omega}_{nl}^2} \tag{14.25}$$

For driving at resonance, $\overline{\omega}_0 = \overline{\omega} = \overline{\omega}_{nl}$ and the denominator reduces to $\overline{\omega}_{nl}^2$; on the other hand, because of the decorrelation of the antenna, the driving frequency drifts such that $(\overline{\omega}_{nl} - \overline{\omega}_0) \simeq \overline{\omega}_{nl}^2$, and therefore the denominator will be $2\overline{\omega}_{nl}^2$. To cover both of these limits, we introduce a parameter $\delta \in [1,2]$, and we have the result

$$|\hat{A}_\parallel(t\to\infty)|^2 = \frac{\hat{A}_{\parallel 0}^2}{\delta\overline{\omega}_{nl}^2}. \tag{14.26}$$

Taking into account that we stir with N_{stir} driving modes, the total energy is increased by N_{stir}, and thus the saturation amplitude is increased by $\sqrt{N_{stir}}$. Hence we find a final result connecting the driving amplitude of the antenna $\hat{A}_{\parallel 0}$ to the saturation amplitude $\hat{A}_\parallel$

$$|\hat{A}_\parallel(t\to\infty)| = \frac{\hat{A}_{\parallel 0}\sqrt{N_{stir}}}{\delta^{1/2}\overline{\omega}}. \tag{14.27}$$

Now we set these two values equal to determine the correct driving amplitude in order to be in critical balance. This gives

$$\hat{A}_{\|0} \sim \frac{2\hat{k}_{\|}\overline{\omega}^2}{\hat{k}_{\perp}^2\sqrt{N_{stir}}}\frac{\delta^{1/2}}{C_2\alpha(k_{\perp})} \tag{14.28}$$

Chapter 15

$\mathbf{E} \times \mathbf{B}$ Shear

15.1

Chapter 16

Distribution Function Advance

16.1 periodic boundary condition

Let us describe how the periodic boundary condition is implemented in (12.27). We will consider $v_\parallel > 0$ case here. The case with $v_\parallel < 0$ goes in the similar way with opposite sweep along z.

First the distribution function is split into homogeneous and inhomogeneous parts:

$$g^{n+1} = g^{n+1}_{(\mathrm{inh})} + g^{n+1}_{(\mathrm{h})}, \tag{16.1}$$

where inhomogeneous part includes all effects from g^n and other terms and homogeneous part includes only g^{n+1} terms. Then (12.27) can be split into two equations:

$$C_1 g^n_i + C_2 g^n_{i+1} + D_1 g^{n+1}_{(\mathrm{inh}),i} + D_2 g^{n+1}_{(\mathrm{inh}),\mathrm{i}+1} = F_1 \phi^n_i + F_2 \phi^n_{i+1} + G_1 \phi^{n+1}_i + G_2 \phi^{n+1}_{i+1} + \text{other terms}, \tag{16.2}$$

$$D_1 g^{n+1}_{(\mathrm{h}),i} + D_2 g_{(\mathrm{h}),i+1} = 0, \tag{16.3}$$

the sum of which recovers the original equation (12.27). It is noted that the inhomogeneous and homogeneous split here is to do with boundary condition and nothing to do with the split in the overview chapter. Thus they are denoted with the bracketed subscript.

Assuming $g_{(\mathrm{inh}),-\mathtt{ntgrid}} = 0$, we can solve (16.2) by a sweep from $i = -\mathtt{ntgrid}$ to $i = \mathtt{ntgrid} - 1$:

$$g^{n+1}_{(\mathrm{inh}),\mathrm{i}+1} = -\frac{1}{D_2}(C_1 g^n_i + C_2 g^n_{i+1} + D_1 g^{n+1}_{(\mathrm{inh}),i} - F_1 \phi^n_i - F_2 \phi^n_{i+1} - G_1 \phi^{n+1}_i - G_2 \phi^{n+1}_{i+1} - \text{other terms}). \tag{16.4}$$

Next by assuming $g'_{(\mathrm{h}),-\mathtt{ntgrid}} = 1$, we can solve (16.3) by another sweep from $i = -\mathtt{ntgrid}$ to $i = \mathtt{ntgrid} - 1$:

$$g'_{(\mathrm{h}),i+1} = -\frac{D_1}{D_2} g'_{(\mathrm{h}),i}. \tag{16.5}$$

By rescaling the homogeneous solution as

$$g_{(\mathrm{h})} = \frac{g_{(\mathrm{inh}),\mathtt{ntgrid}}}{1 - g'_{(\mathrm{h}),\mathtt{ntgrid}}} g'_{(\mathrm{h})}, \tag{16.6}$$

we obtain the periodic solution (16.1).

16.2 Subroutines

Subroutines section from Tomo notes.tex

$\boxed{\texttt{init\_vpar}}$

In `dist_fn.f90`. For untrapped particle

$$\texttt{vpa(ig)} = \sigma\sqrt{E(1-\lambda B(\texttt{ig}))} \tag{16.7}$$

$$\texttt{vpac(ig)} = \frac{1}{2}[\texttt{vpa(ig)} + \texttt{vpa(ig+1)}] \tag{16.8}$$

$$\tag{16.9}$$

and for nonpassing zone

$$\texttt{vpa} = 0 \tag{16.10}$$

$$\texttt{vpac} = \sigma \tag{16.11}$$

where $\sigma = \pm 1$ denotes the coordinate for the sign of $v_\parallel$. And then `vpar` is defined by

$$\texttt{vpar(ig)} = \frac{Z}{\sqrt{mT}}\texttt{tunits}\frac{\Delta t}{2\Delta\theta}\frac{1}{2}[\texttt{gradpar(ig)} + \texttt{gradpar(ig+1)}] * \texttt{vpac(ig)} \tag{16.12}$$

where $\texttt{gradpar}(:) = \texttt{kp}$ in `salpha` option.

$\boxed{\texttt{init\_wstar}}$

$$\texttt{wstar} = \texttt{delt} * \texttt{wunits} * [\texttt{fprim} + \texttt{tprim} * (E-1.5)] \tag{16.13}$$

$\boxed{\texttt{init\_bessel}}$

In `dist_fn.f90`. We define Bessel functions

$$\texttt{aj0} = J_0(\texttt{arg}), \quad \texttt{aj1} = J_1(\texttt{arg})/\texttt{arg}, \tag{16.14}$$

with a formula taken from Abramovitz & Stegun (page 369, 9.4), where

$$\texttt{arg} = \frac{\sqrt{mT}}{|Z|}\sqrt{\texttt{kperp2} * \check{\lambda}\check{E}}, \tag{16.15}$$

$$\texttt{kperp2} = k_x^2 + k_y^2. \tag{16.16}$$

$\boxed{\texttt{init\_invert\_rhs}}$

$$\texttt{ainv}(ntg, \texttt{glo}) = \frac{1}{1 + \texttt{bd} + (1 - \texttt{fexp})\frac{T}{Z}[i\texttt{wd}(1 + \texttt{bd}) + 2\texttt{vp}]} \tag{16.17}$$

$$\sim \left[1 + (1 - f_{\rm exp})\Delta t \left(i\omega_{\rm d} + \frac{2\hat{v}_\parallel}{\Delta\theta}\right)\right]^{-1} \tag{16.18}$$

$$= (\text{coeff. of } f_{i+1}^{n+1} \text{ in the lhs})^{-1} = (D_4 \text{ plus } \texttt{bd} \text{ factor})^{-1} \tag{16.19}$$

$$\texttt{r}(ntg, \texttt{glo}) = \frac{1 - \texttt{bd} + (1 - \texttt{fexp})\frac{T}{Z}[i\texttt{wd}(1 - \texttt{bd}) - 2\texttt{vp}]}{1 + \texttt{bd} + (1 - \texttt{fexp})\frac{T}{Z}[i\texttt{wd}(1 + \texttt{bd}) + 2\texttt{vp}]} \tag{16.20}$$

$$= \left[1 + (1 - f_{\rm exp})\Delta t \left(i\omega_{\rm d} - \frac{2\hat{v}_\parallel}{\Delta\theta}\right)\right] * \texttt{ainv} \tag{16.21}$$

$$= (\text{coeff. of } f_i^{n+1} \text{ in the lhs}) * \texttt{ainv} = (D_3 \text{ plus } \texttt{bd} \text{ factor}) * \texttt{ainv} \tag{16.22}$$

$$\texttt{a}(ntg, \texttt{glo}) = 1 + \texttt{bd} + \texttt{fexp}\frac{T}{Z}\left[-i\texttt{wd}(1 + \texttt{bd}) - 2\texttt{vp}\right] \tag{16.23}$$

$$= 1 - f_{\rm exp}\Delta t \left(i\omega_{\rm d} + \frac{2}{\Delta\theta}\hat{v}_\parallel\right) ?(\text{haven't checked } \omega_{\rm d} \text{ factor}) \tag{16.24}$$

$$= D_2 \text{ (coeff. of } f_i^{n+1} \text{ term) plus } \texttt{bd} \text{ factor} \tag{16.25}$$

$$\texttt{b}(ntg, \texttt{glo}) = 1 - \texttt{bd} + \texttt{fexp}\frac{T}{Z}\left[-i\texttt{wd}(1 - \texttt{bd}) + 2\texttt{vp}\right] \tag{16.26}$$

$$= 1 - f_{\rm exp}\Delta t \left(i\omega_{\rm d} - \frac{2}{\Delta\theta}\hat{v}_\parallel\right) ?(\text{haven't checked } \omega_{\rm d} \text{ factor}) \tag{16.27}$$

$$= D_1 \text{ (coeff. of } f_i^{n} \text{ term) plus } \texttt{bd} \text{ factor} \tag{16.28}$$

where `wd` = `wdrift` (= 0 in reconnection) and `vp` = `vpar`$(ntg, 1, \texttt{glo})$. Note that they are all defined for positive $v_\parallel$.

`fexp` is a complex number. What's the meaning of the imaginary part? The meaning of the real part of `fexp` and `bd` (= `bakdif`) is explained later (in `get_source_term`?).

$\boxed{\texttt{init\_fieldeq}}$

$$\texttt{gamtot} = \sum_s \frac{nZ_s^2}{T_s} \iint (1 - J_0^2) * \texttt{anon}\, d\lambda\, dE + \texttt{kperp2} * \texttt{poisfac} \tag{16.29}$$

$$\texttt{gamtot1} = \sum_s nZ_s \iint 2v_\perp^2 J_0 J_1 * \texttt{anon}\, d\lambda\, dE \tag{16.30}$$

$$\texttt{gamtot2} = \sum_s nT \iint 2v_\perp^4 J_1^2 * \texttt{anon}\, d\lambda\, dE \tag{16.31}$$

Of course these integrations are done with proper weights and Jacobians described elsewhere.

$\boxed{\texttt{invert\_rhs}}$

In `dist_fn.f90`. Add source term

$$\texttt{sourcefac} = \begin{cases} s_0 \exp[(-i\omega_0 + \gamma_0)t] & (t > t_0) \\ \dfrac{1}{2}\left(1 - \cos\dfrac{\pi t}{t_0}\right)\exp[(-i\omega_0 + \gamma_0)t] & (t \le t_0) \end{cases}, \tag{16.32}$$

where s_0 (`source0`), ω_0 (`omega0`), γ_0 (`gamma0`), and t_0 (`t0`) are given parameters specified in `source\_knobs`.

$\boxed{\texttt{get\_source\_term}}$

Writing $f_\phi =$ `fphi`, $f_{\rm exp} =$ `fexp` ($= 1 - \delta$ in Kotschenreuther paper)

$$\texttt{phigavg} = f_\phi J_0 \left[f_{\rm exp}\phi^n + (1 - f_{\rm exp})\phi^{n+1}\right] + f_{A_\perp}\frac{T}{Z}v_\perp^2 J_1 \left[f_{\rm exp}A_\perp^n + (1 - f_{\rm exp})A_\perp^{n+1}\right] \tag{16.33}$$

$$\texttt{apargavg} = f_{A_\parallel} J_0 \left[f_{\rm exp}A_\parallel^n + (1 - f_{\rm exp})A_\parallel^{n+1}\right] \tag{16.34}$$

$$\texttt{ufac} = 2 * \texttt{uprim} + \frac{\sqrt{\pi}}{4}E^{3/2} * \texttt{uprim2} \tag{16.35}$$

The following is for reconnection problem:

$$\texttt{source(ig)} = -2\texttt{vpar(ig)}\phi_m - \frac{Z}{\sqrt{mT}}\texttt{vpac(ig)}\frac{J_0\texttt{(ig)} + J_0\texttt{(ig+1)}}{2}A_{\parallel m} \tag{16.36}$$

where

$$\phi_m = \texttt{phigavg(ig+1)} - \texttt{phigavg(ig)} \sim \Delta\theta\frac{\partial(J_0\phi)}{\partial\theta} \tag{16.37}$$

$$\begin{aligned} A_{\parallel m} &= A_\parallel^{n+1}\texttt{(ig+1)} + A_\parallel^{n+1}\texttt{(ig)} - A_\parallel^n\texttt{(ig+1)} - A_\parallel^n\texttt{(ig)} \\ &\sim 2\Delta t\frac{\partial A_\parallel}{\partial t} \end{aligned} \tag{16.38}$$

$$\texttt{phigavg} = J_0\texttt{(ig)}\left[f_{\rm exp}\phi^n\texttt{(ig)} + (1 - f_{\rm exp})\phi^{n+1}\texttt{(ig)}\right] \tag{16.39}$$

$$\texttt{vpar(ig)} = \frac{Z}{\sqrt{mT}}\frac{\Delta t}{\Delta\theta}\texttt{kp} * \frac{\check{v}_\parallel\texttt{(ig)} + \check{v}_\parallel\texttt{(ig+1)}}{2} \sim \frac{\hat{Z}}{\hat{T}}\frac{\Delta t}{\Delta\theta}\hat{v}_\parallel \tag{16.40}$$

$$\texttt{vpac(ig)} = \frac{\check{v}_\parallel\texttt{(ig)} + \check{v}_\parallel\texttt{(ig+1)}}{2} \tag{16.41}$$

Thus,

$$\texttt{source(ig)} \sim -\frac{\hat{Z}}{\hat{T}}\hat{v}_\parallel\left[\frac{\partial(J_0\phi)}{\partial\theta} + J_0\frac{\partial A_\parallel}{\partial t}\right](2\Delta t) \tag{16.42}$$

where ϕ and $A_\parallel$ are evaluated from both time steps of n and $n+1$. This is the expression you get at `set_source`. Is the sign of the first term in $[\dots]$ okay??

Moreover, if `nonlin` = `.true.`, then add nonlinear terms

$$\texttt{source} = (16.42) + \frac{1}{2}\frac{\texttt{delt}}{\texttt{tnorm}} \times (\text{nonlinear terms}), \tag{16.43}$$

in Euler scheme at the first time step and in second order Adams-Bashforth scheme for the rest. `tnorm` $= \sqrt{2}$ in reconnection runs, and `delt` is multiplied by `tnorm` in `init_run_parameters`. So the factor `delt`/`tnorm` corresponds to the real `delt` specified in the input file. The Δt (= `delt`) in the linear terms is $\sqrt{2}$ times larger than that. The precise form of the nonlinear terms is described in `add_nl`.

Next, we go back to `get_source_term` and around the place where `Do matrix multiplications`... For $\sigma = 1$

$$\texttt{b}(ig, iglo) * g(ig, 1, iglo) + \texttt{a}(ig, iglo) * g(ig + 1, 1, iglo) \tag{16.44}$$

is added to `source(ig)`, which corresponds to the $\texttt{g}^n$ terms arising from the finite difference form of the lhs:

$$\begin{aligned}\frac{\partial f}{\partial t} + i\omega_{\rm d} f + \hat{v}_\parallel \frac{\partial f}{\partial \theta} \sim \frac{1}{2\Delta t}&\left[(f_i^{n+1} + f_{i+1}^{n+1}) - (f_i^n + f_{i+1}^n)\right] \\ +\frac{i\omega_{\rm d}}{2}&\left[(1 - f_{\rm exp})(f_i^{n+1} + f_{i+1}^{n+1}) + f_{\rm exp}(f_i^n + f_{i+1}^n)\right] \\ +\frac{\hat{v}_\parallel}{\Delta\theta}&\left[(1 - f_{\rm exp})(f_{i+1}^{n+1} - f_i^{n+1}) + f_{\rm exp}(f_{i+1}^n - f_i^n)\right]\end{aligned} \tag{16.45}$$

where $\omega_{\rm d} = 0$ in the reconnection problem. For $\sigma = -1$, the sign change of $\hat{v}_\parallel$ is taken care of by multiplying a and b oppositely on $g(ig)$ and $g(ig+1)$, respectively, because the definition of `a` and `b` uses `vpar`(ntg, 1, `glo`) which is the positive part of $\hat{v}_\parallel$.

Okay, let's think about `bakdif` now. It is introduced in order to make $\hat{v}_\parallel \partial_\theta f$ term an upwind difference scheme. As is described in (16.45), everything is evaluated at grid point $i + 1/2$ in θ. Instead of changing the finite differencing of $\partial_\theta f$, we shift the grid point for other terms to be evaluated a little bit forward. Then, the scheme is going to be upwind finite difference.

Let's work on the terms appearing in (16.45), and we write β = `bakdif` for simplicity. Any term evaluated at $i+1/2$ is expressed as follows:

$$f_{i+1/2} = \frac{1}{2}(f_{i+1} + f_i). \tag{16.46}$$

By shifting it forward, we may write it as

$$f_{i+(1+\beta)/2} = \frac{1}{2}[(1+\beta) f_{i+1} + (1-\beta) f_i], \tag{16.47}$$

where $0 \leq \beta \leq 1$ and $\beta = 0$ corresponds to second order centered difference scheme (may β be larger than unity?).

Thus, for the terms in (16.45), they are finite differenced as

$$\left(\frac{\partial f}{\partial t} + i\omega_{\rm d} f\right)_{i+(1+\beta)/2} + \left(\hat{v}_\parallel \frac{\partial f}{\partial \theta}\right)_{i+1/2}$$

$$\sim \frac{1}{2}\left[(1+\beta)\frac{f_{i+1}^{n+1} - f_{i+1}^{n}}{\Delta t} + (1-\beta)\frac{f_i^{n+1} - f_i^n}{\Delta t}\right]$$

$$+\frac{i\omega_{\rm d}}{2}\left\{(1-f_{\rm exp})\left[(1+\beta)f_{i+1}^{n+1} + (1-\beta)f_i^{n+1}\right] + f_{\rm exp}\left[(1+\beta)f_{i+1}^{n} + (1-\beta)f_i^{n}\right]\right\}$$

$$+\frac{\hat{v}_\parallel}{\Delta\theta}\left[(1-f_{\rm exp})(f_{i+1}^{n+1} - f_i^{n+1}) + f_{\rm exp}(f_{i+1}^{n} - f_i^{n})\right]$$

$$= \frac{1}{2\Delta t}\left[\frac{1}{\texttt{ainv}}f_{i+1}^{n+1} + \frac{\texttt{r}}{\texttt{ainv}}f_i^{n+1} - \texttt{a}f_{i+1}^{n} - \texttt{b}f_i^{n}\right]. \tag{16.48}$$

Here comes the question. When you use finite `bakdif`, do you not need to implement it in the source term either?

$\boxed{\texttt{invert\_rhs\_1}}$

Is it okay to use `r` and `ainv` defined for $\hat{v}_\parallel > 0$ in the calculation of `gnew` for `vpar` < 0?

$\boxed{\texttt{getan}}$

This is in `dist_fn.f90`.

$$\texttt{antot} = \sum_s nZ \iint J_0 * \texttt{gnew}\, d\lambda\, dE \tag{16.49}$$

$$\texttt{antota} = \sum_s 2 * \texttt{beta} * nZ\sqrt{\frac{T}{m}} \iint J_0 \check{v}_\parallel * \texttt{gnew}\, d\lambda\, dE$$

$$= \sum_s 2 * \texttt{beta} * nZ \iint J_0 \hat{v}_\parallel * \texttt{gnew}\, d\lambda\, dE \tag{16.50}$$

$$\texttt{antotp} = \sum_s nT \iint J_1 \check{v}_\perp^2 * \texttt{gnew}\, d\lambda\, dE \tag{16.51}$$

$\boxed{\texttt{getfieldeq1}}$

In `dist_fn.f90`.

$$\texttt{fieldeq} = \texttt{antot} - \texttt{gamtot} * \phi + \texttt{gamtot1} * B_\parallel \tag{16.52}$$

$$\texttt{fieldeqa} = \texttt{antota} - k_\perp^2 A_\parallel \tag{16.53}$$

$$\texttt{fieldeqp} = \beta_0\texttt{antotp} + (\beta_0\texttt{gamtot2} + 1)B_\parallel + \frac{\beta_0}{2}\texttt{gamtot1} * \phi \tag{16.54}$$

$\boxed{\texttt{get\_field\_vector}}$

In `fields_implicit.f90`.

$$\texttt{fl}[\texttt{nfield} * (2 * \texttt{ntgrid} + 1), \texttt{nakx}, \texttt{naky}] = \begin{pmatrix} \texttt{fieldeq} \\ \texttt{fieldeqa} \\ \texttt{fieldeqp} \end{pmatrix} \tag{16.55}$$

$\boxed{\texttt{getfield}}$

In `fields_implicit.f90`.

$$\texttt{call get\_field\_vector} \tag{16.56}$$

$$\texttt{u} = -\texttt{aminv} * \texttt{fl} \tag{16.57}$$

$$\texttt{call get\_field\_solution} \tag{16.58}$$

$\boxed{\texttt{get\_field\_solution}}$

In `fields_implicit.f90`.

$$\begin{pmatrix} \texttt{phinew} \\ \texttt{aparnew} \\ \texttt{bparnew} \end{pmatrix} = \texttt{u} \tag{16.59}$$

$\boxed{\texttt{add\_nl}}$

If the time step (`istep`) is advanced from the last call (`istep_last`),

$$\texttt{g2} = \texttt{g1} \tag{16.60}$$

$$\texttt{g1} = ik_x \left[J_0 \left(f_\phi \phi - \hat{v}_\parallel f_{A_\parallel} A_\parallel^n \right) + \frac{2m}{Z} J_1 \hat{v}_\perp^2 f_{A_\perp} A_\perp^n \right] \tag{16.61}$$

$$\texttt{ba} = \mathcal{F}\left(\texttt{g1}\right) \tag{16.62}$$

$$\texttt{g1} = \frac{Z}{T} \left[ik_y \left(J_0 f_\phi \phi + \frac{2m}{Z} J_1 \hat{v}_\perp^2 f_{A_\perp} A_\perp^n \right) \right] + ik_y \texttt{g} \tag{16.63}$$

$$\texttt{gb} = \mathcal{F}\left(\texttt{g1}\right) \tag{16.64}$$

$$\texttt{bracket} = \texttt{ba} * \texttt{gb} * \texttt{kxfac} \tag{16.65}$$

$$\texttt{g1} = ik_y \left[J_0 \left(f_\phi \phi - \hat{v}_\parallel f_{A_\parallel} A_\parallel^n \right) + \frac{2m}{Z} J_1 \hat{v}_\perp^2 f_{A_\perp} A_\perp^n \right] \tag{16.66}$$

$$\texttt{ba} = \mathcal{F}\left(\texttt{g1}\right) \tag{16.67}$$

$$\texttt{g1} = \frac{Z}{T} \left[ik_x \left(J_0 f_\phi \phi + \frac{2m}{Z} J_1 \hat{v}_\perp^2 f_{A_\perp} A_\perp^n \right) \right] + ik_x \texttt{g} \tag{16.68}$$

$$\texttt{gb} = \mathcal{F}\left(\texttt{g1}\right) \tag{16.69}$$

$$\texttt{bracket} = \texttt{bracket} - \texttt{ba} * \texttt{gb} * \texttt{kxfac} \tag{16.70}$$

$$\texttt{g1} = \mathcal{F}\left(\texttt{bracket}\right) \tag{16.71}$$

where $\texttt{kxfac} = 1$ when $\texttt{equilibrium\_option} = \texttt{s} - \texttt{alpha}$.

Chapter 17

Nonlinear Terms

17.1

Chapter 18

Matrix Inversion

18.1

Chapter 19

Collisions

19.1 `AstroGK`'s Implementation of Collisions

Let us be specific about the implementation of collisions and hypercollisionality in `GS2`. This is well described in a note by Greg Hammett dated June 26, 2003; this note also includes insightful discussion that is not included here. I will summarize his results here and update them to include the classical diffusion operator that arises from taking pitch angle derivative with at constant guiding center $\mathbf{R}$ rather than constant position $\mathbf{r}$.

The electron collision operator includes both electron-electron and electron-ion collisions with the equilibrium Maxwellian distribution,

$$\langle \mathcal{C}_e(h_e)\rangle_{\mathbf{R}_e} = \langle \mathcal{C}_{ee}(h_e, F_{0e}) + \mathcal{C}_{ei}(h_e, F_{0i})\rangle_{\mathbf{R}_e} \tag{19.1}$$

while the ion collision operator includes only ion-ion collisions with the equilibrium Maxwellian distribution,

$$\langle \mathcal{C}_i(h_i)\rangle_{\mathbf{R}_i} = \langle \mathcal{C}_{ii}(h_i, F_{0i})\rangle_{\mathbf{R}_i}. \tag{19.2}$$

`Question: What happens to the adiabatic piece of the distribution function in the collision operator?`

19.1.1 Electron Collision Operator

The electron collision operator is given by

$$\langle \mathcal{C}_e(h_e)\rangle_{\mathbf{R}_e} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_e} \nu_e(v)\frac{1}{2}\left\{\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}e}}{\partial \xi}\right] - \frac{v^2}{v_{te}^2}(1+\xi^2)\frac{k_\perp^2\rho_e^2}{2}h_{\mathbf{k}e}\right\} \tag{19.3}$$

where the coefficient, dependent only on the magnitude of velocity, is given by

$$\nu_e(v) = \nu_{ei}\left(\frac{v_{te}}{v}\right)^3\left[Z_i^2 + H_{ss}\left(\frac{v}{v_{te}}\right)\right] \tag{19.4}$$

(NOTE here that in the `GS2`code, the factor of 1/2 in (19.3) is actually absorbed into the constant $\nu_e(v)$ and so actually appears only in (19.4)) with the like-particle collision coefficient is given by

$$H_{ss}(x) = \left(1 - \frac{1}{2x^2}\right)\frac{2}{\sqrt{\pi}}\int_0^x dte^{-t^2} + \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x} \tag{19.5}$$

and with

$$\nu_{ei} = \frac{4\pi e^4 n_e \lambda}{m_e^{1/2}(2T_e)^{3/2}}. \tag{19.6}$$

Here λ denotes the Coulomb logarithm $\lambda = \ln(\Lambda_{ee}) \simeq \ln(\Lambda_{ei})$, the pitch angle coordinate is defined by $\xi \equiv v_\parallel/v$, and the thermal velocity is defined by $v_{te}^2 = 2T_e/m_e$. The first term in (19.3) is the standard pitch-angle collision operator and the second term is a classical-diffusion correction due to the fact that the derivatives $\partial/\partial\xi$ are evaluated at constant guiding center $\mathbf{R}_s$ rather than at constant position $\mathbf{r}$. At the moment the second term is included when `cfac=1.0` in the collisions namelist; this is the default behavior (a change from earlier versions of the code). This correction term is motivated and derived in Schekochihin et al. (2006).

Figure 19.1 plots the variation of the functions $H_{ss}(v/v_{ts})$ and $\nu_e(v/v_{te})$ with energy, or magnitude of velocity.

19.1.2 Ion Collision Operator

Analogous to the electron collision operator, but neglecting ion-electron collisions, the ion collision operator is given by

$$\langle \mathcal{C}_i(h_i)\rangle_{\mathbf{R}_i} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_i}\nu_i(v)\frac{1}{2}\left\{\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}i}}{\partial\xi}\right] - \frac{v^2}{v_{ti}^2}(1+\xi^2)\frac{k_\perp^2\rho_i^2}{2}h_{\mathbf{k}i}\right\} \tag{19.7}$$

where

$$\nu_i(v) = \nu_{ii}\left(\frac{v_{ti}}{v}\right)^3 H_{ss}\left(\frac{v}{v_{ti}}\right) \tag{19.8}$$

and

$$\nu_{ii} = \frac{4\pi Z_i^4 e^4 n_i \lambda}{m_i^{1/2}(2T_i)^{3/2}}. \tag{19.9}$$

Figure 19.1 plots the variation of the function $\nu_i(v/v_{ti})$ with energy, or magnitude of velocity.

19.1.3 Collisional Coefficients

In the `GS2` input file, the collisional coefficients $\texttt{vnewk}_s$ are used as the values of ν_{ei} and ν_{ii} in the electron and ion collision operators above. If we denote these user supplied values as $\texttt{vnewk}_e = \hat{\nu}_{ce}$ and $\texttt{vnewk}_i = \hat{\nu}_{ci}$, then the normalization is given by

$$\hat{\nu}_{cs} = \frac{\nu_{cs}a_0}{v_{t0}}. \tag{19.10}$$

19.1.4 Collisional Heating

To estimate the collisional heating, we take the gyrokinetic equation,

$$\frac{\partial h_s}{\partial t} + v_\parallel\frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \langle\mathcal{C}(h_s)\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t}F_{0s}, \tag{19.11}$$

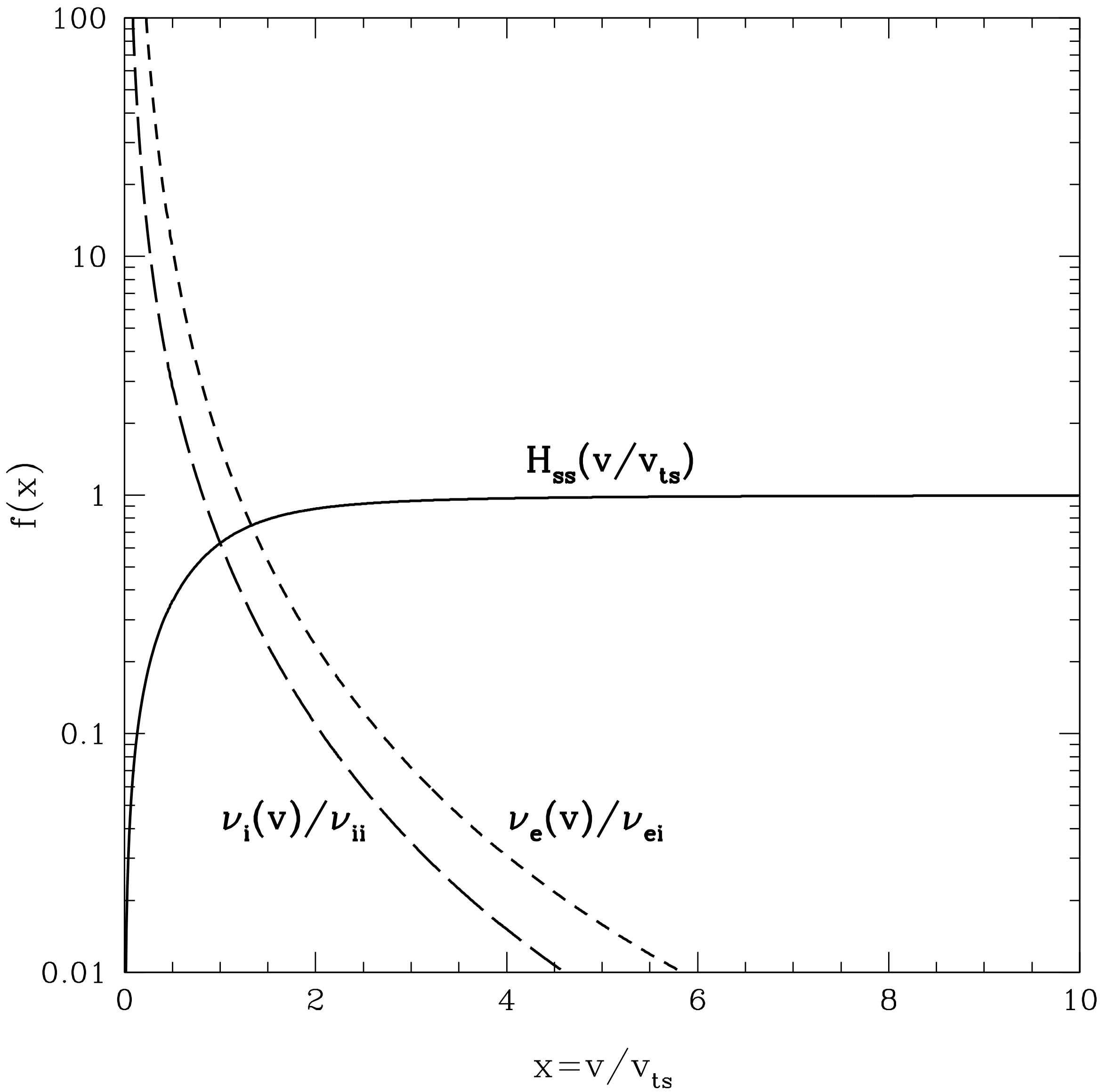

Figure 19.1: Plot of the functions $H_{ss}(v/v_{ts})$, $\nu_e(v/v_{te})/\nu_{ei}$, and $\nu_i(v/v_{ti})/\nu_{ii}$ to demonstrate how these coefficients vary with energy (magnitude of velocity).

multiply by $T_s h_s/F_{0s}$ and integrate over all space and velocity to give the entropy-balance equation. (In fact, to be more precise, in complex space we must actually take $\int_{\mathbf{R}}\int_{\mathbf{v}} T_s h_s^* \mathrm{GK}/F_{0s} + \int_{\mathbf{R}}\int_{\mathbf{v}} T_s h_s \mathrm{GK}^*/F_{0s}$, where GK denotes the gyrokinetic equation; but we will be somewhat loose with notation here.) The second and third terms on the left-hand side give nothing when integrated over all space for periodic boundary conditions, leaving the result

$$\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s \frac{\partial \langle\chi\rangle_{\mathbf{R}_s}}{\partial t} h_s - \frac{d}{dt}\int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{2F_{0s}} h_s^2 + \int \frac{d^3\mathbf{R}_s}{V}\int d^3\mathbf{v}\frac{T_{0s}}{F_{0s}} \langle h_s \mathcal{C}(h_s)\rangle_{\mathbf{R}_s} = 0 \tag{19.12}$$

Let us now take a closer look at the heating from the collisional term. Writing the gyrokinetic distribution function as a Fourier series

$$h_s(\mathbf{R}_s, \mathbf{v}, t) = \sum_{\mathbf{k}} h_{\mathbf{k}s}(\mathbf{k}, \mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{R}_s} \tag{19.13}$$

and using a generalization of (19.3) and (19.7), we find

$$\begin{aligned} &\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s \mathcal{C}(h_s) \rangle_{\mathbf{R}_s} \qquad (19.14) \\ = &\sum_{\mathbf{k}} \sum_{\mathbf{k}'} \int \frac{d^3\mathbf{R}_s}{V} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_s} T_s \int d^3\mathbf{v} \frac{\nu_s(v)}{2F_{0s}} h_{\mathbf{k}'s} \left\{ \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}s}}{\partial \xi} \right] - \frac{v^2}{v_{ts}^2} (1+\xi^2) \frac{k_\perp^2 \rho_s^2}{2} h_{\mathbf{k}s} \right\} \end{aligned}$$

Using the property

$$\int d^3\mathbf{R}_s e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_s} = \delta(\mathbf{k}+\mathbf{k}'), \tag{19.15}$$

the reality condition that

$$h_{-\mathbf{k}s} = h_{\mathbf{k}s}^*, \tag{19.16}$$

and transforming to velocity v, pitch angle $\xi = v_\parallel / v$, and gyrophase angle θ coordinates

$$\int d^3\mathbf{v} = \int_0^\infty v^2 dv \int_{-1}^1 d\xi \int_0^{2\pi} d\theta, \tag{19.17}$$

this simplifies to

$$\begin{aligned} &\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s \mathcal{C}(h_s) \rangle_{\mathbf{R}_s} \qquad (19.18) \\ = &\sum_{\mathbf{k}} \pi T_s \int_0^\infty v^2 dv \frac{\nu_s(v)}{F_{0s}(v)} \left\{ \int_{-1}^1 d\xi h_{\mathbf{k}s}^* \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial h_{\mathbf{k}s}}{\partial \xi} \right] - \frac{v^2}{v_{ts}^2} \frac{k_\perp^2 \rho_s^2}{2} \int_{-1}^1 d\xi (1+\xi^2) |h_{\mathbf{k}s}|^2 \right\}. \end{aligned}$$

We may now perform an integration by parts in pitch angle on the pitch angle scattering term to obtain the final, sign-definite result

$$\begin{aligned} &\int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} \frac{T_{0s}}{F_{0s}} \langle h_s \mathcal{C}(h_s) \rangle_{\mathbf{R}_s} \qquad (19.19) \\ = &-\sum_{\mathbf{k}} \pi T_s \int_0^\infty v^2 dv \frac{\nu_s(v)}{F_{0s}(v)} \left\{ \int_{-1}^1 d\xi (1-\xi^2) \left| \frac{\partial h_{\mathbf{k}s}}{\partial \xi} \right|^2 + \frac{v^2}{v_{ts}^2} \frac{k_\perp^2 \rho_s^2}{2} \int_{-1}^1 d\xi (1+\xi^2) \left| h_{\mathbf{k}s} \right|^2 \right\}. \end{aligned}$$

19.2 Hypercollisionality

The hypercollisionality operator for a species s is defined by

$$\langle \mathcal{C}_{ns}(h_s)\rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_s}\nu_{Hs}(k_\perp\rho_i)^n\frac{1}{2}\left\{\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial\xi}\right] - \frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\} \tag{19.20}$$

where ν_{Hs} is a constant coefficient, independent of velocity.

19.2.1 Goal of Hypercollisionality

The aim of any hyperdamping term is to model the transfer of energy from the smallest resolved scales in the box to smaller, unresolved scales. Without such a mechanism, energy builds up at the smallest scales because there are no smaller scale modes with which to couple nonlinearly. Hence, a bottleneck in the energy spectrum results.

One task to be completed is to estimate the nonlinear energy transfer rate (as a function of amplitude) and determine the minimum necessary hyperdamping to remove that energy. Although, at the moment the magnitude of linear hypercollisional damping is not well understood (in relation to the coefficient supplied in the input file), in principle we can at least connect the required effective hyperdamping to the nonlinear energy transfer rate. Due to *critical balance*, the nonlinear transfer rate should simply be of order the linear wave frequency, $k_\perp v_\perp \sim k_\parallel v_A$. Hence, the requirement for hyperdamping is to achieve $\gamma/\omega > 1$ at the end of the cascade, meaning that the energy will be damped out in roughly one wave period.

A more elegant technique, rather than to just choose some constant coefficient for the hypercollisional damping in any given run, is to allow the coefficient for the hypercollisional damping to vary as a function of simulation quantities (such as $|\chi|^2$), so that the effective damping supplies just the right amount of energy transfer for a given amplitude at the smallest resolves scales in the box. Hence, rather than specifying two parameters, the driving amplitude and hyperdamping amplitude, we simply specify the driving amplitude and allow the hyperdamping to vary in such a way as to always provide at least the minimum required damping rate.

19.2.2 Hypercollisionality in the Reduced MHD Limit

To find the what form a hypercollisional term takes in the Reduced MHD Limit will shed light on the effect of hypercollisionality.

We begin with the gyronkinetic equation written in terms of g_s,

$$\frac{\partial g_s}{\partial t} + v_\parallel\frac{\partial g_s}{\partial z} + \frac{q_s}{T_s}v_\parallel F_{0s}\frac{\partial\tilde{\phi}}{\partial z} + \frac{c}{B}\left[\tilde{\phi}-A, h_s\right] - \langle\mathcal{C}(h_s)\rangle - \langle\mathcal{C}_n(h_s)\rangle = -\frac{q_s}{T_s}F_{0s}\frac{\partial A}{\partial t} \tag{19.21}$$

where we have used following definitions

$$g_s \equiv h_s - \frac{q_s\langle\phi\rangle}{T_s}F_{0s} + \frac{q_s\langle\mathbf{v}_\perp\cdot\mathbf{A}_\perp\rangle}{cT_s}F_{0s}, \tag{19.22}$$

$$\tilde{\phi} \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s})\hat{\phi} + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\frac{k_\perp v_\perp}{\Omega_s}}\frac{m_s v_\perp^2}{q_s}\frac{\delta\hat{B}_\parallel}{B_0}, \tag{19.23}$$

and

$$A \equiv J_0(\frac{k_\perp v_\perp}{\Omega_s})\frac{v_\| A_\|}{c} \tag{19.24}$$

To derive the vorticity equation, the first step is to multiply the gyrokinetic equation by q_s, ring average at constant position $\mathbf{r}$, integrate the equation over velocity, and sum over species. We will consider each term in turn. Using the quasineutrality condition in terms of g_s, integrating over velocity for the potential terms, and solving for the integral of g_s gives us a form for the first term

$$\frac{\partial}{\partial t}\sum_s \int_{\mathbf{v}} q_s \langle g_s \rangle_{\mathbf{r}} = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}(1-\Gamma_{0s})\frac{\partial \hat{\phi}}{\partial t} - \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\Gamma_{1s}\frac{\partial}{\partial t}\left(\frac{T_s}{q_s}\frac{\delta \hat{B}_\|}{B_0}\right). \tag{19.25}$$

The second term, after using the Parallel Ampere's Law in terms of g_s, becomes

$$\frac{\partial}{\partial z}\sum_s \int_{\mathbf{v}} q_s v_\| \langle g_s \rangle_{\mathbf{r}} = -\frac{c}{4\pi}\frac{\partial}{\partial z}\nabla_\perp^2 A_\|. \tag{19.26}$$

The third term integrates to zero because it is odd in $v_\|$.

The fourth term, the nonlinear term, we neglect because it does not affect the form of the hypercollisional term.

The fifth term is the physical collisional term. Since we are generally interested in collisionless problems for which $\omega > \nu_c$, we neglect this term as well.

The term on the right-hand side is also odd in $v_\|$ and so contributes nothing after integration over velocity.

The sixth term on the left-hand side is the hypercollisional term. The ring average at constant position $\mathbf{r}$ of the hypercollisional operator becomes

$$\langle\langle \mathcal{C}_{ns}(h_s)\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} J_0(\alpha_s)\nu_{Hs}(k_\perp \rho_i)^n \frac{1}{2}\left\{\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial \xi}\right] - \frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\} \tag{19.27}$$

where $\alpha_s = k_\perp v_\perp/\Omega_s$ Summing over species and integrating over velocity gives

$$\sum_s \int_{\mathbf{v}} q_s \langle\langle \mathcal{C}_{ns}(h_s)\rangle_{\mathbf{R}_s}\rangle_{\mathbf{r}} \tag{19.28}$$

$$= \sum_s \int_{\mathbf{v}} q_s \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} J_0(\alpha_s)\nu_{Hs}(k_\perp \rho_i)^n \frac{1}{2}\left\{\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial h_{\mathbf{k}s}}{\partial \xi}\right] - \frac{v^2}{v_{ts}^2}(1+\xi^2)\frac{k_\perp^2\rho_s^2}{2}h_{\mathbf{k}s}\right\} \tag{19.29}$$

19.3 Adaptive Hypercollisionality

Because the effectiveness of the hypercollisionality in damping the turbulent cascade is dependent on the structure of the distribution function in velocity space, determining the value of the hypercollisional coefficient needed to achieve the desired amount of damping is difficult. Thus, an algorithm for adaptively adjusting the hypercollisional coefficient to yield just the right amount of damping is extremely valuable. But even this requires a diagnostic of the effective damping of the cascade. Here we review a model-dependent derivation of this diagnostic and explain the algorithm for adjusting the hypercollisionality.

19.3.1 Estimation of γ/ω from `AstroGK`

To dissipate the turbulent cascade at the highest resolved wavenumbers—and thus avoid a bottleneck of energy—we want the effective normalized dissipation in the nonlinear simulation $\gamma_{nl}(k_\perp)/\omega_{nl}(k_\perp)$ to take on a large enough value to damp the cascade, where $\gamma_{nl}(k_\perp)$ is the damping rate of the nonlinear fluctuations at some value $k_\perp$ and $\omega_{nl}(k_\perp)$ is the nonlinear energy transfer rate to higher wavenumbers. At the same time, we do not want γ_{nl}/ω_{nl} to be too large or the hypercollisional damping will affect too much of the dynamic range of the simulation. A good estimate for the value needed is $\gamma_{nl}/\omega_{nl} \simeq 1/(2\pi) \simeq 0.16$ at $k_\perp = k_{\perp\max} 1/\sqrt{2}$.

We can use the heating diagnostics in `AstroGK` along with a model for the nonlinear transfer rate to calculate a value of $\gamma_{nl}(k_\perp)/\omega_{nl}(k_\perp)$. An estimate of the damping rate is $\gamma_{nl}(k_\perp) = P_k(k_\perp)/E_k(k_\perp)$, where $P_k(k_\perp)$ is the heating power within a given wavenumber band centered at $k_\perp$ and $E_k(k_\perp)$ is the total energy contained in that band. We must normalize this damping rate by a nonlinear energy transfer frequency; we use the cascade model from [Howes et al., 2007] to determine $\omega_{nl}(k_\perp)$. The nonlinear cascade rate from this model is

$$\omega_{nl} = C_2 k_\perp v_\perp(k_\perp) = C_2 k_\perp \alpha \delta B_\perp(k_\perp)/\sqrt{4\pi n_i m_i} \tag{19.30}$$

where

$$\alpha(k_\perp) = \begin{cases} 1, & k_\perp\rho_i \ll 1 \\ k_\perp\rho_i/\sqrt{\beta_i + 2/(1+T_e/T_i)}, & k_\perp\rho_i \gg 1 \end{cases} \tag{19.31}$$

and we can use

$$\alpha = \overline{\omega} \simeq \left[1 + \frac{(k_\perp\rho_i)^2}{\beta_i + 2/(1+T_e/T_i)}\right]^{1/2} \tag{19.32}$$

over all values of $k_\perp$. Thus we find

$$\omega_{nl} = C_2 k_\perp \delta B_\perp(k_\perp)/\sqrt{4\pi n_i m_i}\left[1 + \frac{(k_\perp\rho_i)^2}{\beta_i + 2/(1+T_e/T_i)}\right]^{1/2} \tag{19.33}$$

where the value of the critical balance Kolmogorov constant is $C_2 \in [1,2]$. Normalizing this to the `AstroGK` normalization,

$$\left(\frac{\omega_{nl} a_0}{v_{t0}}\right) = C_2 \frac{(k_\perp\rho_0)^2}{2}\left(\frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0}\right)\left(\frac{B_0}{v_{t0}\sqrt{4\pi n_0 m_0}}\right)\left(\frac{2T_0}{m0}\frac{m_0 c}{q_0 B_0}\frac{1}{\rho_0 v_{t0}}\right)\left[1 + \frac{(k_\perp\rho_i)^2}{\beta_i + 2/(1+T_e/T_i)}\right]^{1/2} \tag{19.34}$$

which gives

$$\hat{\omega}_{nl} = C_2 \frac{\hat{k}_\perp^2 \hat{A}_\parallel}{2\sqrt{\beta_0}} \left[1 + \frac{\hat{k}_\perp^2}{\beta_i + 2/(1+T_e/T_i)} \right]^{1/2} \tag{19.35}$$

19.3.2 Connection to Code Diagnostics

One can use the code heating and energy diagnostics to determine the magnitude of the magnetic field fluctuations at a given $k_\perp$. For a given mode (k_x, k_y), the magnetic energy in fluctuations perpendicular to the mean field is

$$E_{B_\perp}(k_x, k_y) = \frac{k_\perp^2 |A_\parallel(k_x, k_y)|^2}{8\pi} \tag{19.36}$$

In normalized units this becomes

$$\hat{E}_{B_\perp}(k_x, k_y) = \frac{\hat{k}_\perp^2 |\hat{A}_\parallel(k_x, k_y)|^2}{4\beta_0} \tag{19.37}$$

which is calculated in `hk(it,ik)%eapar` in the code.

Summing over all of the N_{mode} modes in the band from $\hat{k}_\perp - \Delta\hat{k}_\perp/2$ to $\hat{k}_\perp + \Delta\hat{k}_\perp/2$, we find

$$\hat{E}_{B_\perp}^{\Delta k_\perp} = \frac{\hat{k}_\perp^2 |\hat{A}_\parallel|^2}{4\beta_0} N_{mode} \tag{19.38}$$

assuming that all of the modes have statistically the same value of $\hat{A}_\parallel$. To connect with the formula for $\hat{\omega}_{nl}$ above, I can use this to find

$$\frac{\hat{k}_\perp \hat{A}_\parallel}{2\sqrt{\beta_0}} = \sqrt{\frac{\hat{E}_{B_\perp}^{\Delta k_\perp}}{N_{mode}}} \tag{19.39}$$

so the nonlinear frequency can be determined by

$$\hat{\omega}_{nl} = C_2 \hat{k}_\perp \sqrt{\frac{\hat{E}_{B_\perp}^{\Delta k_\perp}}{N_{mode}}} \left[1 + \frac{\hat{k}_\perp^2}{\beta_i + 2/(1+T_e/T_i)} \right]^{1/2} \tag{19.40}$$

19.3.3 Rules for Adaptivity

The rules for adapting the hypercollisional coefficient are outlined here. First, we define a few quantities. The measures for

$$f_1 = \frac{\gamma_{nl}}{\omega_{nl}} = \frac{P_1}{E_k \omega_{nl}} \quad f_2 = \frac{\gamma_{nl}}{\omega_{nl}} = \frac{P_2}{E_k \omega_{nl}} \tag{19.41}$$

where for the ions

$$P_1 = P_{ci} + P_{Hci} \quad P_2 = P_{Hci} \tag{19.42}$$

and for the electrons

$$P_1 = P_{ci} + P_{Hci} + P_{ce} + P_{Hce} \quad P_2 = P_{Hce}. \tag{19.43}$$

For example, for the electrons, f_1 is the total normalized damping rate and f_2 is the normalized damping due only to the electron hypercollisionality.

The coefficient is only adjusted adaptively if there is enough energy in the band compared to the total turbulent energy

$$\frac{E_k}{E_{tot}} > 10^{-6} \tag{19.44}$$

and adaptive hypercollisionality is specified for the species `spec(is)%adapt_hc = T`. If these conditions are specified, then the following tests are performed to determine if the hypercollisionality coefficient should be increased or decreased.

The threshold normalized damping rate is f_t =`spec(is)%gw_hc`; a typical value is $f_t = 1/2\pi \simeq 0.16$. When the hypercollisional coefficient is adjusted, the collisional matrix must be recalculated, so one wants to adjust the value as infrequently as possible—thus, we define a buffer around the threshold value for which the hypercollisional damping is acceptable. We define this as q =`spec(is)%gw_frac`; a typical value is $q = 0.1$. The test is simply that when $|f_n - f_t|/(f_n + f_t) > q$, the damping rate is beyond the threshold buffer and the hypercollisional value *may* need to be adjusted. For the moment, only the electron hypercollisionality is adaptive—preliminary tests suggests adaptive *both* ion and electron hypercollisionality is unstable.

FIRST TEST: Damping too strong If total damping is outside the buffer and too strong

$$\frac{|f_1 - f_t|}{f_1 + f_t} > q \text{ and } f_1 > f_t \tag{19.45}$$

then we need to determine if reducing the electron hypercollisionality will help (if real collisional damping dominates and is too strong, then reducing electron hypercollisionality will not make any difference). If the electron hypercollisional damping is also outside the buffer and too strong

$$\frac{|f_2 - f_t|}{f_2 + f_t} > q \text{ and } f_2 > f_t \tag{19.46}$$

and if the electron hypercollisionality is greater than the minimum value

$$\nu_{He} > \nu_{He\mathrm{min}} \tag{19.47}$$

then the electron hypercollisionality is adjusted according to

$$\nu_{He} = \max\left(\nu_{He}\frac{f_t}{f_2}, \nu_{He\mathrm{min}}\right) \tag{19.48}$$

The flag is set to `nuh_changed = T` so that the collisional matrix will be recalculated.

SECOND TEST: Damping too weak Else if total damping is outside the buffer and too weak

$$\frac{|f_1 - f_t|}{f_1 + f_t} > q \text{ and } f_1 < f_t \tag{19.49}$$

then increase electron hypercollisionality up to the maximum value

$$\nu_{He} = \min\left(\nu_{He}\frac{f_t}{f_1}, \nu_{He}\mathrm{max}\right) \tag{19.50}$$

and the flag is set to `nuh_changed = T` so that the collisional matrix will be recalculated.

Chapter 20

Fields Advance

20.1 Normalized field equations

Here is the list of normalized field equations in terms of `gs2` variables:

$$Q_\phi \phi_{\boldsymbol{k}} - Q_B \delta B_{\parallel \boldsymbol{k}} = \mathcal{Q}_d(g_{s\boldsymbol{k}}), \tag{20.1}$$

$$k_\perp^2 A_{\parallel \boldsymbol{k}} = \mathcal{P}_d(g_{s\boldsymbol{k}}), \tag{20.2}$$

$$R_\phi \phi_{\boldsymbol{k}} + R_B \delta B_{\parallel \boldsymbol{k}} = -\mathcal{R}_d(g_{s\boldsymbol{k}}), \tag{20.3}$$

where

$$\begin{aligned}
Q_\phi &= \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}[1-\Gamma_0(\alpha_s)], & Q_B &= \sum_s q_s n_{0s}\Gamma_1(\alpha_s), \\
\mathcal{Q}_d &= \sum_s q_s n_{0s} \int \mathcal{J} J_0(a_s) \cdot d\lambda\, dE, \\
\mathcal{P}_d &= 2\beta_0 \sum_s q_s n_{0s} \int \mathcal{J} v_\parallel \cdot d\lambda\ dE, \\
R_\phi &= \frac{\beta_0}{2} \sum_s q_s n_{0s} \Gamma_1(\alpha_s), & R_B &= 1 + \frac{\beta_0}{2}\sum_s n_{0s} T_{0s} \Gamma_2(\alpha_s), \\
\mathcal{R}_d &= \sum_s n_{0s} T_{0s} \beta_0 \int \mathcal{J} \check{v}_\perp^2 \frac{J_1(a_s)}{a_s} \cdot d\lambda\, dE,
\end{aligned} \tag{20.4}$$

$\check{v}$ is under species dependent normalization and $\mathcal{J}$ is the Jacobian of the velocity integral:

$$\mathcal{J} = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{E}{1-\lambda}} e^{-E}. \tag{20.5}$$

Notice that $\check{v}_\perp$ is normalized species-dependently while $v_\parallel$ uses general thermal velocity v_{t0}.

In GS2, the following variables are defined:

$$\texttt{gamtot} = Q_\phi = \sum_s \frac{q_s^2 n_{0s}}{T_{0s}}[1-\Gamma_0(\alpha_s)], \tag{20.6}$$

$$\texttt{gamtot1} = Q_B = \frac{2}{\beta_0} R_\phi = \sum_s q_s n_{0s} \Gamma_1(\alpha_s), \tag{20.7}$$

$$\texttt{gamtot2} = \frac{1}{\beta_0}(R_B - 1) = \frac{1}{2}\sum_s n_{0s} T_{0s} \Gamma_2(\alpha_s), \tag{20.8}$$

$$\texttt{antot} = \mathcal{Q}_d(g_{s\boldsymbol{k}}) = \sum_s q_s n_{0s} \int \mathcal{J} J_0(a_s) g_{s\boldsymbol{k}}\, d\lambda\, dE, \tag{20.9}$$

$$\texttt{antota} = \mathcal{P}_d(g_{s\boldsymbol{k}}) = 2\beta_0 \sum_s q_s n_{0s} \int \mathcal{J} v_\parallel g_{s\boldsymbol{k}}\, d\lambda\, dE, \tag{20.10}$$

$$\texttt{antotp} = \frac{1}{\beta_0}\mathcal{R}_d(g_{s\boldsymbol{k}}) = \sum_s n_{0s} T_{0s} \int \mathcal{J} \check{v}_\perp^2 \frac{J_1(a_s)}{a_s} g_{s\boldsymbol{k}}\, d\lambda\, dE. \tag{20.11}$$

20.2 Discretization

Let's discuss about the difference between g_* and g_{inh} or g_{h}. We may introduce g_* by

$$g_*^{n+1} = g_{\mathrm{inh}}^{n+1} + g_{\mathrm{h}}^n. \tag{20.12}$$

The discretization of gyrokinetic eqn is straightforward, and the equivalence of field equations is explained for quasi-neutrality condition in the electrostatic case for simplicity. The discretization of the electrostatic quasi-neutrality condition at the future time step

$$\begin{aligned} Q_\phi \phi_i^{n+1} &= \mathcal{Q}_d(g_i^{n+1}) \\ &= \mathcal{Q}_d(g_{\mathrm{inh},i}^{n+1}) + \mathcal{Q}_d\left(\frac{\delta g_i}{\delta \phi_j}\right)(\phi_{*,j}^{n+1} + \phi_j^n) \\ &= \mathcal{Q}_d(g_{*,i}^{n+1}) + \mathcal{Q}_d\left(\frac{\delta g_i}{\delta \phi_j}\right)\phi_{*,j}^{n+1}, \end{aligned} \tag{20.13}$$

where we used $(\delta g/\delta\phi)\phi^n = g_{\mathrm{h}}^n$, is solved with respect to ϕ_*^{n+1} and yields

$$\left[-Q_\phi \delta_{ij} + \mathcal{Q}_d\left(\frac{\delta g_i}{\delta \phi_j}\right)\right]\phi_{*,j}^{n+1} = Q_\phi \phi_i^n - \mathcal{Q}_d(g_{*,i}^{n+1}). \tag{20.14}$$

Since the form of the operator on both sides are exactly the same, the matrix on the left hand side can actually be obtained by computing the right hand side with δ_{ij} and $\delta g_i/\delta\phi_j$ for ϕ_i^n and $g_{*,i}^{n+1}$.

General form of the discretized field equations are summarized as follows.

$$\left[-Q_\phi\delta_{ij}+\mathcal{Q}_d\left(\frac{\delta g_i}{\delta\phi_j}\right)\right]\phi^{n+1}_{*,j}+\mathcal{Q}_d\left(\frac{\delta g_i}{\delta A_j}\right)A^{n+1}_{*,j}+\left[Q_B\delta_{ij}+\mathcal{Q}_d\left(\frac{\delta g_i}{\delta B_j}\right)\right]B^{n+1}_{*,j}=Q_\phi\phi^n_i-Q_BB^n_i-\mathcal{Q}_d(g^{n+1}_{*,i}),\quad(20.15)$$

$$\mathcal{P}_d\left(\frac{\delta g_i}{\delta\phi_j}\right)\phi^{n+1}_{*,j}+\left[-k_\perp^2\delta_{ij}+\mathcal{P}_d\left(\frac{\delta g_i}{\delta A_j}\right)\right]A^{n+1}_{*,j}+\mathcal{P}_d\left(\frac{\delta g_i}{\delta B_j}\right)B^{n+1}_{*,j}=k_\perp^2A^n_i-\mathcal{P}_d(g^{n+1}_{*,i}),\quad(20.16)$$

$$\left[R_\phi\delta_{ij}+\mathcal{R}_d\left(\frac{\delta g_i}{\delta\phi_j}\right)\right]\phi^{n+1}_{*,j}+\mathcal{R}_d\left(\frac{\delta g_i}{\delta A_j}\right)A^{n+1}_{*,j}+\left[R_B\delta_{ij}+\mathcal{R}_d\left(\frac{\delta g_i}{\delta B_j}\right)\right]B^{n+1}_{*,j}=-R_\phi\phi^n_i-R_BB^n_i-\mathcal{R}_d(g^{n+1}_{*,i}).\quad(20.17)$$

Since these field equations are coupled together, we build a big matrix containing the information of all z components for all three field equations, where the each component of z appears alternatively in the matrix. Namely, the first component is for the first z component of (20.15), the next is for that of (20.16), the next is for that of (20.17), and then we move along z axis and start from (20.15) again. This is for the ease of obtaining inverted matrix. Since the plasma response must be mostly concentrated around the same point where you put the source, this way makes the matrix to have large values around the diagonal components while they scatter when we separate equations and collect with z components.

20.3 Adiabatic option

When we don't want to solve GK eqn for electrons, we may simply choose to not define electron species in the `species_parameters` namelist. In this case, we are allowed to choose `adiabatic_option` in the namelist `dist_fn_knobs`. There are four choices for the treatment of adiabatic electron dynamics.

`default` This choice changes the coefficient Q_ϕ as follows:

$$Q_\phi=\frac{q_e^2n_{0e}}{T_{0e}}+\sum_s\frac{q_s^2n_{0s}}{T_{0s}}[1-\Gamma_0(\alpha_s)],\tag{20.18}$$

where the whole term $q_e^2n_{0e}/T_{0e}$ should be put in the parameter `tite` in the namelist `parameters`. Species sum may only represent ion species.

`iphi00=3` This choice applies the `default` option for $k_y\neq0$ only.

`field-line-average-term` In addition to the `default` choice, it also changes the right hand side of the quasi-neutrality condition in a complicated manner. I don't know the details.

`zero` This option cannot be chosen by the input file.

Chapter 21

Diagnostics

21.1 Polar Energy Spectrum Calculation

The polar energy spectra in $k_\perp$ are calculated in the module `gs2_diagnostics` in the routines `init_polar_spectrum`, `get_polar_spectrum`, and `finish_polar_spectrum`.

The energies calculated are:

$$E_{B_\perp} = \int_{\mathbf{r}} \frac{k_\perp^2 |A_{\parallel\mathbf{k}}|^2}{8\pi} \tag{21.1}$$

$$E_{B_\parallel} = \int_{\mathbf{r}} \frac{|\delta B_{\parallel\mathbf{k}}|^2}{8\pi} \tag{21.2}$$

$$E_{\phi} = \int_{\mathbf{r}} n_s T_s \frac{|q_s \phi / T_s|^2}{2} \tag{21.3}$$

$$E_{\delta f_s} = \int_{\mathbf{R}_s} \int_{\mathbf{v}} T_s F_{0s} \frac{|\delta f_s / F_{0s}|^2}{2} \tag{21.4}$$

$$E_{h_s} = \int_{\mathbf{R}_s} \int_{\mathbf{v}} T_s F_{0s} \frac{|h_s / F_{0s}|^2}{2} \tag{21.5}$$

where $\delta f_s = h_s - q_s \phi F_{0s}/T_s$.

NOTE: The equations above are not quite correct. Really it is a sum over (k_x, k_y) modes, not an integration over space, that gives you the energy.

21.1.1 Energy Spectrum Calculation in `AstroGK`

The perpendicular energy spectrum calculation in `AstroGK` is performed by calculating raw and log-averaged spectra using an average of the energies of each mode at a given $k_\perp$.

First, some notes on the indexing of perpendicular Fourier modes in GS2. In the input file, we specifiy nx and ny; these values are used in the calculation for the number of k_x modes

$$\texttt{ntheta0} = 2\mathrm{int}\left(\frac{\texttt{nx}-1}{3}\right)+1 \tag{21.6}$$

and the number of k_y modes

$$\texttt{naky} = \mathrm{int}\left(\frac{\texttt{ny}-1}{3}\right)+1. \tag{21.7}$$

Choosing y0 specifies the minimum wavenumber in the box, $k_0 = 1/\texttt{y0}$.

As an example, for $\texttt{nx} = \texttt{ny} = 24$, we get the number of k_x modes $\texttt{ntheta0} = 15$ and the number of k_y modes $\texttt{naky} = 8$. The akx, ikx, aky, and iky arrays become

$$\begin{array}{lcl} \texttt{ikx} & = & 0,1,2,3,4,5,6,7,-7,-6,-5,-4,-3,-2,-1 \\ \texttt{akx} & = & 0,k_0,2k_0,3k_0,4k_0,5k_0,6k_0,7k_0,-7k_0,-6k_0,-5k_0,-4k_0,-3k_0,-2k_0,-k_0 \\ \texttt{iky} & = & 0,1,2,3,4,5,6,7 \\ \texttt{aky} & = & 0,k_0,2k_0,3k_0,4k_0,5k_0,6k_0,7k_0 \end{array} \tag{21.8}$$

Raw Spectrum

The grid of k_x and k_y values gives a set of $k_\perp = \sqrt{k_x^2 + k_y^2}$ values, each with at least two modes at that $k_\perp$. For each of the $k_\perp$ values in this set, the energy of all the modes with that $k_\perp$ value is averaged. This averaged value is then multiplied by $\pi \frac{k_\perp}{k_0}$ to give the proper weighting that would occur by integrating over $\int_0^\pi k_\perp d\theta$. The resulting spectrum is equivalent to the the 1-dimensional energy spectrum, but is typically very noisy. The output is written to runname.kspec_raw.

Averaged Spectrum

The raw spectrum is then logarithmically averaged in both $k_\perp$ and energy in linear-spaced bins in $k_\perp$. This is clearly the correct approach if one hopes to find a power-law behaviour, which produces a linear relation on a log-log plot.

The number of bins for averaging (if nkpolar is not set in the kt_grids namelist) is calculated by

$$\texttt{nkpolar} = \mathrm{int}\left(\sqrt{2}\mathrm{real}(\texttt{naky}-1)\right). \tag{21.9}$$

Effectively, this gives bins of width k_0.

Within each bin, the averages of $\log(k_\perp)$ and $\log(E)$ are taken. The average values produce a much smoother spectrum, more suitable for fitting to a power-law. The output is written to runname.kspec_avg.

NOTE: This log-averaged spectrum is currently broken.

Tests of Spectrum Calculation

Here I present the spectrum calculated by four different methods: raw, averaged, directly binned, and directly binned with correction. Each of these methods is described below.

The raw spectrum calculation finds the energy at each possible $k_\perp$ arising from the set of k_x and k_y modes by

$$E_{raw}(k_\perp) = \pi \frac{k_\perp}{k_0} \frac{\sum_N E(k_x, k_y)}{N} \tag{21.10}$$

where the sum is over the N modes satisfying $k_\perp = \sqrt{k_x^2 + k_y^2}$.

The averaged spectrum begins with the raw spectrum values as calculated above, averaging over bin of width $\Delta k_\perp$. For all N_{bin} values of $E_raw(k_\perp)$ with $k_\perp$ falling within the bin, the logarithmically averaged value is calculated by

$$E_{avg}(k_{\perp avg}) = \exp\left[\frac{\sum_{N_{bin}} \log E_{raw}(k_\perp)}{N_{bin}}\right] \tag{21.11}$$

where the value of ($k_{\perp avg}$ is calculated in a similar way using

$$k_{\perp avg} = \exp\left(\frac{\sum_{N_{bin}} \log k_\perp}{N_{bin}}\right). \tag{21.12}$$

The directly binned results simply sum the energy of all (k_x, k_y) modes with $k_\perp$ falling within bins of width $\Delta k_\perp$,

$$E_{bin}(k_{\perp cen}) = \sum_{N_{bin}} E(k_x, k_y) \tag{21.13}$$

where $k_{\perp cen}$ is taken at the bin center.

Finally a corrected, directly binned result is calculated from the directly binned results above. The correction is intended to compensate for the missing modes in the corner of the box in k_x-k_y space with $k_\perp > \min(k_x, k_y)$. The correction calculates the area of ring corresponding to the bins of width $\Delta k_\perp$ on the k_x-k_y upper half-plane; this area is given by

$$A_{bin} = \frac{\pi}{2} \frac{k_{\perp up}^2 - k_{\perp low}^2}{k_0^2} \tag{21.14}$$

where the values $k_{\perp up}$and $k_{\perp low}$ are the upper and lower $k_\perp$ values of each bin. The corrected energy is given by

$$E_{bincorr}(k_{\perp cen}) = E_{bin} \frac{A_{bin}}{N_{bin}}. \tag{21.15}$$

This correction performs a similar average (although not logarithmic) as the average spectrum described above, but beginning instead with the directly binned results; it accounts for the decreasing number of modes in the corner of the box because the possible modes at that $k_\perp$ do not span the entire upper half-plane of k_x-k_y space.

To compare these methods, we have initialized the energy at each (k_x, k_y) mode by the formula

$$E(k_x, k_y) = v_0^2 \left(\frac{k_0}{\sqrt{k_x^2 + k_y^2}}\right)^{8/3} (1 + f_{noise} w) \tag{21.16}$$

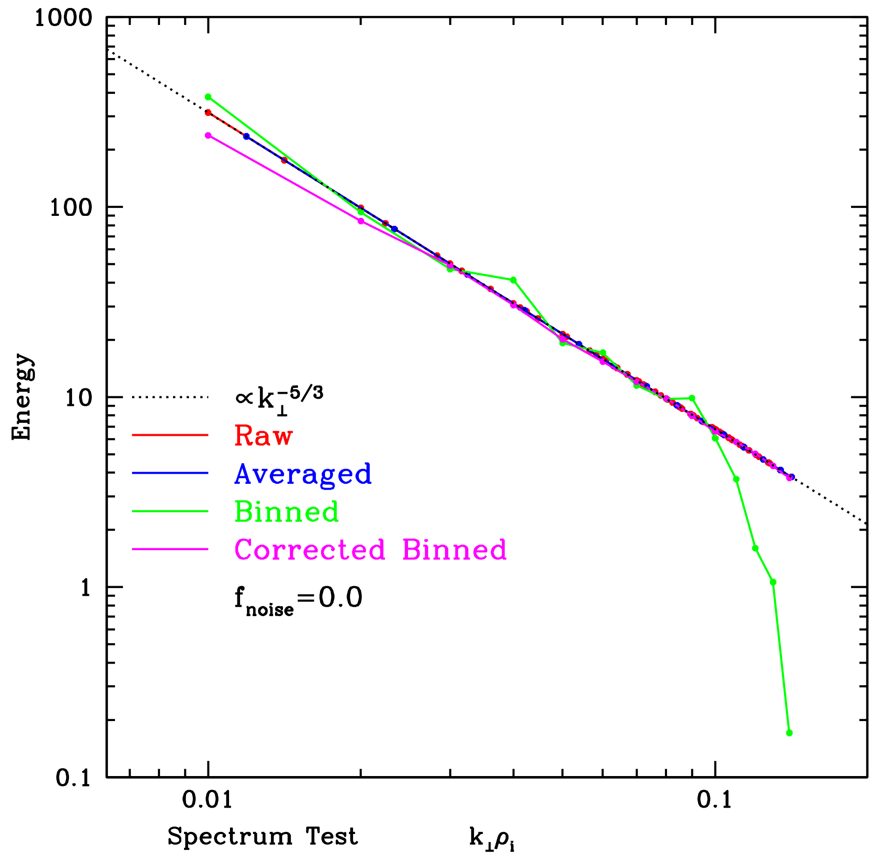

Figure 21.1: Comparison of the four methods for energy spectrum calculation with noise-free data.

where f_{noise} gives the fractional noise and the random number $w \in [-1, 1]$. Figure 21.1 and Figure 21.2 compare the results of these methods for different levels of noise.

21.1.2 N-dimensional Energy Spectra

The definition of the the N-dimensional energy spectrum $E^{(N)}$ is not clearly defined in much of the literature on magnetized turbulence. Here I will try to sort out the story starting from the single point of the Goldreich-Sridhar (GS) theory of strong incompressible MHD turbulence.

Magnetohydrodynamic Goldreich-Sridhar Turbulence

Here I review the heuristic argument behind the Goldreich-Sridhar (GS) theory of strong incompressible MHD turbulence. We begin with turbulence that is stirred isotropically at some wavenumber k_0 with a velocity v_0 (there will be constraints on the magnitude of v_0 to ensure that we are in the regime of strong turbulence). Three basic conjectures underlie the GS strong turbulence theory:

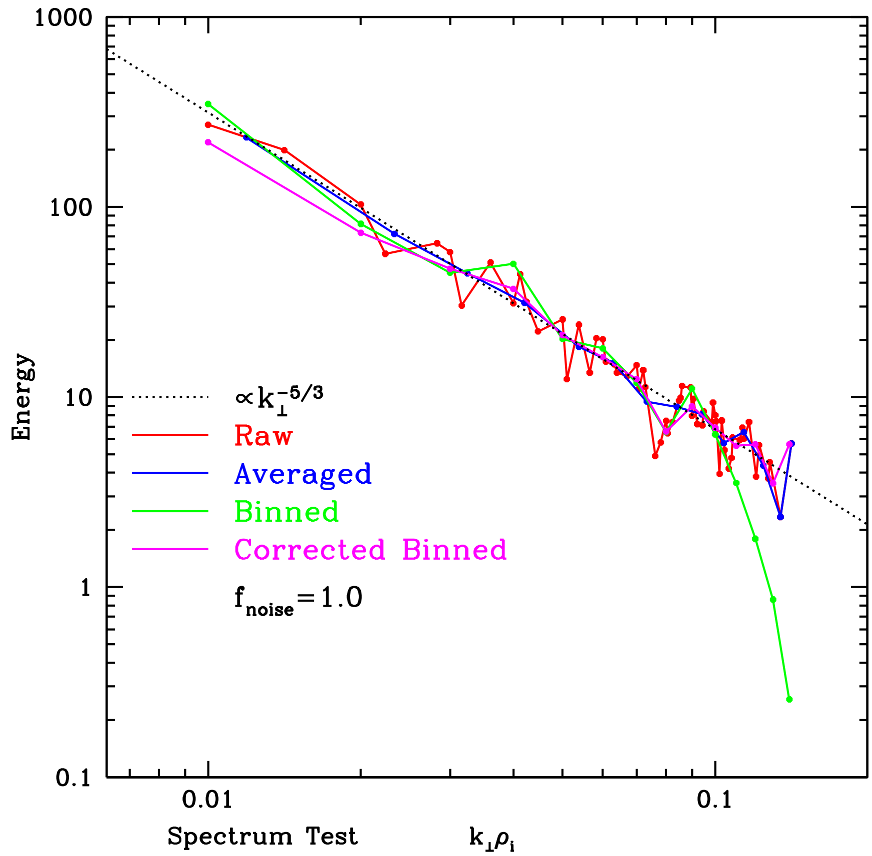

Figure 21.2: Comparison of the four methods for energy spectrum calculation with a noise level of 100%, or $f_{noise} = 1$.

1. The frequency of nonlinear energy transfer to higher wavenumbers in strong turbulence is quasi-two dimensional, governed by the eddy turn-around frequency in the plane perpendicular to the mean magnetic field

$$\omega_{NL} \sim k_\perp v_\perp. \tag{21.17}$$

   This effectively is an assumption that the interactions are local in wavenumber space—that eddies on a given scale only interact with eddies on nearby scales.

2. For strong turbulence, the cascade remains in *critical balance*, where the nonlinear energy transfer frequency balances the linear frequency of Alfvén waves

$$\omega = k_\parallel v_A \sim k_\perp v_\perp. \tag{21.18}$$

3. Kolmogorov's hypothesis that the rate of energy transfer in the inertial range is constant

$$\frac{\partial E}{\partial t} = \text{constant}. \tag{21.19}$$

Combining conjectures (1) and (3) allows us to derive the scaling of velocity $v_\perp$ with wavenumber.

$$\frac{\partial E}{\partial t} = \omega_{NL} v_\perp^2 = k_\perp v_\perp^3 = k_0 v_0^3 \tag{21.20}$$

Solving for the velocity as a function of the scale k gives

$$v_\perp = \left(\frac{k_0}{k_\perp}\right)^{1/3} v_0. \tag{21.21}$$

Hence, the energy as a function of scale $k_\perp$ is

$$E = \left(\frac{k_0}{k_\perp}\right)^{2/3} v_0^2. \tag{21.22}$$

Combining the critical balance conjecture (2) with the solution for $v_\perp$ above allows us to relate the characteristic parallel wavenumber of the turbulence $k_\parallel$ to the perpendicular wavenumber $k_\perp$.

$$k_\parallel = k_\perp^{2/3} k_0^{1/3} \frac{v_0}{v_A}. \tag{21.23}$$

If this constraint is precisely conserved, it means that turbulence exists only on a the surface of a cone in 3-dimensional wavenumber space defined by the equation above. For any value of $k_\perp$, only one value of $k_\parallel$ containing any energy; a δ-function can be used to incorporate this constraint into the full 3-dimensional energy spectrum. The turbulence is then, in fact, entirely two-dimensional on the surface of this GS cone. In practice, this constraint is only as exact as this heuristic theory.

Constructing the N-dimensional Energy Spectra

To determine the correct form of the N-dimensional energy spectra, we will use the Goldreich-Sridhar result for the 1-dimensional energy spectrum as a starting point and normalize all spectra using the total integrated energy. To simplify the final form of the integrated energy, we will integrate from k_0 to ∞ in $k_\perp$; since the smallest scales contain the most energy, this will not change our determinations of the N-dimensional energy spectra.

We begin with the result that the 1-dimensional energy spectrum predicted by the GS theory has the form

$$E^{(1)} = \frac{k_0^{2/3}}{k_\perp^{5/3}} v_0^2. \tag{21.24}$$

We'll see that this energy spectrum is the energy *after* integration over $k_\parallel$ and angle θ. We know that the total energy E can be calculated from the 1-dimensional energy spectrum by

$$E = \int E^{(1)} dk_\perp. \tag{21.25}$$

Therefore, we find that the total energy is

$$E = \int_{k_0}^{\infty} \frac{k_0^{2/3}}{k_\perp^{5/3}} v_0^2 d\mathbf{k}_\perp = \frac{3}{2} v_0^2. \tag{21.26}$$

Now, I define the 2-dimensional energy spectrum by

$$E = \int\int E^{(2)} k_\perp d\theta dk_\perp; \tag{21.27}$$

this is the energy spectrum integrated only over all values of $k_\parallel$. To yield the result that the total energy $E = \frac{3}{2}v_0^2$, we discover the 2-dimensional energy spectrum must be defined by

$$E^{(2)} = \frac{k_0^{2/3}}{2\pi k_\perp^{8/3}} v_0^2. \tag{21.28}$$

Similarly, we define the 3-dimensional energy spectrum by

$$E = \int\int\int E^{(3)} dk_\parallel k_\perp d\theta dk_\perp \tag{21.29}$$

To achieve a consistent result for the total energy, the 3-dimensional energy spectrum must be defined by

$$E^{(3)} = \frac{k_0^{2/3}}{2\pi k_\perp^{8/3}} v_0^2 \delta\left(k_\parallel - k_\perp^{2/3} k_0^{1/3} \frac{v_0}{v_A}\right). \tag{21.30}$$

Here the δ-function enforces the critical balance constraint, maintaining turbulence that is two-dimensional on the surface of the GS cone. Note that this is not what actually occurs in magnetized turbulence. Within the cone, at $k_\parallel v_A < k_\perp v_\perp$, turbulence can exist; in this case, the timescale for nonlinear transfer of energy to higher $k_\perp$ is faster than the linear timescale, so the cascade progresses in a manner similar to hydrodynamic turbulence as if there were no characteristic linear frequency in the medium. But, for the moment I will keep the δ-function because it simplifies this argument and I like it.

Summarizing the results, we find

$$\begin{array}{ll} \text{1-D} & E^{(1)} = E(k_\perp) = \frac{k_0^{2/3}}{k_\perp^{5/3}} v_0^2 \\ \text{2-D} & E^{(2)} = E(k_\perp, \theta) = \frac{k_0^{2/3}}{2\pi k_\perp^{8/3}} v_0^2 \\ \text{3-D} & E^{(3)} = E(k_\parallel, k_\perp, \theta) = \frac{k_0^{2/3}}{2\pi k_\perp^{8/3}} v_0^2 \delta\left(k_\parallel - k_\perp^{2/3} k_0^{1/3} \frac{v_0}{v_A}\right). \end{array} \tag{21.31}$$

21.1.3 Connection to `GS2` Diagnostics

The described above in Section (21.1.1) produces the angle-integrated energy spectrum—this is equivalent to the 1-dimensional energy spectrum. Hence, the slopes of the output from the `GS2` polar spectrum diagnostics should be directly comparable to the 1-dimensional energy spectra predicted by the Golreich-Sridhar theory.

21.1.4 `.kspec_raw` and `.kspec_avg`

If `write_ascii` is true (the default), then if `write_Epolar` is true, the polar spectrum vs. $k_\perp$ is output to `runname.kspec_raw` and `runname.kspec_avg`. The output columns are:

Column	Value
2	t
4	$k_\perp$
6	E_{tot}
8	$E_{A_\parallel}$
10	$E_{B_\parallel}$
12	$E_{\phi_i^2}$
14	$E_{h_i^2}$
16	$E_{\delta f_i^2}$
18	$E_{\phi_e^2}$
20	$E_{h_e^2}$
22	$E_{\delta f_e^2}$

21.2 Discrete Fourier Mode Energy Calculation

There is some subtlety involved in the calculation of the turbulent energy and it's connection to analytical theory. In this section, I will attempt to describe precisely the calculation of the energy and make this connection to theory clear.

The energy density in gyrokinetic theory can be expressed as

$$E = \int \frac{d^3\mathbf{r}}{V} \left[\frac{|\delta \mathbf{B}|^2}{8\pi} + \int d^3\mathbf{v} \sum_s \frac{T_{0s}}{2F_{0s}} \left(\langle h_s \rangle_{\mathbf{r}} - \frac{q_s \phi}{T_s} F_{0s} \right)^2 \right] \tag{21.32}$$

For the moment, we'll focus on the energy in the perpendicular magnetic field perturbation $\delta B_\perp(\mathbf{r}, t)$. Suppressing the dependence on time for notational simplicity, the perpendicular magnetic field perturbation in a periodic box of size (L_x, L_y, L_z) can be expressed as the Fourier series

$$\delta B_\perp(\mathbf{r}) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta B_{\perp ij}(z) e^{i(k_{xi}x + k_{yj}y)} \tag{21.33}$$

where $k_{xi} = 2\pi i/L_x$ and $k_{yj} = 2\pi j/L_y$. We may then express

$$|\delta B_\perp(\mathbf{r})|^2 = \delta B_\perp(\mathbf{r}) \delta B_\perp^*(\mathbf{r}) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \delta B_{\perp ij}(z) \delta B^*_{\perp i'j'}(z) e^{i(k_{xi} - k_{xi'})x} e^{i(k_{yj} - k_{yj'})y} \tag{21.34}$$

We can thus write the energy density as

$$E_{B_\perp} = \frac{1}{8\pi} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \int \frac{dz}{L_z} \delta B_{\perp ij}(z) \delta B^*_{\perp i'j'}(z) \int \frac{dx}{L_x} e^{i2\pi(i-i')x/L_x} \int \frac{dy}{L_y} e^{i2\pi(j-j')y/L_y}. \tag{21.35}$$

Now, we make use of the identity

$$\frac{1}{L} \int_{L/2}^{L/2} dx e^{i2\pi(n-n')x/L} = \delta_{nn'} \tag{21.36}$$

where $\delta_{nn'}$ is the Kronecker delta. Thus, the energy simplifies to

$$E_{B_\perp} = \frac{1}{8\pi} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \int \frac{dz}{L_z} \delta B_{\perp ij}(z) \delta B^*_{\perp i'j'}(z) \delta_{ii'} \delta_{jj'} \tag{21.37}$$

and we obtain the final result

$$E_{B_\perp} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int \frac{dz}{L_z} \frac{|\delta B_{\perp ij}(z)|^2}{8\pi}. \tag{21.38}$$

21.3 Ascii output files

Everything turned on by the flag `write_ascii`.

`(runname).moments` Controlled by `write_final_moments` and each column means

$$\theta \quad k_y \quad k_x \quad \texttt{ntot} \quad \texttt{dens} \quad u_{\|} \quad T_{\|} \quad T_{\perp} \quad \theta - \theta_0 \quad \texttt{is}$$

middle 5 normalized by `phi0`

`(runname).mom2` Controlled by `write_final_moments` and each column means

$$\theta \quad k_y \quad k_x \quad \texttt{ntot} \quad \texttt{dens} \quad u_{\|} \quad T_{\|} \quad T_{\perp} \quad \theta - \theta_0 \quad \texttt{is}$$

`(runname).fields` Controlled by `write_final_fields` and each column means

$$\theta \quad k_y \quad k_x \quad \phi_{\mathrm{r}} \quad \phi_{\mathrm{i}} \quad A_{\|,\mathrm{r}} \quad A_{\|,\mathrm{i}} \quad A_{\perp,\mathrm{r}} \quad A_{\perp,\mathrm{i}} \quad \theta - \theta_0 \quad |\phi|$$

21.4 .fields

If `write_ascii` is true (the default), then if `write_final_fields` is true, the fields as functions the parallel coordinate (`theta`, or `ig`) are output into the file `runname.fields` at the end of the run. The output columns are:

Column	Value
1	`theta(ig)`
2	`aky_out(ik)`
3	`akx_out(it)`
4	$\Re$(`phi`)
5	$\Im$(`phi`)
6	$\Re$(`apar`)
7	$\Im$(`apar`)
8	$\Re$(`bpar`)
9	$\Im$(`bpar`)
10	`theta(ig)`−`theta0(it,ik)`
11	$\lvert\phi\rvert$?

Chapter 22

Parallelization

22.1

Chapter 23

Input and Output with NetCDF

23.1 Netcdf file and gs2.pro

`phi2` in `(runname).out` is a volume average.

`phi0`, `apar0`, `aperp0` written out in the subroutine `nc_loop` are all 3D arrays including time at the slice with $\texttt{ig} = \texttt{igomega}$ ($\theta = \texttt{igomega} * 2\pi$) where `igomega` is an input variable in `gs2_diagnostics_knobs` namelist.

`phi` and `apar` are 3D arrays of the electrostatic field and parallel vector potential at the last timestep written out by the subroutine `nc_final_fields`. Their arguments are (k_y, k_x, θ, ri) as seen in the `ncdump` command, but in `gs2` and IDL routine, they are accessed as (ri, θ, k_x, k_y).

Variables `md` and `nd` are valid number of modes after truncation by 2/3-rule in k_y and k_x direction, respectively. `malias`, and `nalias` are the full number of modes, or the number of grid points in y and x directions. Making the connection to `gs2` variables, we obtain the following correspondence:

$$\texttt{md} = \texttt{naky}, \quad \texttt{nd} = \texttt{ntheta0} \tag{23.1}$$

$$\texttt{malias} = \texttt{ny}, \quad \texttt{nalias} = \texttt{nx}, \tag{23.2}$$

where the left hand sides are the variables in `gs2.pro` and the rhs are those in `gs2`. `malias` and `nalias` were defined in `gs2.pro` as

$$\texttt{malias} = 3 * \texttt{md}, \quad \texttt{nalias} = 3 * \texttt{nd}/2 + 1, \tag{23.3}$$

but I changed them to the followings:

$$\texttt{malias} = (\texttt{md} - 1) * 3 + 1, \tag{23.4}$$

$$\texttt{malias} = \texttt{malias} + (\texttt{malias} \mod 2) \tag{23.5}$$

$$\texttt{nalias} = (\texttt{nd} - 1)/2 * 3 + 1 \tag{23.6}$$

$$\texttt{nalias} = \texttt{nalias} + (\texttt{nalias} \mod 2) \tag{23.7}$$

which are the exact inverse of the aliasing expressions found in `gs2` when `nx` and `ny` are exact powers of 2 and larger than 2.

Here is a list of changes I made on `gs2.pro`.

1. recovered exact number of grid points as explained above
2. added one more grid in both x and y directions to take care of the periodicity
3. added `phi`, `apar`, and `apar_1` in the 'Field Plot' section. They are the 2D real-space values of each quantity at the final step. So, if the run stops in the linear phase, they give the eigenfunctions. They only work with the axes of 'x,y', and `apar_1` is obtained by eliminating the equilibrium component out of `apar`. The value of θ is controlled by 'Active l' slidebar in the right.
4. added `phi`, `apar`, and `apar_1` in the 'Line Plot' section. They are the 1D real-space values of the above. The plane you slice in y is determined by 'Active M' slidebar.

Chapter 24

Isothermal Electron Fluid Equations

24.1 dispersion relation of IEF equation

Here we work on the Fourier space without putting the subscript $\boldsymbol{k}$. The linearized ion gyrokinetic equation without collision term is

24.2 multiple ion species

Equations (88), (93) and (96) do not change. Changes should be made in the field equations. Quasi-neutrality equation yields

$$q_e \delta n_e = \sum_{\text{ions}} \left(\frac{q_i^2 n_{0i}}{T_{0i}} \phi - \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} q_i \int J_0(a_i) h_{i\boldsymbol{k}} \, d^3\boldsymbol{v} \right). \tag{24.1}$$

Parallel Ampère's law yields

$$q_e n_{0e} u_{\parallel e} = -\frac{c}{4\pi} \nabla_\perp^2 A_\parallel - \sum_{\text{ions}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} q_i \int v_\parallel J_0(a_i) h_{i\boldsymbol{k}} \, d^3\boldsymbol{v}. \tag{24.2}$$

Perpendicular Ampère's law yields

$$\frac{\delta B_\parallel}{B_0} = -\frac{\beta_e}{2} \left(\frac{\delta n_e}{n_{0e}} + \frac{q_e \phi}{T_{0e}} \right) - \sum_{\text{ions}} \frac{\beta_i}{2 n_{0i}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \int \frac{2 v_\perp^2}{v_{\text{th}i}^2} \frac{J_1(a_i)}{a_i} h_{i\boldsymbol{k}} \, d^3\boldsymbol{v}, \tag{24.3}$$

where $\beta = 8\pi n_0 T_0 / B_0^2$ and β_i is defined for individual ion species.

24.3 normalization

The normalization of IEF equations are summarized as follows.

Part III

AstroGK Operation

Chapter 25

AstroGK Operation

25.1 Compiling AstroGK

1. To compile, use the Makefile included in the src directory.
2. Run the script test_os to assign the environmental variable CPU. On Dawson, this can just be set to Dawson by the command
   `export CPU=Dawson`
3. Create a temporary directory called /tmp/ghowes. At the moment, you need to change the line in the Makefiles that say
   `F90FLAGS = -qmoddir=/tmp/ghowes -I/tmp/ghowes -qsuffix=f=f90 -I` $(UTILS)-I$`(GEO) -I /u/local/apps/netcdf/i`
   `-I/u/local/mpi/mpilam/include`
   so that all the /tmp directories reflect the correct name.
4. Also, for Dawson, do not us `FC5` or `FLIBS5` as the compiler or libraries for any compilation because the Dawson option does not recognize it.
5. To compile on Bassi, you must add to the `.login.ext` file in your home directory:

```
module load GNU emacs netcdf fftw totalview gnuplot hdf5_par
setenv TMPDIR $SCRATCH
setenv MP_SINGLE_THREAD no
setenv TERM xterm
umask 022
```

25.2 AstroGK Makefile Notes

1. To use `aperp_r` instead of the new `bpar_r` in the NetCDF files, create `Makefile.local` in the `src` directory, and put in the line
   `USE_NETCDF=old`
2. NOTE: This is probably not useful and better done by expand.

25.3 General Operation

1. Running agk: To run agk, type: `./agk file.in`
2. To stop a run, type: `touch file.stop`

25.4 Nonlinear Runs

Here is an example of some of the important input parameters used in nonlinear runs. The file used for this example is `trans1.in`

1. `&parameters: beta = 40.0`
2. `&kt_grids_knobs:`
   (a) `grid_option='box'`
   (b) `norm_option='mtk'`
3. `&kt_grids_box_parameters:`
   (a) `ny = 24`
   (b) `nx = 24`
   (c) `y0 = 4.0` Note that the box size is $L_y = 2\pi y_0$ so that the $k_{y_{min}} = 1/y_0$.
   (d) `rtwist = -1.` Square box with no twist: $L_x/L_y =$ `rtwist`.
4. `&theta_grid_parameters: ntheta= 48` Rule of thumb: use `ntheta`$= 2(n_x$ or $n_y)$
5. `&le_grids_knobs:`
   (a) `ngauss = 16`
   (b) `negrid = 8`
   (c) `ecut= 6.0`
6. `&dist_fn_knobs: test = .true.` Will not run a timestep, but will allow you to look at the energy grid for testing.
7. `&knobs:`
   (a) `delt = 1.0e-2` This is the maximum timestep
   (b) `nstep = 4000`
8. `&reinit_knobs: delt_minimum = 5.e-7` Choose this to be small enough to be out of the way; it is useful to stop the run if something goes wrong, because usually a problem will cause the timestep to shrink rapidly.
9. `&hyper_knobs: hyper_option='none'` For $\beta = 40$, there should be plenty of damping.
10. `&nonlinear_terms_knobs:`

(a) `nonlinear_mode='on'`

(b) `cfl = 0.2`

11. Various `&species` namelists

    (a) $m_i/m_e = 1846$

    (b) $T_i/T_e = 100$

    (c) $\nu_i = \nu_e = 0.05$ Generally, we probably want $\nu_e/\nu_i \sim \sqrt{m_i/m_e}$

    (d) Decentering: `fexpr= 0.4` Good for temporal implicitness to be slightly upwinded.

    (e) Decentering: `bakdif= 0.` Spatial implicitness decentering does not work well for electromagnetic runs—best left at 0.

12. `&init_g_knobs:`

    (a) `ginit_option= "gs"` Goldreich-Sridhar cascade

    (b) `restart_file = "nc/trans1.nc"`

    (c) `phiinit= 1.e-1` This option might be turned off?

13. `&driver:`

    (a) `amplitude = 200.` Strong/Weak turbulence depends on this

    (b) `w_antenna = (0.6364, -0.5657)` Slightly off resonsance

    (c) `nk_stir = 2`

    (d) `write_antenna=.true.`

14. Driven modes:

    (a) `&stir_1:  kx = 1, ky = 0, kz = 1`

    (b) `&stir_2:  kx = 0, ky = 1, kz = 1`

15. `&gs2_diagnostics_knobs:`

    (a) `write_Epolar = T` Write out polar spectra

    (b) `write_hrate = T` Write out heating data

    (c) `write_final_epar = .true.` May have a sign error?

    (d) `nsave= 1000` Write restart files every `nsave` steps.

    (e) `save_for_restart = .true.`

    (f) `omegatol= -1.0e-3` (`omegatol`<0 means it won't halt simulation if frequency converges)

    (g) OPTION: `make_movie = .true.` In development.

25.5 Restarts

Restarting is an important step in performing nonlinear runs. Here is a step by step guide on how to do restarts.

1. Your first run used the input file `run1.in`. Specified in this file were:
   (a) `&gs2_diagnostics_knobs: save_for_restart = .true.`
   (b) `&init_g_knobs: restart_file = "nc/trans1.nc"`
2. To restart this file, copy `trans1.in` to `trans1b.in`
3. In `trans1b.in`, make the following changes
   (a) `&init_g_knobs: ginit_option="many"` (Changed from `ginit_option= "gs"`)
   (b) `&knobs: delt_option='check_restart'`
   (c) In the antenna namelist, set `&driver: restarting=.true.`. This imports from the netCDF file the last known antenna amplitudes so the restart connects smoothly to the last run.
   (d) The old way to specify the antenna amplitudes for restart by hand is described below for historical reasons: Copy the last antenna amplitudes to from `trans1.antenna` to the driving modes. The example here is for two stirring modes. The final lines of `trans1.antenna` will look like this (a number of significant figures have been dropped to fit this on one line):
   ```
   0.4E+01 0.2E+02 0.5E+02 0.7E+02 0.3E+02 0.1E+03 0.4E+02 0.6E+00 1
   0.4E+01 0.3E+02 0.2E+03 0.1E+03 0.1E+03 0.9E+02 0.1E+03 0.6E+00 2
   ```
   Columns 4–7 contain the real and complex parts of the driving mode coefficients a and b:
   - Column 4: Re[a]
   - Column 5: Im[a]
   - Column 6: Re[b]
   - Column 7: Im[b]

   These values should be copied into `trans1b.in` to read:
   ```
   &stir_1
   kx = 1
   ky = 0
   kz = 1
   a = ( 0.765263E+02, 0.387651E+02 )
   b = ( -0.112786E+03, 0.420840E+02 )
   /
   &stir_2
   kx = 0
   ky = 1
   kz = 1
   a = ( -0.148025E+03, 0.181600E+03 )
   b = ( 0.922631E+02, 0.186526E+03 )
   /
   ```
   Soon this step will be automated, with the final antenna amplitudes incorporated into the NetCDF restart file.
   (e) OPTIONAL: You can change the maximum timestep, `&knobs: delt = 1.0e-2`, if you want here.

25.6 Advanced Options

Special Diagnostics are listed and described in this section.

Output of distribution function over velocity space.

(a) Set `write_f = .true.` in `diagnostics` namelist to use the routine `write_f` which is found in `dist_fn.f90`.

(b) Output appears in `runname.dist`.

(c) For each total energy, all of the possible values of $v_\perp$ and $v_\|$ are plotted, with a space between energies.

(d) The columns of the output are:

  i. 1: $v_\perp$

  ii. 2: $+v_\|$

  iii. 3: g (or maybe h) for $\sigma = +1$

  iv. 4: $-v_\|$

  v. 5: g (or maybe h) for $\sigma = -1$

(e) I believe this diagnostic is only called at the final timestep.

25.7 Obsolete input parameters

Here is the list of obsolete parameters, which must be useful when one imports an input file from `GS2`.

namelist	paramater	description
init_g_knobs	width0	disappeared
knobs	wstar_units	disappeared
	fphi	use_Phi (logical)
	fapar	use_Apar (logical)
	faperp	use_Bpar (logical)
dist_fn_knobs	gridfac	disappeared
	boundary_option	disappeared
species_parameters_i	vnewk	nu
dist_fn_species_knobs_i	fexpr	fexp
kt_grids_knobs		kgrids
	norm_option	disappeared
kt_grids_single_parameters	aky	akperp
theta_grid_parameters		theta_grid
	eps	disappeared
	epsl	disappeared
	shat	disappeared
theta_grid_knobs		disappeared
gs2_diagnostics_knobs		diagnostics
	write_line	disappeared
	write_phi	disappeared
	write_apar	disappeared
	write_aperp	disappeared
	write_qheat	disappeared
	write_tavg	disappeared
	write_pflux	disappeared
	write_vlux	disappeared
	write_qmheat	disappeared
	write_pmflux	disappeared
	write_vmflux	disappeared
	write_dmix	disappeared
	write_kperpnorm	disappeared
	write_phitot	disappeared
	write_eigenfunc	disappeared
	write_avg_moments	disappeared
	write_fieldline_avg_phi	disappeared
	dump_check1	disappeared
	write_fieldcheck	disappeared
	write_fcheck	disappeared
	print_old_units	disappeared
	dump_final_xfields	disappeared
	dump_neoclassical_flux	disappeared
	write_neoclassical_flux	disappeared
fields_knobs		disappeared
layouts_knobs	layouts_fields_solve	disappeared

25.8 Choosing initial condition

Here is the list of available initial conditions. The values are specified for the input variable `ginit_option` in the `init_g_knobs` namelist.

`default` This doesn't seem to make any initialization. Probably local variable `phi` has to be initialized at least.

`gs` Random phase perturbations

$$\cos(\theta+\delta)*(\texttt{refac},\texttt{imfac}) \tag{25.1}$$

for all k_x and k_y.

`kpar` Homogeneous in all k_x and k_y components and have some sinusoidal dependence along z.

`kz0` This puts homogeneous initial condition on g defined by

$$g = \texttt{phiinit} * \hat{Z} * (1+i) \tag{25.2}$$

except for $k_y = 0$ mode.

`nl` This option puts perturbation for 3 wave numbers chosen in the 3 dimensional array `ikkk` for k_y and `ittt` for k_x with a random amplitude

$$\texttt{dfac} = \texttt{den0} + \texttt{den1}\cos\theta + \texttt{den2}\cos 2\theta \tag{25.3}$$

$$\texttt{ufac} = \texttt{upar0} + \texttt{upar1}\sin\theta + \texttt{upar2}\sin 2\theta \tag{25.4}$$

$$\texttt{tparfac} = \texttt{tpar0} + \texttt{tpar1}\cos\theta + \texttt{tpar2}\cos 2\theta \tag{25.5}$$

$$\texttt{tperpfac} = \texttt{tperp0} + \texttt{tperp1}\cos\theta + \texttt{tperp2}\cos 2\theta \tag{25.6}$$

where $\theta = z/\texttt{z0} - \pi$ and

$$g(\texttt{ittt},\texttt{ikkk}) = \texttt{phiinit} * \texttt{dfac} * n_s * \mathrm{Ra}, \tag{25.7}$$

where Ra denotes the complex random amplitudes of their values in $(-0.5, 0.5)$. This amplitude is random only for different k_x and k_y, but homogeneous along z and velocity space.

`nl3r` This option puts perturbation for 2 wave numbers chosen in the 2 dimensional array `ikk` for k_y and `itt` for k_x with a fixed amplitude $(\texttt{refac},\texttt{imfac})$ on g and $A_\parallel$. The perturbation in g is same as `nl`, but it also adds perturbation on $A_\parallel$, which is

$$A_\parallel(\texttt{itt},\texttt{ikk}) = \texttt{apar0} * (\texttt{refac},\texttt{imfac}). \tag{25.8}$$

`noise` This option puts noise for g in all 3 spatial directions but homogeneous in velocity space.

`recon`, `recon2`, `restart` Restart runs...

`zero` Zero initial conditions for all fields and distribution functions. Good for driven simulation.

25.9 General Notes for Collaborative Work

1. To allow others to view your files at supercomputing centers, in your home directory issue the command
   `chmod 755 .`
   and then, if files within are not world readable, issue
   `chmod 644 *`
   within each directory you want people to read.

2. However, be careful not to issue `chmod 644` for executable files.

25.10 Version Control with Subversion

1. We use Subversion 1.4 for version control of `AstroGK`. These are general notes on its use, assuming the Subversion repository has already been created.

2. The subversion repository is hosted at Maryland on `spare.cscamm.umd.edu`. To find out what exists in the repository, you can get a list using
   `svn list svn://spare.cscamm.umd.edu/`

3. To create a working copy of the repository on the local machine, you must check it out using
   `svn checkout svn://spare.cscamm.umd.edu/agk/trunk agk_dir`
   where `agk_dir` is a local directory in which you want the working copy to be created.

4. To see the status of the local working copy compared to the latest revision in the repository,
   `svn -v status`.

5. To update your local working copy with the latest revision,
   `svn update`
   Doing this merges any changes you have made with those in the latest version. If the status of a file comes up as `C`, then you have a conflict and you must look at the file to see what to do to resolve the conflict. When you have resolved the conflict,
   `svn resolved file.f90`
   will removed the conflict flag and allow the new changes to be committed.

6. To commit your changes to the repository,
   `svn commit -m ''Put notes on this revision here''`

25.11 Running on Bassi at NERSC

1. `ssh -X bassi.nersc.gov`

2. Repository number `m414`; this is CMPD, with Jean-Noel Lebouef as the project P.I.

3. Bassi has 888 processors (111 nodes with 8 processors on each node).

4. Job control commands for Load Leveller:
   `llqs` returns the list of all jobs in the queue.
   `llqs -u ghowes` just lists my jobs.
   `llsubmit hc01.nqs` submits the `.nqs` file to the job manager.
   `llcancel b0201.54486.0` cancels the job with the `Step Id`.

5. Bassi has `gnuplot`, so you can look at your data on the fly.

6. Batch submission file (`.nqs` file)

   - `class` can be regular or interactive
   - `tasks_per_node` can be up to 8
   - `nodes` is the number of nodes you want
   - `wall_clock_limit` has 00:30:00 maximum for interactive; look on website for other limits. This is important.
   - `account_no` is `m414` for me.
   - Follow all the specifications by the simple command
     `./gs2.x runname.in`

7. Environment setup in `bdorland/.login.ext` file:
   `module load GNU emacs netcdf fftw totalview gnuplot hdf5_par`
   `umask 022` makes my directories world readable.

8. Bassi password is changed at `http://nim.nersc.gov`

9. To run from the shell (not in batch mode), type
   `poe ./gs2 runname.in -tasks_per_node 8 -nodes 1` Here `poe` stands for Parallel Operating Environment.

25.12 Running on Seaborg at NERSC

1. `ssh -X seaborg.nersc.gov` Password changes are done on `sadmin.nersc.gov`.

2. Repository number `m414`; this is CMPD, with Jean-Noel Lebouef as the project P.I.

3. Bassi has 6080 processors (380 nodes with 16 processors on each node).

4. 16 to 64 GB of shared memory on each node

5. Job control commands for Load Leveller:
   `llqs` returns the list of all jobs in the queue.
   `llqs -u ghowes` just lists my jobs.
   `llsubmit hc01.nqs` submits the `.nqs` file to the job manager.
   `llcancel b0201.54486.0` cancels the job with the `Step Id`.

6. Batch submission file (`.nqs` file)
   - `class` can be regular or interactive
   - `tasks_per_node` can be up to 8
   - `nodes` is the number of nodes you want
   - `wall_clock_limit` has 00:30:00 maximum for interactive; look on website for other limits. This is important.
   - `account_no` is `m414` for me.
   - Follow all the specifications by the simple command
     `./gs2.x runname.in`

7. Environment setup in `bdorland/.login.ext` file:
   `module load GNU emacs netcdf fftw totalview gnuplot hdf5_par`
   `umask 022` makes my directories world readable.

8. Seaborg password is changed using `ssh sadmin.nersc.gov`.

9. To run from the shell (not in batch mode), type
   `poe ./gs2 runname.in -tasks_per_node 8 -nodes 1` Here `poe` stands for Parallel Operating Environment.

10. For some reason, I have to use `gmake` to make `GS2` on Seaborg.

11. HORRIBLE MEMORY PROBLEMS: Default sizes:
    `datasize 131072 kbytes`, 128 MB
    `stacksize 32768 kbytes` 32 MB

    To avoid these problems, I must compile using the following flags:
    `-bmaxdata:0x70000000`, sets to 1792 MB
    `-bmaxstack:0x10000000`, sets to 256 MB

12. Another way of avoiding the memory problems is to compile it in 64 bits using compiler flag `-q64`; this can be difficult because then everything (including the libraries) has to be 64-bit.

13. Compiling 64-bit version of `AstroGK`
    (a) Specify compiler flag `-q64`
    (b) To compile a 64-bit version, you must link to 64-bit versions of all libraries used.
    (c) The mpi wrapper for the compiler `mpxlf90_r` will use the 64-bit MPI libraries when compiler flag `-q64` is used.

(d) Switch modules for NetCDF and FFTW to 64 bit versions using
`module switch fftw fftw_64`
and
`module switch netcdf netcdf_64`
Doing this will automatically set up necessary paths, so other than adding the `-q64` compiler flag the Makefile does not need to be changed

(e) You *must* keep physical memory limitations in mind when using 64-bit version because it can start to page and dramatically slow down performance if you exceed physical memory.

(f) NERSC website has lots of information on using and changing modules

25.13 HPSS File Server at NERSC

The HPSS (High Performance Storage System) mass storage system is a backed up server designed to archive high-performance computing output. The useful clients for accessing HPSS are `hsi` and `htar`. Once the system is set up, use is simple. The following are examples:

1. Logging in: `hsi`

2. Once running `hsi`, one can use obvious commands to maneuver and view files on HPSS and your local machine. Commands such as `cd`, `ls`, and `mkdir` act on the HPSS system; `lcd`, `lls`, and `lmkdir` act on the local filesystem.

3. To copy files from the local system to HPSS, use
`put local_file : hpss_file`

4. To copy files from HPSS to the local system , use
`get local_file : hpss_file`

5. `htar` is a handy utility that performs both `tar` archive creation/extraction and transfer to/from HPSS simultaneously, avoiding the need for making a local tar file before transfer.

   - To create a tar file on HPSS:
   `htar -cvf archive_filename.tar content_files*`

   - To extract from a tar archive on HPSS to the local machine:
   `htar -xvf archive_filename.tar`

For external computers, the procedure is different and is outlined in detail on the NERSC webpage.

1. Basically, for ftp access, an encrypted username/password pair is put into a .netrc file for authentication.

2. Then type `ftp archive.nersc.gov`

3. The access is automatically authenticated and you can send many files using `mput swt*tar`.

4. Be sure to use `lcd` to get into the correct local directory and `cd` to get to the correct remote directory.

25.14 Running on IBM Blue Gene/L ANL

1. `ssh -X bgl.mcs.anl.gov`
2. `cqstat -f` to query the job queue
3. 2048 processors (1024 nodes with 2 processors each)
4. Running Jobs
   (a) `cqsub -q short -n 16 -c 32 -t 30 -N ghowes@astro.berkeley.edu -O agk.lap2v agk.lap2v lap2v.in`
   (b) Here, `-n` specifies number of nodes, `-c` is the number of processors if not equal to the number of nodes, `-t` is time in minutes, `-q` is used to specify the short queue for jobs of less than 30 minutes and 64 nodes or less, `-O` is used to specify the prefix of the `.output` and `.error` files, `-N` is the e-mail address to send notification of job start/stop.
   (c) `cqstat -f` to see list of jobs
   (d) `cqdel 121362` to delete job 121362. If it will not die, then you can force with `cqdel -f 121362`, but send a message to support@bgl.mcs.anl.gov so they can clean up the mess.

25.15 Running on Franklin Cray XT4 at NERSC

1. `ssh -l ghowes franklin.nersc.gov`
2. `myquota` command gives information on the limits of the `$HOME` directory.
3. Compilation: `ftn -fast -o mpi-hello mpi-hello.f`, compiler runs on login node
4. Running: `aprun -n 2 ./mpi-hello`, launch parallel job on compute nodes
5. Batch System Script:
```
#PBS -N hellojob
#PBS -q debug
#PBS -l size=2
#PBS -l walltime=00:01:00
#PBS -e hellojob.out
#PBS -j eo

cd $PBS_O_WORKDIR
ftn -fast -o mpi-hello mpi-hello.f #compiler runs on login node
aprun -n2 ./mpi-hello #launch parallel job on compute nodes
qsub runhello
```
6. To specify using a single core on each node, `aprun -n 4 -N 1 a.out`
   To specify using both cores on each node (default), `aprun -n 4 -N 2 a.out`
7. 9672 compute nodes with dual core processors (19344 processors)

8. Memory: 4 GB per dual core node, 3.66 GB per node available when overhead is accounted for.
9. Run batch jobs out of `$SCRATCH` directory rather than `$HOME` directory for better performance.
10.

25.15.1 Module Software User Environment

1. To compile `AstroGK` we need to ensure that the correct modules are loaded for third party software such as the FFTW, NetCDF, and HDF libraries. This is accomplished by running the command
   `module load netcdf fftw/2.1.5 hdf5/1.6.7_par`
   to load the correct version of those libraries. Similarly, you can swap out current modules for other modules, for example to remove the fftw/3.1.1 library and add the fftw/2.1.5 library (which is the version used by `AstroGK`, you run
   `module switch fftw/3.1.1 fftw/2.1.5`

25.16 Computational Resources

1. Dawson: at least 540 processors (270 nodes with 2 processors on each node) At least 256 of these processors are 2.3 GHz G5 processors.
2. Bassi: 888 processors (111 nodes with 8 processors on each node)

25.17 Notes on Porting

1. Read the README file
2. The UNIX command `uname` is useful for determining the computer on which you are operating:
   `uname -s` gives the system: `Darwin`
   `uname -p` gives the processor type: `i386`
   `uname -n` gives the nodename: `wave.physics.uiowa.edu`

Chapter 26

Normalization in `AstroGK`

26.1 `AstroGK` Normalization

This section defines the normalization of all quantities used in `GS2` and of supplemental quantities that arise in the analytical theory.

26.1.1 Normalization of all quantities

In this section, the normalization of each quantity is summarized. There are several notes of importance:

1. The subscript 0 refers to the species independent reference quantity.
2. The subscript s signifies that a quantity is species dependent.
3. Note that all fluctuating, first-order quantities are multiplied by the factor a_0/ρ_0.
4. Note that the distribution function, velocity, energy, pitch angle, and magnetic moment are species dependent quantities so that integrations over velocity space can be efficient even when each plasma species has a different temperature. Because of this dependence, certain factors arise in the normalized equations that otherwise would not be there. Hence, going through the normalization carefully is necessary to avoid any subtle errors.
5. Naturally, temperature, mass, and charge are also species dependent, but in a less complicated way.

Quantities Stored by the Code

Note that any normalized quantity with a subscript s is a species dependent quantity.

Parallel Position

$$\hat{z} = \frac{z}{a_0} \tag{26.1}$$

Perpendicular Position (x or y)

$$\hat{x} = \frac{x}{\rho_0} \tag{26.2}$$

Parallel Wavenumber

$$\hat{k}_{\parallel} = k_{\parallel} a_0 \tag{26.3}$$

Perpendicular Wavenumber

$$\hat{k}_{\perp} = k_{\perp} \rho_0 \tag{26.4}$$

Time

$$\hat{t} = \frac{t v_{t0}}{a_0} \tag{26.5}$$

Species Temperature

$$\hat{T}_s = \frac{T_s}{T_0} \tag{26.6}$$

Species Mass

$$\hat{m}_s = \frac{m_s}{m_0} \tag{26.7}$$

Species Charge

$$\hat{q}_s = \frac{q_s}{q_0} \tag{26.8}$$

Distribution Function

$$\hat{h}_s = \frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \tag{26.9}$$

Velocity

$$\hat{\mathbf{v}}_s = \frac{v}{v_{ts}} \tag{26.10}$$

Energy

$$\hat{E}_s = \frac{1/2 m_s v_s^2}{1/2 m_s v_{ts}^2} = \frac{v_s^2}{v_{ts}^2} = \hat{v}_s^2 = \frac{E_s}{T_s}. \tag{26.11}$$

Pitch Angle

$$\hat{\lambda}_s = \frac{(v_{\perp s}^2 / v_{ts}^2)}{(v_s^2 / v_{ts}^2)(B/B_0)} = \frac{\hat{v}_{\perp s}^2}{\hat{v}_s^2 \hat{B}} \tag{26.12}$$

Magnetic Moment

$$\hat{\mu}_s = \frac{\frac{m_s v_{\perp s}^2}{2B}}{\frac{m_s v_{ts}^2}{2B_0}} = \frac{\hat{v}_{\perp s}^2}{\hat{B}} \tag{26.13}$$

Magnetic Field

$$\hat{B} = \frac{B}{B_0} \tag{26.14}$$

Scalar Potential

$$\hat{\phi} = \frac{q_0 \phi}{T_0} \frac{a_0}{\rho_0} \tag{26.15}$$

Parallel Vector Potential

$$\hat{A}_\parallel = \frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0} \tag{26.16}$$

Parallel Magnetic Field

$$\delta\hat{B}_\parallel = \frac{\delta B_\parallel}{B_0}\frac{a_0}{\rho_0} \tag{26.17}$$

Current

$$\hat{\mathbf{J}} = \frac{\mathbf{J}}{n_0 q_0 v_{t0}}\frac{a_0}{\rho_0} \tag{26.18}$$

Bessel function J_0

$$\hat{J}_0 = J_0(\gamma_s) \tag{26.19}$$

Bessel function J_1

$$\hat{J}_1 = \frac{J_1(\gamma_s)}{\gamma_s} \tag{26.20}$$

Bessel function argument

$$\gamma_s = \frac{k_\perp v_{\perp s}}{\Omega_s} = \frac{\sqrt{\hat{m}_s \hat{T}_s}}{\hat{q}_s}\frac{\hat{k}_\perp \hat{v}_{\perp s}}{\hat{B}} \tag{26.21}$$

Hyperviscosity

$$\hat{\nu}_H = \frac{\nu_H a_0}{\rho_0^4 v_{t0}} \tag{26.22}$$

Hyperresistivity

$$\hat{\eta}_H = \frac{\eta_H a_0}{\rho_0^4 v_{t0}} \tag{26.23}$$

Supplemental Quantities Not Explicitly in the Code

Gyrokinetic potential

$$\hat{\chi} = \frac{q_0 \chi}{T_0}\frac{a_0}{\rho_0} \tag{26.24}$$

Electric Field

$$\hat{\mathbf{E}} = \frac{q_0 \mathbf{E} a_0}{T_0}\frac{a_0}{\rho_0} \tag{26.25}$$

Distribution Function Normalization

As noted above, the distribution function h_s is normalized by

$$\hat{h}_s = \frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}.$$

The normalizing factor F_{0s} is a species-dependent quantity, whereas a species-independent normalization is given by

$$F_{00} \equiv \frac{n_0}{v_{t0}^3}. \tag{26.26}$$

Note that the normalized version of the species-dependent distibution function normalization can be written

$$\frac{F_{0s}}{F_{00}} = \frac{\frac{n_s}{\pi^{3/2} v_{ts}^3} e^{-v^2/v_{ts}^2}}{\frac{n_0}{v_{t0}^3}} = \frac{\hat{n}_s}{\hat{v}_{ts}^3} \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \tag{26.27}$$

How this is done in the code, when the exponential is combined in the integration over velocity space, is explained in section (26.7.1).

Other Useful Relations

Of prime importance is the definition of the thermal velocity v_t used in this document. Everything in this document assumes the convention

$$v_t = \sqrt{\frac{2T}{m}} \tag{26.28}$$

Note that any quantities with the subscript 0 signify quantities calculated using reference values. For example, the reference thermal velocity is

$$v_{t0} = \sqrt{\frac{2T_0}{m_0}} \tag{26.29}$$

but the species dependent thermal velocity is given by

$$v_{ts} = \sqrt{\frac{2T_s}{m_s}}. \tag{26.30}$$

Thus, a quantity which depends on velocity will be ultimately normalized by the reference value, but within integrals the velocity is species dependent. Thus, the parallel velocity $v_\parallel$ is normalized as follows

$$\hat{v}_{\parallel s} = \frac{v_\parallel}{v_{ts}} \tag{26.31}$$

but the sum over species to get a total quantity (total energy, for example) must be normalized by the reference value. Thus,

$$\frac{1}{v_{t0}} \sum_s v_\parallel = \sum_s \frac{v_\parallel}{v_{ts}} \frac{v_{ts}}{v_{t0}} = \sum_s \hat{v}_{\parallel s} \sqrt{\frac{\hat{T}_s}{\hat{m}_s}} \tag{26.32}$$

Other useful definitions, defined here for the reference values, are:
Cyclotron frequency

$$\Omega_0 = \frac{q_0 B_0}{m_0 c} \tag{26.33}$$

Gyroradius

$$\rho_0 = \frac{v_{t0}}{\Omega_0} \tag{26.34}$$

Plasma Beta

$$\beta_0 = \frac{8\pi n_0 T_0}{B_0^2} \tag{26.35}$$

Alfven Velocity

$$v_A^2 = \frac{B_0^2}{4\pi n_0 m_0} \tag{26.36}$$

26.1.2 Gyrokinetic Potential

The unnormalized version of the gyrokinetic potential is given by

$$\langle\chi\rangle_{\mathbf{R}} = J_0(\frac{k_\perp v_\perp}{\Omega})(\phi - \frac{v_\| A_\|}{c}) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} \frac{m v_\perp^2}{q} \frac{\delta B_\|}{B_0} \tag{26.37}$$

The Bessel functions, already dimensionless, are normalized in the code by

$$\hat{J}_0 = J_0(\gamma) \tag{26.38}$$

and

$$\hat{J}_1 = \frac{J_1(\gamma)}{\gamma} \tag{26.39}$$

where the argument is

$$\gamma = \frac{k_\perp v_\perp}{\Omega}. \tag{26.40}$$

Multiplying both sides by $q_0 a_0/(T_0 \rho_0)$ gives

$$\langle\frac{q_0 \chi}{T_0}\frac{a_0}{\rho_0}\rangle_{\mathbf{R}} = J_0(\gamma)\left(\frac{q_0 \phi}{T_0}\frac{a_0}{\rho_0}\right) - J_0(\gamma)\frac{v_{\|s}}{v_{ts}}\frac{v_{ts}}{v_{t0}}\left(\frac{v_{t0}}{c}\frac{q_0 A_\|}{T_0}\frac{a_0}{\rho_0}\right) + \frac{J_1(\gamma)}{\gamma}\frac{m_s}{q_s}\frac{v_{\perp s}^2}{v_{ts}^2}\frac{v_{ts}^2}{v_{t0}^2}v_{t0}^2\frac{q_0}{T_0}\left(\frac{\delta B_\|}{B_0}\frac{a_0}{\rho_0}\right) \tag{26.41}$$

This can be simplified to

$$\langle\hat{\chi}\rangle_{\mathbf{R}} = \hat{J}_0\left(\hat{\phi} - \hat{v}_{\|s}\sqrt{\frac{T_s m_0}{m_s T_0}}\hat{A}_\|\right) + \hat{J}_1\frac{m_s}{q_s}\hat{v}_{\perp s}^2\frac{T_s m_0}{m_s T_0}\frac{2T_0}{m_0}\frac{q_0}{T_0}\delta\hat{B}_\| \tag{26.42}$$

Further simplification leads to the final result

$$\langle\hat{\chi}\rangle_{\mathbf{R}} = \hat{J}_0\left(\hat{\phi} - \hat{v}_{\|s}\sqrt{\frac{\hat{T}_s}{\hat{m}_s}}\hat{A}_\|\right) + \hat{J}_1 2\hat{v}_{\perp s}^2\frac{\hat{T}_s}{\hat{q}_s}\delta\hat{B}_\|. \tag{26.43}$$

Besssel Function Argument

The Bessel function argument is a bit tricky:

$$\frac{k_\perp v_{\perp s}}{\Omega_s} = (k_\perp \rho_0)\left(\frac{v_{\perp s}}{v_{ts}}\right)\frac{q_0 m_s B_0}{q_s m_0 B}\frac{v_{ts}}{\rho_0}\left(\frac{m_0 c}{q_0 B_0}\right) = \hat{k}_\perp \hat{v}_{\perp s}\frac{\hat{m}_s}{\hat{q}_s \hat{B}}\sqrt{\frac{\hat{T}_s}{\hat{m}_s}} = \frac{\sqrt{\hat{m}_s \hat{T}_s}}{\hat{q}_s}\frac{\hat{k}_\perp \hat{v}_{\perp s}}{\hat{B}} \tag{26.44}$$

where we used $\rho_0 \Omega_0 = v_{t0}$ to simplify the expression. Note also that

$$\Omega_s = \frac{q_s B}{m_s c}, \tag{26.45}$$

where B is the local (not reference) value of the magnetic field.

26.1.3 Normalization of velocity space

Velocity space is defined in terms of energy and pitch angle (E, λ) instead of $(v, v_\perp)$ and is always a species dependent quantity, so we will append the species subscript s to all such quantities to make this explicit. In unnormalized units, the quantities as defined as follows:

Energy

$$E_s = \frac{1}{2} m_s v_s^2 \tag{26.46}$$

Pitch angle

$$\lambda_s = \frac{v_{\perp s}^2}{v_s^2 B} \tag{26.47}$$

Magnetic Moment

$$\mu_s = \frac{m_s v_{\perp s}^2}{2B} \tag{26.48}$$

The relation between these quantities is

$$\lambda_s = \frac{\mu_s}{E_s}. \tag{26.49}$$

The energy is normalized by the species dependent thermal energy $1/2 m_s v_{ts}^2$,

$$\hat{E}_s = \frac{1/2 m_s v_s^2}{1/2 m_s v_{ts}^2} = \frac{v_s^2}{v_{ts}^2} = \hat{v}_s^2 \tag{26.50}$$

which can alternatively be expressed as

$$\hat{E} = \frac{E}{T_s}. \tag{26.51}$$

The pitch angle is normalized in terms of the reference magnetic field B_0 (and the thermal velocities which cancel),

$$\hat{\lambda}_s = \frac{(v_{\perp s}^2 / v_{ts}^2)}{(v_s^2/v_{ts}^2)(B/B_0)} = \frac{\hat{v}_{\perp s}^2}{\hat{v}_s^2 \hat{B}} \tag{26.52}$$

The magnetic moment is normalized in terms of the species dependent thermal energy $1/2m_s v_{ts}^2$ and the reference magnetic field B_0,

$$\hat{\mu}_s = \frac{\frac{m_s v_{\perp s}^2}{2B}}{\frac{m_s v_{ts}^2}{2B_0}} = \frac{\hat{v}_{\perp s}^2}{\hat{B}} \tag{26.53}$$

Thus, it is clear that the fundamental relation for the normalized quantities is unchanged The relation between these quantities is

$$\hat{\lambda}_s = \frac{\hat{\mu}_s}{\hat{E}_s}. \tag{26.54}$$

26.2 Normalized Equations

This section derives the equations of gyrokinetics in the normalization system of the code.

26.2.1 Parallel Ampere's Law

$$-\nabla_{\perp}^{2} A_{\|} = \frac{4\pi}{c}\delta J_{\|} = \sum_{s} \frac{4\pi}{c} q_s \int d^3\mathbf{v} v_{\|} \langle h_s \rangle_{\mathbf{r}}. \tag{26.55}$$

After Fourier transform this can be written

$$k_{\perp}^{2} A_{\|} = \frac{4\pi}{c}\delta J_{\|}. \tag{26.56}$$

Normalizing these two quantities as above,

$$\frac{T_0 c}{v_{t0} q_0 \rho_0^2} \frac{\rho_0}{a_0} (k_{\perp}^2 \rho_0^2) \left(\frac{v_{t0}}{c} \frac{q_0 A_{\|}}{T_0} \frac{a_0}{\rho_0} \right) = \frac{4\pi n_0 q_0 v_{t0}}{c} \frac{\rho_0}{a_0} \left(\frac{\delta J_{\|}}{n_0 q_0 v_{t0}} \frac{a_0}{\rho_0} \right) \tag{26.57}$$

Expressing $k_{\perp}$, $A_{\|}$, and $\delta J_{\|}$ as normalized quantities and simplifying this expression gives

$$\hat{k}_{\perp}^{2} \hat{A}_{\|} = \frac{4\pi n_0 q_0^2 v_{t0}^2 \rho_0^2}{T_0 c^2} \delta \hat{J}_{\|} \tag{26.58}$$

$$\hat{k}_{\perp}^{2} \hat{A}_{\|} = 2 \left(\frac{8\pi n_0 T_0}{B_0^2} \right) \left(\frac{q_0^2 B_0^2}{m_0^2 c^2} \right) \left(\frac{m_0^2}{4 T_0^2} \right) v_{t0}^2 \rho_0^2 \delta \hat{J}_{\|} \tag{26.59}$$

This can be expressed as

$$\hat{k}_{\perp}^{2} \hat{A}_{\|} = 2\beta_0 \frac{\rho_0^2 \Omega_0^2 v_{t0}^2}{v_{t0}^4} \delta \hat{J}_{\|} \tag{26.60}$$

Noting that $\rho_0^2 \Omega_0^2 = v_{t0}^2$ leaves

$$\hat{k}_{\perp}^{2} \hat{A}_{\|} = 2\beta_0 \delta \hat{J}_{\|} \tag{26.61}$$

The $2\beta_0$ factor exists to eliminate the coefficients in the parallel Ampere's Law (and is also there because of the v_{t0} in the parallel vector potential).

26.2.2 The Gyrokinetic Equation

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B}\left[\langle\chi\rangle_{\mathbf{R}_s}, h_s\right] - \left\langle \left(\frac{\partial h_s}{\partial t}\right)_{\text{coll}} \right\rangle_{\mathbf{R}_s} = \frac{q_s}{T_{0s}} \frac{\partial \langle\chi\rangle_{\mathbf{R}_s}}{\partial t} F_{0s}. \tag{26.62}$$

Multiplying by $a_0^2/(F_{0s}\rho_0 v_{t0})$ we get

$$\begin{aligned} \frac{\partial}{\partial(tv_{t0}/a_0)} & \left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right) + \frac{v_\parallel}{v_{ts}}\frac{v_{ts}}{v_{t0}}\frac{\partial}{\partial(z/a_0)}\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right) - \frac{a_0}{v_{t0}}\langle \mathcal{C}\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\rangle \\ + & \frac{c}{B}\frac{T_0 a_0}{q_0 v_{t0}}\frac{\rho_0}{a_0}\frac{1}{\rho_0^2}\left[\langle\frac{q_0\chi}{T_0}\frac{a_0}{\rho_0}\rangle, \left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\right]_N = \frac{q_s T_0}{q_0 T_s}\frac{\partial\langle\frac{q_0\chi}{T_0}\frac{a_0}{\rho_0}\rangle}{\partial(tv_{t0}/a_0)}\frac{F_{0s}}{F_{0s}} \end{aligned} \tag{26.63}$$

Simplifying this to the normalized quantities gives

$$\frac{\partial \hat{h}_s}{\partial \hat{t}} + \hat{v}_\parallel \sqrt{\frac{\hat{T}_s}{\hat{m}_s}} \frac{\partial \hat{h}_s}{\partial \hat{z}} + \frac{B_0}{2B}\left(\frac{m_0 c}{q_0 B_0}\right)\left(\frac{2T_0}{m_0}\right)\frac{1}{\rho_0 v_{t0}}\left[\langle\hat{\chi}\rangle, \hat{h}_s\right]_N - \frac{a_0}{v_{t0}}\langle\mathcal{C}(\hat{h}_s)\rangle = \frac{\hat{q}_s}{\hat{T}_s}\frac{\partial\langle\hat{\chi}\rangle}{\partial\hat{t}} \tag{26.64}$$

And then we finally find the normalized gyrokinetic equation

$$\frac{\partial \hat{h}_s}{\partial \hat{t}} + \hat{v}_\parallel \sqrt{\frac{\hat{T}_s}{\hat{m}_s}} \frac{\partial \hat{h}_s}{\partial \hat{z}} + \frac{1}{2\hat{B}}\left[\langle\hat{\chi}\rangle, \hat{h}_s\right]_N - \frac{a_0}{v_{t0}}\langle\mathcal{C}(\hat{h}_s)\rangle = \frac{\hat{q}_s}{\hat{T}_s}\frac{\partial\langle\hat{\chi}\rangle}{\partial\hat{t}} \tag{26.65}$$

Note that for the pitch angle collision operator, defined by

$$\mathcal{C}(f_a) = \sum_b \mathcal{C}_{ab}(f_a) = \sum_b \nu_{\perp ab}\frac{1}{4}\frac{\partial}{\partial\xi}\left[(1-\psi^2)\frac{\partial f_a}{\partial \xi}\right] \tag{26.66}$$

where $\xi \equiv v_\parallel/v$. The quantities ξ and ψ are dimensionless and thus needs no normalization, and so we need only scale the pitch-angle scattering rate by

$$\hat{\nu}_{\perp ab} = \frac{\nu_{\perp ab} a_0}{v_{t0}}. \tag{26.67}$$

We can write shorthand for this normalized collision operator by $\hat{\mathcal{C}}$, so we get

$$\frac{\partial \hat{h}_s}{\partial \hat{t}} + \hat{v}_\parallel \sqrt{\frac{\hat{T}_s}{\hat{m}_s}} \frac{\partial \hat{h}_s}{\partial \hat{z}} + \frac{1}{2\hat{B}}\left[\langle\hat{\chi}\rangle, \hat{h}_s\right]_N - \langle\hat{\mathcal{C}}(\hat{h}_s)\rangle = \frac{\hat{q}_s}{\hat{T}_s}\frac{\partial\langle\hat{\chi}\rangle}{\partial\hat{t}} \tag{26.68}$$

26.2.3 The Power Balance Equation (B19)

Equation (B19) is given by

$$\frac{d}{dt}\int\frac{d^3\mathbf{r}}{V}\qquad\left[\int d^3\mathbf{v}\sum_s\frac{T_s}{2F_{0s}}\left(h_s-\frac{q_s\phi}{T_s}F_{0s}\right)^2+\frac{|\delta\mathbf{B}|^2}{8\pi}\right] \tag{26.69}$$
$$=\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\sum_s\frac{T_s}{F_{0s}}\left\langle h_s\left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{coll}}\right\rangle_{\mathbf{R}_s}-\int\frac{d^3\mathbf{r}}{V}\mathbf{J}_a\cdot\mathbf{E}$$

We define the species-indepedent normalization of the distribution function by

$$F_{00}\equiv\frac{n_0}{v_{t0}^3} \tag{26.70}$$

and note that the normalized version of the species-dependent distibution function normalization can be written

$$\frac{F_{0s}}{F_{00}}=\frac{\frac{n_s}{\pi^{3/2}v_{ts}^3}e^{v^2/v_{ts}^2}}{\frac{n_0}{v_{t0}^3}}=\frac{\hat{n}_s}{\hat{v}_{ts}^3}\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \tag{26.71}$$

First, we multiply the equation by $\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{n_0T_0}$ where we can substitute $n_0=v_{t0}^3F_{00}$ when necessary.

$$\frac{d}{d(tv_{t0}/a_0)}\int\frac{d^3\mathbf{r}/(\rho_0^2a_0)}{V/(\rho_0^2a_0)}\sum_s\int\frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3\frac{T_s}{2T_0}\frac{F_{0s}}{F_{00}}\left[\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)-\frac{q_sT_0}{q_0T_s}\left(\frac{q_0\phi}{T_0}\frac{a_0}{\rho_0}\right)\right]^2 \tag{26.72}$$
$$+\quad\frac{B_0^2}{8\pi n_0T_0}\frac{d}{d(tv_{t0}/a_0)}\int\frac{d^3\mathbf{r}/(\rho_0^2a_0)}{V/(\rho_0^2a_0)}\left|\frac{\delta\mathbf{B}}{B_0}\frac{a_0}{\rho_0}\right|^2$$
$$=\quad\int\frac{d^3\mathbf{r}/(\rho_0^2a_0)}{V/(\rho_0^2a_0)}\sum_s\int\frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3\frac{T_s}{T_0}\frac{F_{0s}}{F_{00}}\left\langle\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\frac{\mathcal{C}v_{t0}}{a_0}\left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\right\rangle$$
$$-\quad\int\frac{d^3\mathbf{r}/(\rho_0^2a_0)}{V/(\rho_0^2a_0)}\left(\frac{\mathbf{J}_a}{n_0q_0v_{t0}}\frac{a_0}{\rho_0}\right)\cdot\left(\frac{q_0a_0\mathbf{E}}{T_0}\frac{a_0}{\rho_0}\right)$$

Thus in normalized quantities, this becomes

$$\frac{d}{d\hat{t}}\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\frac{\hat{n}_s\hat{T}_s}{2}\left(\hat{h}_s-\frac{\hat{q}_s}{\hat{T}_s}\hat{\phi}\right)^2+\frac{1}{\beta_0}\frac{d}{d\hat{t}}\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\left|\delta\hat{\mathbf{B}}\right|^2 \tag{26.73}$$
$$=\quad\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left\langle\hat{h}_s\hat{\mathcal{C}}(\hat{h}_s)\right\rangle-\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\hat{\mathbf{J}}_a\cdot\hat{\mathbf{E}}$$

26.2.4 The Entropy Equation (B10)

Equation (B10), written in an alternative order, is given by

$$-\int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{F_{0s}} \left\langle h_s \left(\frac{\partial h_s}{\partial t} \right)_{\rm coll} \right\rangle_{\mathbf{R}_s} = \int \frac{d^3\mathbf{R}_s}{V} \int d^3\mathbf{v} q_s \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} h_s - \frac{\partial}{\partial t} \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_s}{2F_{0s}} h_s^2. \tag{26.74}$$

As with equation (B19) above, we begin by multiplying the equation by $\frac{a_0}{v_{t0}} \left(\frac{a_0}{\rho_0}\right)^2 \frac{1}{v_{t0}^3 F_{00} T_0}$ to find

$$\begin{aligned} & -\int \frac{d^3\mathbf{r}/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{T_s}{T_0} \frac{F_{0s}}{F_{00}} \left\langle \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \right) \frac{\mathcal{C} a_0}{v_{t0}} \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \right) \right\rangle \\ = & \int \frac{d^3\mathbf{R}_s/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{q_s}{q_0} \frac{F_{0s}}{F_{00}} \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \right) \frac{\partial}{\partial(t v_{t0}/a_0)} \left\langle \frac{q_0 \chi}{T_0} \frac{a_0}{\rho_0} \right\rangle \\ - & \frac{\partial}{\partial(t v_{t0}/a_0)} \int \frac{d^3\mathbf{r}/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}} \frac{T_s}{2T_0} \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0} \right)^2 . \end{aligned} \tag{26.75}$$

In normalized quantities this simplifies to

$$\begin{aligned} & -\int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{T}_s \left\langle \hat{h}_s \hat{\mathcal{C}} \hat{h}_s \right\rangle \\ = & \int \frac{d^3\hat{\mathbf{R}}_s}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{q}_s \frac{\partial \langle \hat{\chi} \rangle}{\partial \hat{t}} \hat{h}_s - \frac{\partial}{\partial \hat{t}} \int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \frac{\hat{n}_s \hat{T}_s}{2} \hat{h}_s^2. \end{aligned} \tag{26.76}$$

26.3 The Power Balance Equation (B19) Term by Term

26.3.1 Antenna Power

The power due to the driving antenna is given in the power equation (B19), in normalized units, by

$$-\int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \hat{\mathbf{J}}_a \cdot \hat{\mathbf{E}} \tag{26.77}$$

To evaluate this we need to determine $\hat{\mathbf{J}}_a \cdot \hat{\mathbf{E}}$, the normalized antenna power, in terms of the quantities we know: $\hat{\phi}$, $\hat{A}_\parallel$, and $\hat{k}_\parallel$.

First, we consider here only the case when we drive a parallel current (which drives only the Alfven mode), so $\mathbf{J}_a = J_{a\parallel}\hat{\mathbf{z}}$. Using the definition of the electric field in terms of the potentials,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{26.78}$$

we obtain

$$\mathbf{J}_a \cdot \mathbf{E} = J_{a\parallel}\left(-ik_\parallel \phi - \frac{1}{c}\frac{\partial A_\parallel}{\partial t}\right) \tag{26.79}$$

To normalize $\mathbf{J}_a \cdot \mathbf{E}$, we multiply by

$$\frac{a_0}{n_0 T_0 v_{t0}}\frac{a_0^2}{\rho_0^2} \tag{26.80}$$

to get

$$\left(\frac{\mathbf{J}_a}{n_0 q_0 v_{t0}}\frac{a_0}{\rho_0}\right)\cdot\left(\frac{q_0 \mathbf{E} a_0}{T_0}\frac{a_0}{\rho_0}\right) = \left(\frac{J_{a\parallel}}{n_0 q_0 v_{t0}}\frac{a_0}{\rho_0}\right)\left[-i\left(k_\parallel a_0\right)\left(\frac{q_0\phi}{T_0}\frac{a_0}{\rho_0}\right) - \frac{\partial}{\partial(t v_{t0}/a_0)}\left(\frac{v_{t0}}{c}\frac{q_0 A_\parallel}{T_0}\frac{a_0}{\rho_0}\right)\right] \tag{26.81}$$

Thus, in normalized units we have

$$\hat{\mathbf{J}}_a \cdot \hat{\mathbf{E}} = \hat{J}_{a\parallel}\left(-i\hat{k}_\parallel \hat{\phi} - \frac{\partial \hat{A}_\parallel}{\partial \hat{t}}\right) \tag{26.82}$$

Substituting, we find the driving antenna power is given by

$$+\int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \hat{J}_{a\parallel}\left(i\hat{k}_\parallel \hat{\phi} + \frac{\partial \hat{A}_\parallel}{\partial \hat{t}}\right) \tag{26.83}$$

Since the gyrokinetic approximation requires that the parallel wavenumber is small, we may drop the first term to find

$$+\int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \hat{J}_{a\parallel}\frac{\partial \hat{A}_\parallel}{\partial \hat{t}} \tag{26.84}$$

26.3.2 Change of Magnetic Field Energy

First, a short note on the magneitc field in gyrokinetics. The fluctuating magnetic field is given by

$$\delta\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times(A_\parallel\hat{\mathbf{z}} + \mathbf{A}_\perp) = \nabla A_\parallel\times\hat{\mathbf{z}} + \nabla\times\mathbf{A}_\perp = \nabla A_\parallel\times\hat{\mathbf{z}} + \left(\frac{\partial}{\partial z}\hat{\mathbf{z}} + \nabla_\perp\right)\times\mathbf{A}_\perp \tag{26.85}$$

This can be written

$$\delta\mathbf{B} = \nabla A_\parallel\times\hat{\mathbf{z}} + \hat{\mathbf{z}}\times\frac{\partial}{\partial z}\mathbf{A}_\perp + \nabla_\perp\times\mathbf{A}_\perp \tag{26.86}$$

Note that the term $\nabla_\perp\times\mathbf{A}_\perp$ gives a component only in the $\hat{\mathbf{z}}$ direction, while the other two terms are only in the perpendicular direction. Thus we can replace $\nabla_\perp\times\mathbf{A}_\perp$ with $\delta B_\parallel\hat{\mathbf{z}}$. Also, note that only the perpendicular derivative of the first term survives after the cross product with $\hat{\mathbf{z}}$, so are left with

$$\delta\mathbf{B} = \nabla_\perp A_\parallel\times\hat{\mathbf{z}} + \delta B_\parallel\hat{\mathbf{z}} + \hat{\mathbf{z}}\times\frac{\partial}{\partial z}\mathbf{A}_\perp \tag{26.87}$$

Finally, under the gyrokinetic approximation, the parallel derivative is small compared to perpendicular derivatives, so we may drop the final term to achieve the final result

$$\delta\mathbf{B} = \nabla_\perp A_\parallel\times\hat{\mathbf{z}} + \delta B_\parallel\hat{\mathbf{z}} \tag{26.88}$$

Now, lets find the magnitude $|\delta\mathbf{B}|^2$ after taking the Fourier transform, so

$$\delta\mathbf{B} = ik_\perp A_\parallel(\hat{k}_\perp\times\hat{\mathbf{z}}) + \delta B_\parallel\hat{\mathbf{z}} \tag{26.89}$$

Taking the magnitude,

$$|\delta\mathbf{B}|^2 = k_\perp^2|A_\parallel|^2 + |\delta B_\parallel|^2 \tag{26.90}$$

We normalize this by multiplying by

$$\frac{1}{B_0^2}\frac{a_0^2}{\rho_0^2} \tag{26.91}$$

to find

$$\left|\frac{\delta\mathbf{B}}{B_0}\frac{a_0}{\rho_0}\right|^2 = \frac{k_\perp^2\rho_0^2}{\rho_0^2}\left|\frac{v_{t0}}{c}\frac{q_0A_\parallel}{T_0}\frac{a_0}{\rho_0}\right|^2\frac{c^2}{v_{t0}^2}\frac{T_0^2}{q_0^2B_0^2} + \left|\frac{\delta B_\parallel}{B_0}\frac{a_0}{\rho_0}\right|^2 \tag{26.92}$$

This can be simplified to

$$\left|\delta\hat{\mathbf{B}}\right|^2 = \hat{k}_\perp^2\left|\hat{A}_\parallel\right|^2\left(\frac{m_0^2c^2}{q_0^2B_0^2}\right)\frac{1}{4v_{t0}^2\rho_0^2}\left(\frac{4T_0^2}{m_0^2}\right) + \left|\delta\hat{B}_\parallel\right|^2 \tag{26.93}$$

Finally, we get

$$\left|\delta\hat{\mathbf{B}}\right|^2 = \frac{1}{4}\hat{k}_\perp^2\left|\hat{A}_\parallel\right|^2 + \left|\delta\hat{B}_\parallel\right|^2 \tag{26.94}$$

The magnetic energy term of equation (B19) is

$$\frac{1}{\beta_0}\frac{d}{d\hat{t}}\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\left|\delta\hat{\mathbf{B}}\right|^2 \tag{26.95}$$

Thus we should have

$$\frac{1}{\beta_0}\frac{d}{d\hat{t}}\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\left(\frac{1}{4}\hat{k}_\perp^2\left|\hat{A}_\parallel\right|^2+\left|\delta\hat{B}_\parallel\right|^2\right) \tag{26.96}$$

ERROR: The code does NOT have this factor of 1/4.

26.3.3 Fluctuating Energy

The term due to the fluctuating energy in the distribution function is given by

$$\frac{d}{d\hat{t}}\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\frac{\hat{n}_s\hat{T}_s}{2}\left(\hat{h}_s-\frac{\hat{q}_s}{\hat{T}_s}\hat{\phi}\right)^2 \tag{26.97}$$

If we take the derivative with respect to time, we get

$$\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left(\hat{h}_s-\frac{\hat{q}_s}{\hat{T}_s}\hat{\phi}\right)\frac{d}{d\hat{t}}\left(\hat{h}_s-\frac{\hat{q}_s}{\hat{T}_s}\hat{\phi}\right) \tag{26.98}$$

which can be written out to yield

$$\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left[\hat{h}_s\frac{d\hat{h}_s}{d\hat{t}}-\frac{\hat{q}_s}{\hat{T}_s}\left(\hat{h}_s\frac{d\hat{\phi}}{d\hat{t}}+\hat{\phi}\frac{d\hat{h}_s}{d\hat{t}}\right)+\frac{\hat{q}_s^2}{\hat{T}_s^2}\hat{\phi}\frac{d\hat{\phi}}{d\hat{t}}\right] \tag{26.99}$$

NOTE: This agrees with the code except for a factor of J_0 multiplying the middle term. I think that the code is correct, but I need to think about it more.

26.3.4 Collisional Heating

The collisional term in the power equation (B19) and heating equation (B10) is

$$-\int\frac{d^3\hat{\mathbf{r}}}{\hat{V}}\sum_s\int d^3\hat{\mathbf{v}}_s\frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left\langle\hat{h}_s\hat{\mathcal{C}}(\hat{h}_s)\right\rangle \tag{26.100}$$

NOTE: The collisional heating is difficult to extract from the code, so more work must be done to figure this out.

26.4 The Entropy Equation (B10) Term by Term

26.4.1 Collisional Heating

This term is identical to the collisional heating term in the Power Balance Equation (B19), so it is discussed in section (26.3.4).

26.4.2 Wave-Particle Heating (Cowley Form)

In the heating equation (B10), the Cowley form of the heating is

$$\int \frac{d^3\hat{\mathbf{R}}_s}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{q}_s \frac{\partial \langle \hat{\chi} \rangle}{\partial \hat{t}} \hat{h}_s - \frac{\partial}{\partial \hat{t}} \int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \frac{\hat{n}_s \hat{T}_s}{2} \hat{h}_s^2. \tag{26.101}$$

Writing out the time derivative on the second term, we have

$$\int \frac{d^3\hat{\mathbf{R}}_s}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{q}_s \frac{\partial \langle \hat{\chi} \rangle}{\partial \hat{t}} \hat{h}_s - \int \frac{d^3\hat{\mathbf{r}}}{\hat{V}} \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{T}_s \hat{h}_s \frac{\partial \hat{h}_s}{\partial \hat{t}}. \tag{26.102}$$

NOTE: In the code, the term $\hat{n}_s$ is not present. Although for a pure electron-proton plasma this term will cancel, it is present in the other terms in the Power Balance Equation and should be kept here for consistency.

26.5 Normalization of Collisional Heating

We begin with (19.19) and multiply by $\frac{a_0}{v_{t0}}\left(\frac{a_0}{\rho_0}\right)^2\frac{1}{v_{t0}^3 F_{00} T_0}$ to find

$$\int \frac{d^3\mathbf{R}_s/(\rho_0^2 a_0)}{V/(\rho_0^2 a_0)} \int \frac{d^3\mathbf{v}}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{T_s}{T_0}\frac{F_{0s}}{F_{00}} \left\langle \left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right) \frac{\mathcal{C}a_0}{v_{t0}} \left(\frac{h_s}{F_{0s}}\frac{a_0}{\rho_0}\right)\right\rangle_{\mathbf{R}_s} \tag{26.103}$$

$$= \quad -\sum_{\mathbf{k}} \pi \frac{T_s}{T_0}\int_0^\infty \frac{v^2 dv}{v_{ts}^3}\left(\frac{v_{ts}}{v_{t0}}\right)^3 \frac{F_{0s}}{F_{00}}\left(\frac{\nu_s a_0}{v_{t0}}\right)\left\{\int_{-1}^1 d\xi(1-\xi^2)\left|\frac{\partial}{\partial \xi}\left(\frac{h_{\mathbf{k}s}}{F_{0s}}\frac{a_0}{\rho_0}\right)\right|^2\right.$$

$$\left. + \quad \frac{v^2}{v_{ts}^2}\frac{(k_\perp \rho_0)^2}{2}\left(\frac{\rho_s}{\rho_0}\right)^2 \int_{-1}^1 d\xi(1+\xi^2)\left|\frac{h_{\mathbf{k}s}}{F_{0s}}\frac{a_0}{\rho_0}\right|^2\right\}.$$

Note here that the variable ξ is already dimensionless

$$\hat{\xi} = \xi = \frac{v_\parallel}{v} \tag{26.104}$$

and the collision frequency $\nu_s(v)$ is normalized by

$$\hat{\nu}_s = \frac{\nu_s a_0}{v_{t0}}. \tag{26.105}$$

We also write the

$$\frac{\rho_s}{\rho_0} = \frac{v_{ts}}{\Omega_s}\frac{\Omega_0}{v_{t0}} = \left(\frac{2T_s}{m_s}\right)^{1/2}\frac{m_s c}{q_s B_0}\frac{q_0 B_0}{m_0 c}\left(\frac{m_0}{2T_0}\right)^{1/2} = \left(\frac{T_s}{T_0}\right)^{1/2}\left(\frac{m_s}{m_0}\right)^{1/2}\frac{q_0}{q_s} = \frac{(\hat{T}_s \hat{m}_s)^{1/2}}{\hat{q}_s} \tag{26.106}$$

Writing all of this in normalized, dimensionless variables yields

$$\int \frac{d^3\hat{\mathbf{R}}_s}{\hat{V}}\int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{T}_s\left\langle \hat{h}_s \hat{\mathcal{C}}\hat{h}_s\right\rangle \tag{26.107}$$

$$= \quad -\sum_{\mathbf{k}}\frac{1}{\pi^{1/2}}\hat{n}_s\hat{T}_s\int_0^\infty \hat{v}_s^2 d\hat{v}_s e^{-\hat{v}_s^2}\hat{\nu}_s\left\{\int_{-1}^1 d\hat{\xi}(1-\hat{\xi}^2)\left|\frac{\partial \hat{h}_{\mathbf{k}s}}{\partial\hat{\xi}}\right|^2 + \hat{v}_s^2\frac{\hat{k}_\perp^2}{2}\frac{\hat{T}_s\hat{m}_s}{\hat{q}_s^2}\int_{-1}^1 d\hat{\xi}(1+\hat{\xi}^2)\left|\hat{h}_{\mathbf{k}s}\right|^2\right\}.$$

26.6 Thermal Velocity Definition

The input parameters of GS2 allow several choices for the definition of the definition. We concetrate here on two options for the thermal velocity, the Bill Dorland convention (`norm_option='bd'`)

$$v_{tb}^2 = \frac{T}{m} \tag{26.108}$$

and the Mike Kotschenreuther convention (`norm_option='mtk'`)

$$v_{tm}^2 = \frac{2T}{m} \tag{26.109}$$

Thus, we see the difference in the two conventions is

$$v_{tm}^2 = 2v_{tb}^2 \tag{26.110}$$

We will use the variable $\alpha = \sqrt{2}$ to keep track of factors when switching from one definition to the next. This choice is made far more convoluted because I the internal normalization always has the `mtk` definition. Thus, only in some places will this factor of alpha appear; I believe these places are where values are output to the user, such as time, wavenumber, and energy. Just how this works is not yet completely apparent.

Note the following useful transformations:

$$v_{t0m} = \alpha v_{t0b} \tag{26.111}$$

$$\rho_{0m} = \frac{v_{t0m}}{\Omega_0} = \frac{\alpha v_{t0b}}{\Omega_0} = \alpha \rho_{0b} \tag{26.112}$$

$$\hat{k}_{\perp m} = k_\perp \rho_{0m} = k_\perp \alpha \rho_{0b} = \alpha \hat{k}_{\perp b} \tag{26.113}$$

$$\hat{t}_{0m} = \frac{t v_{t0m}}{a_0} = \frac{t \alpha v_{t0b}}{a_0} = \alpha \hat{t}_{0b} \tag{26.114}$$

$$\hat{\omega}_{0m} = \frac{\omega_0 a_0}{v_{t0m}} = \frac{\omega_0 a_0}{\alpha v_{t0b}} = \frac{\hat{\omega}_{0b}}{\alpha} \tag{26.115}$$

$$\hat{x}_{0m} = \frac{x_0}{\rho_{0m}} = \frac{x_0}{\alpha \rho_{0b}} = \frac{\hat{x}_{0b}}{\alpha} \tag{26.116}$$

$$\hat{z}_{0m} = \frac{z_0}{a_0} = \frac{z_0}{a_0} = \hat{z}_{0b} \tag{26.117}$$

26.6.1 General relations of normalizations depending on thermal velocity

26.6.2 Velocity integration

I believe the velocity intergration of the distribution function depends only on the unchanging internal normalization of the code. For example, the density fluctuation is defined by

$$\delta n_s = \int d^3\mathbf{v} h_s \tag{26.118}$$

which can be expressed in terms of normalized quantities as

$$\delta\hat{n}_s = \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}\hat{n}_s\hat{h}_s. \tag{26.119}$$

I believe that $\int d^3\hat{\mathbf{v}}_s e^{-\hat{v}_s^2}$ is not changed by the user's choice of normalization, so the only change in the distribution function between thermal velocity conventions is due to the a_0/ρ_0 factor that is present in all fluctuating quantities. Thus,

$$\hat{h}_{sm} = \frac{h_s}{F_{0s}}\frac{a_0}{\rho_{0m}} = \frac{h_s}{F_{0s}}\frac{a_0}{\alpha\rho_{0b}} = \frac{\hat{h}_{sm}}{\alpha} \tag{26.120}$$

26.6.3 Questions

1. The calculations of energy integrated over space are then divided by total volume in the theory paper. If this division is not performed in the code, then the volume multiplying the energy is will have a factor of $V = a_0\rho_0^2$ multiplying it, so because space depends on the definition of thermal velocity so will this volume, and so will the energy over the volume.

2. It seems that the v_{t0} in the normalization of $A_\parallel$ does not change with the definition of thermal velocity or else the balances would be completely wrong. But

3. At what layer do we want to transition from code units to user units? Ideally, it seems this should be done just at output, so that all other calculations are consistent with the internal convention of `GS2`.

26.7 Miscellaneous Notes

26.7.1 Velocity Space Integration

Integration over velocity space includes part of the equilibrium distribution function. Ultimately, we want quantities that have been integrated over velocity to be species independent. Consider the following example of the integration over the perturbed distribution function to get the perturbed density. We begin with

$$\delta n_s = \int d^3\mathbf{v} h_s \tag{26.121}$$

and normalize by multiplying by the factor

$$\frac{1}{v_{t0}^3 F_{00}}\frac{a_0}{\rho_0}. \tag{26.122}$$

Note that our definitions of F_{00} and F_{0s} are

$$F_{00} \equiv \frac{n_0}{v_{t0}^3} \tag{26.123}$$

$$F_{0s} = \frac{n_s}{\pi^{3/2}v_{ts}^3}e^{v^2/v_{ts}^2} \tag{26.124}$$

so that their ratio is

$$\frac{F_{0s}}{F_{00}} = \frac{\frac{n_s}{\pi^{3/2} v_{ts}^3} e^{v^2/v_{ts}^2}}{\frac{n_0}{v_{t0}^3}} = \frac{\hat{n}_s}{\hat{v}_{ts}^3} \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \tag{26.125}$$

and we can use $n_0 = v_{t0}^3 F_{00}$. This gives

$$\left(\frac{\delta n_s}{n_0} \frac{a_0}{\rho_0}\right) = \int \frac{d^3\mathbf{v}}{v_{ts}^3} \left(\frac{v_{ts}}{v_{t0}}\right)^3 \left(\frac{h_s}{F_{0s}} \frac{a_0}{\rho_0}\right) \frac{F_{0s}}{F_{00}} = \int d^3\hat{\mathbf{v}}_s \hat{v}_{ts}^3 \hat{h}_s \frac{\hat{n}_s}{\hat{v}_{ts}^3} \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \tag{26.126}$$

Finally, we obtain the result

$$\delta \hat{n}_s = \int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}} \hat{n}_s \hat{h}_s \tag{26.127}$$

The routine `le_grids:integrate_species` includes the exponential and π factor in the velocity space integration. Hence, it always performs the integral

$$\int d^3\hat{\mathbf{v}}_s \frac{e^{-\hat{v}_s^2}}{\pi^{3/2}}. \tag{26.128}$$

This generally means that any normalized quantity integrated over velocity will have an extra factor of $\hat{n}_s$ in the integrand which arises as shown in equation (26.127).

This can be seen in `le_grids:integrate_species`. The line in the code is:

```
tot(:,it,ik)
=tot(:,it,ik)+weights(is)*w(ie,is)*wl(:,il)*(g(:,1,iglo)+g(:,2,iglo)).
```

Here, `weights(is)` is a species dependent multiplier passed into the routine. For example, to perform the integral $\int d^3\mathbf{v} T_{0s} h_s^2$, the factor T_{0s} would be passed in as the argument `weights(is)`. The factor `w(ie,is)` is a weight factor for integrating over the energy grid. The factor `wl(:,il)` is a weight factor for integrating over the pitch angle λ grid.

26.7.2 Other short notes

1. The routine `collisions:g_adjust` is used to convert from g to h and back. To convert from g to h, call routine with parameters (`fphi`, `fbpar`). To convert from h to g, call routine with parameters ($-$`fphi`, $-$`fbpar`).

Chapter 27

Methods for Analysis of AstroGK Results

27.1 Analysis

27.1.1 General Analysis Methods for Turbulence Runs

1. Energy Spectra: Plot the steady-state spectra of all the fields. How do the slopes of these spectra compare to theoretical expectations?

2. Steady-state at all scales: Check the time evolution of the k by k energy to determine if each scale has reached a quasi-steady state.

3. Movie of Energy Spectra: Compile a quick movie of the evolving spectra to visually see if it appears to reach a steady state. Also, is it possible to see fluctuations in energy injection pass down through the cascade?

4. Steady-State: Look at plots of the evolution of the plasma energy to see if it has reached a steady-state. Is the heating $d\mathcal{E}/dt$ fluctuating about zero (no net energy buildup in plasma)?

5. Heating: How much heating goes to ions vs. electrons?

6. Heating by Mode: Are the channels of heating peaked in the wavenumber ranges we expect? Is there ample dissipation at smaller scales?

7. Composite Spectra: After removing dissipation ranges, assemble spectra from different wavenumber ranges to get a clue about the global spectrum, similar to the Great Power Law in the Sky. This is not quite right because of varying anisotropy, but it will make a nice rough picture before trying a monumental size run (Insight Awards, etc.).

8. Electron Energy vs. Beta: Plot δ_e vs. β_i to see how the ratio P_i/P_e changes with parameters.

9. Entropy Cascade: Can we identify the entropy cascade?

10. Make movies of field lines with existing run data.

27.1.2 Analysis Tools for Development

1. Contour plot of power on the $k_\perp$-$k_\parallel$ plane. Note that this needs to use the facility that derives the actual $k_\parallel$ following the fields lines, not just the k_z as output in the `.fields` file. To get this output, I need to turn on the `write_gs` flag in the diagnostics namelist. The procedure in memory intensive, and should be done only for one timestep to get the output.

2. Polar spectra of heating rates

3. Make movies of field lines.

27.1.3 Notes on GS2 Output Files

Here are some notes on analysis and data files:

1. The file `runname.fields` contains z-structure of all k_x and k_y modes.

2. The file `runname.kspec_raw` contains polar spectra of energies for all possible $k_\perp$ from all of the k_x and k_y modes.

3. The file `runname.kspec_avg` contains binned and log-averaged values of polar spectra of energies using `nkpolar` bins. Default is such that `nkpolar`$=$ int[real(`naky` $+ 1)\sqrt{2}$]

27.2 Transient Fit by Laplace Transform for Linear Runs

The best way to get precise fits of the resonant frequency and damping rate is to fit the transient behavior of the driven, damped system.

27.2.1 Laplace Transform

Assume the linear system behaves as an linear operator with a source term

$$\frac{\partial \phi(t)}{\partial t} = \mathcal{L}\phi(t) + \mathcal{S}(t) \tag{27.1}$$

where the linear operator gives a complex eigenvalue $-i(\omega_0 - i\gamma)$

$$\mathcal{L}\phi(t) = -i(\omega_0 - i\gamma)\phi(t) \tag{27.2}$$

and the source term simply drives a constant frequency ω_s

$$\mathcal{S}(t) = \begin{cases} S_0 e^{-i\omega_s(t-t_0)} \\ 0 \end{cases} . \tag{27.3}$$

Thus, we have the equation

$$\frac{\partial \phi(t)}{\partial t} = -i(\omega_0 - i\gamma)\phi(t) + S_0 e^{-i\omega_s}. \tag{27.4}$$

To solve the the time-dependent behavior if the antenna is turned on at time t_0, we do a Laplace transform of this system.

For the source term, we use the property of Laplace transforms that

$$\int_0^\infty \mathcal{U}(t - t_0) e^{-st} dt = e^{-t_0 s} f(s) \tag{27.5}$$

where the $\mathcal{U}(t - t_0)$ is Heaviside's unit function times $F(t)$

$$\mathcal{U}(t - t_0) = \begin{cases} F(t) \\ 0 \end{cases} \tag{27.6}$$

and $f(s)$ is the Lapace transform of $F(t)$. The Laplace transformed equation becomes

$$s\hat{\phi}(s) - \phi(0) = -i(\omega_0 - i\gamma)\hat{\phi}(s) + \frac{S_0 e^{-as}}{s + i\omega_s} \tag{27.7}$$

Solving for the Laplace transform $\hat{\phi}(s)$ gives

$$\hat{\phi}(s) = \frac{\phi(0)}{s + i\omega_0 + \gamma} + \frac{S_0 e^{-as}}{(s + i\omega_s)(s + i\omega_0 + \gamma)} \tag{27.8}$$

Taking the inverse Laplace transform of this solution gives

$$\phi(t) = \phi(0)e^{-i\omega_0 t}e^{-\gamma t} + S_0 \frac{e^{-i\omega_s(t-t_0)} - e^{-i\omega_0(t-t_0)}e^{-\gamma(t-t_0)}}{i(\omega_0 - \omega_s) + \gamma} \tag{27.9}$$

Let us take the driving antenna to be turned on at $t_0 = 0$ and zero initial amplitude $\phi(0) = 0$, so we get

$$\phi(t) = S_0 \frac{e^{-i\omega_s t} - e^{-i\omega_0 t}e^{-\gamma t}}{i(\omega_0 - \omega_s) + \gamma}, \tag{27.10}$$

For late times $\gamma t \gg 1$, this solution agrees with the Fourier solution

$$\phi(t) = S_0 \frac{e^{-i\omega_s t}}{i(\omega_0 - \omega_s) + \gamma}. \tag{27.11}$$

Solving for the amplitude $|\phi(t)|^2$of the antenna as a function of time, we obtain

$$|\phi(t)|^2 = S_0^2 \frac{1 + e^{-2\gamma t} - 2e^{-\gamma t}\cos[(\omega_0 - \omega_s)t]}{(\omega_0 - \omega_s)^2 + \gamma^2}. \tag{27.12}$$

Again, at long times, we get the usual Lorentzian response

$$|\phi(t)|^2 = \lim_{t\to\infty} \frac{S_0^2}{(\omega_0 - \omega_s)^2 + \gamma^2} \tag{27.13}$$

27.2.2 Fitting the Laplace Transform Solution

There is a simple strategy for obtaining precise fits of the Lapace Transform solution. For a given driving frequency ω_s, there are three parameters to the fit: the amplitude S_0, the resonant frequency ω_0, and the damping rate γ. These parameters can be determine to high precision using the simple precodure below.

1. Run one case (A) off resonance $|\omega_0 - \omega_s| \gg \gamma$. To obtain convergence to a fraction f, the runtime needs to be

   $$t > \frac{-\ln(f/2)}{\gamma}. \tag{27.14}$$

   For a convergence of $f = 0.001$, this is approximately $t > 7.6/\gamma$.

2. From case A, you can determine the resonant frequency quite precisely by fitting the oscillation at the beat frequency $\omega_0 - \omega_s$.

3. After determining the value $\omega_0 - \omega_s$, one can determine the amplitude S_0 by fitting the final amplitude after the transients have settled down. This is only possible if $|\omega_0 - \omega_s| \gg \gamma$.

4. Run a secondd case at resonance (B) such that $|\omega_0 - \omega_s| \ll \gamma$. The required run time is the same as before.

5. From case B, determine the damping rate γ by fitting for the final amplitude.

6. Check the fit of B at all times to see if any further adjustments are necessary. Notice, if the resonant frequency of the fit is adjusted slightly, the fit changes dramatically.

27.3 Frequency Sweep Runs

1. Set $k_\perp \rho_i$, n_λ, n_E for the run.
2. Collisionality of each species C_i, C_e must be less the desired damping rate.
3. Timestep Δt must be much less than the frequency of the resonance (or the resonant frequency will be too low).
4. Guess at resonant frequency ω_0 and set initial frequency ω_i lower.
5. Choose sweeping rate $\dot{\omega}$ such that the sweep is slower than the damping rate (or the Lorentzian will be pushed over and the amplitude will show a lot of oscillations).

To plot the data using Gnuplot

1. First, extract the data from the run_name.out file using grep
   `grep 'w=' swpb1t1k10.out > swpb1t1k10.w`
2. Next use Gnuplot to plot the data against a Lorentzian with three parameters, the peak amplitude A_0, the damping rate γ and the resonant frequency ω_0.
3. To plot, type
   `plot 'swpb1t1k12.w' u 2:4, 'swpb1t1k12.w' u 2:(A*g*(($2-w)**2 + g**2)**(-0.5))`

   This will plot the data against a Lorentzian of the form

$$A(\omega) = \frac{A_0 \gamma}{\sqrt{(\omega - \omega_0)^2 + \gamma^2}}$$

4. If you want to plot the lorentzian and have gunplot automatically fit:

```
f(x)=(A/sqrt((x-w)**2.+g**2.))
fit f(x) 'test.dat' u 2:4 via A, g, w
plot 'test.dat' u 2:4, '' u 2:(f($2))
replot
```

27.4 Ion to Electron Heating Ratio Runs

1. Set $k_\perp \rho_i$, n_λ, n_E for the run. In these cases, a large number of points in velocity space ($n_\lambda = 40$ and $n_E = 40$) is necessary to resolve the very weak damping of the subdominant species.
2. Collisionality of each species C_i, C_e must be less the desired damping rate for that species; for the less damped species this can be VERY small.
3. Run the code until it converges with $\omega_{tol} = 10^{-6}$.
4. Columns 4 and 5 (the first two after heating_rate) in the .h file give the ion and electron power; copy them to the data file to keep.

To plot the data using Gnuplot

1. First, extract the data from the run_name.out file using grep
   `grep heating_rate hrb10t_01k10.out > hrb10t_01k10.h`

2. Next use Gnuplot to plot the data to check that it has stabilized by typing
   `plot 'hrb10t_01k10.h' u 2:4, 'hrb10t_01k10.h' u 2:5`

27.5 Decaying Alfven Wave Runs

1. Two runs are necessary: the first to build up a single eigenmode by driving at a certain frequency and wavenumber; the second to turn off the driving antenna and watch the wave decay

2. Wave set up runs
   (a) nstep= 10000
   (b) $nk_{stir} = 1$
   (c) ant_{off} =F
   (d) $nwrite = 10$
   (e) $navg = 20$
   (f) $omegatol = 1.0 \times 10^{-6}$
   (g) save_for_restart = T

3. Decay runs
   (a) nstep= 5000 and delt_option = check_restart
   (b) $nk_{stir} = 0$
   (c) ant_{off} =T
   (d) $nwrite = 1$
   (e) $navg = 1$
   (f) $omegatol = -1.0$
   (g) save_for_restart = F

4. Set $k_\perp \rho_i$, n_λ, n_E for the run. In these cases, a large number of points in velocity space ($n_\lambda = 40$ and $n_E = 40$) is necessary to resolve the very weak damping of the subdominant species.

5. Collisionality of each species C_i, C_e must be less the desired damping rate for that species; for the less damped species this can be VERY small.

6. Run the code until it converges with $\omega_{tol} = 10^{-6}$.

7. Columns 4 and 5 (the first two after heating_rate) in the .h file give the ion and electron power; copy them to the data file to keep.

To plot the data using Gnuplot

1. First, extract the data from the run_name.out file using grep
   `grep heating_rate hrb10t_01k10.out > hrb10t_01k10.h`

2. Next use Gnuplot to plot the data to check that it has stabilized by typing
   `plot 'hrb10t_01k10.h' u 2:4, 'hrb10t_01k10.h' u 2:5`

27.6 Analysis for Astrophysical Turbulence Runs

This section describes a number of different diagnostics and scripts for analysis of runs simulating astrophysical turbulence.

1. Parallel Fourier spectra: Analyze the `runname.fields` fields by taking each (k_x, k_y) mode and Fourier analyzing the structure along the field. This proves that the run was resolved along the field and did not reach the Nyquist frequency of the parallel grid. We use the `sm` script `/sm/agk/fields_fft.sm` to create a series of postscript files, one for each mode, showing the real space structure and fast Fourier transform. These can be made into an animated gif move using `pstogif.all` and `whirlgif`.

2. Steady State: It is extremely important that turbulence data used for analysis have reached a statistically steady state, where energy injected at large scale is balanced by total energy dissipated (mostly at small scales). There are several methods to check this:

   (a) Movie of energy spectrum: One can compile a movie of the energy spectrum using `/sm/agk/kspec_mov.sm` and inspect visually if some part of the spectrum is monotonically increasing or decreasing in time, rather than just fluctuating about some steady solution.

   (b) Total energy vs. time: A better solution is to plot the evolution of the energy in the plasma over time to see if any components are increasing or decreasing. However, this is dominated by the driving scale; the Driving scale Alfvén waves produce an oscillating behavior in $\delta B_\perp$ and δf_i (or $v_{\perp i}$).

   (c) Total energy by $k_\perp$ vs. time: We can also analyze the `runname.kspec_raw` file using `gs2_analysis.f90`, option 0, to extract the time behavior at each mode, which can then be plotted using `/agk/ek_vs_t.sm`. This is the best diagnsotic to determine if you have reached a steady state at all wavenumbers. Note the h_i seems to adjust most, slowly; you can also look at `etot` for each wavenumber to get the overall picture.

3. Contour plot of Power in $k_\perp$–k_z plane: One can use the Fortran program `kplane.e` to analyze the `runname.fields` file to create a contour map of the power in various fields on the $k_\perp$–k_z plane using the script `/sm/agk/map_kpkz.sm`. It is a bit difficult to interpret this compared to theory since you really should do this along the perturbed field lines to get the power in the $k_\perp$–$k_\parallel$ plane. But this can be a useful diagnostic to ensure that nothing awful has occurred.

27.7 Dissipation Estimates for Nonlinear Runs

1. I can use python script `tnl_hr_sum.py` to read the `runname.out.nc` file and output a `runname.tnl` file.

2. Run `/sm/kspec/swt_hk_diss.sm` to plot the estimated normalized dissipation $P/(E\omega) \sim \gamma/\omega$. Note that this does not take into account the reduction in nonlinear frequency due to dissipation, so we may need a similar estimate based on $P/(E\omega_{nl})$.

27.8 Amplitude Scan for Turbulent Driving

In this section I review methods for analyzing a suite of runs to test the proper amplitude for driving turbulence. In short, I list here the basic methods used to interpret the data.

1. Plot final spectrum for each run to extract the slope of the steady-state spectrum.

2. Plot the energy in the driving mode vs. time to show that we have reached a steady state at different levels.

3. Contour plots of mode energy on the $k_\perp - k_z$ plane. Using the `write_gs` option, I should be able to also produce the same data (or at least for v_x^2 and v_y^2) on the $k_\perp - k_\parallel$ plane; this plot should produce the Goldreich-Sridhar $k_\parallel \propto k_\perp^{2/3}$ law for driving amplitudes in the strong turbulence regime.

4. Plot the antenna power in P_{ant} vs. the heating measures $P_i + P_e$ and $P_{ci} + P_{ce} + P_{Hci} + P_{Hce}$ to show that the net plasma energy change is zero. We should be able to look at the heating data by $k_\perp$ as well to show that energy is indeed cascading to high $k_\perp$ before it is dissipated.

5. We can also plot the steady state P_{ant} vs. the estimated saturated value of the energy $\partial \mathcal{E}_{sat}/\partial t$.

6. Plot the measured value of ω_{nl}/ω as a function of the driving amplitude. Can I explain what appears to be a relation $\omega_{nl}/\omega \propto A^{3/2}$?

Chapter 28

AstroGK Utilities

28.1 Expand

`Expand` is a utility that allows you to increase the number of x and y Fourier modes and z gird points to achieve a higher spatial dynamic range.

28.1.1 Expand Input File

The input file for `expand` is a single namelist `expand`. Below is an example:

```
&expand
nproc_in = 64
nproc_out = 64

layout_in = 'yxles'
layout_out = 'yxles'

file_in = 'exp20.nc'
file_out = 'exp21.nc'

ntgrid_in = 12
nakx_in = 7
naky_in = 4

ntgrid_out = 24
nakx_out = 15
naky_out = 8

negrid_in = 8
negrid_out = 8

nlambda_in = 32
nlambda_out = 32
```

```
nspec_in = 2
nspec_out = 2

tfac = 0.25

use_Phi_in = T
use_Apar_in = T
use_Bpar_in = T

use_Phi_out = T
use_Apar_out = T
use_Bpar_out = T

ant_on = T
/
```

Note that the connection between the expand parameters `ntgrid`, `nakx`, `naky`, `negrid`, and `nlambda` and the `AstroGK` input file parameters `ntheta`, `nx`, `ny`, `negrid`, and `ngauss` are given below:

$$\texttt{ngtrid} = \texttt{ntheta}/2 + (\texttt{nperiod} - 1)\texttt{ntheta} \tag{28.1}$$

$$\texttt{nakx} = 2 \ \mathrm{int}\left(\frac{\texttt{nx} - 1}{3}\right) + 1 \tag{28.2}$$

$$\texttt{naky} = \mathrm{int}\left(\frac{\texttt{ny} - 1}{3}\right) + 1 \tag{28.3}$$

$$\texttt{negrid} = \texttt{negrid} \tag{28.4}$$

$$\texttt{nlambda} = 2 \ \texttt{ngauss} \tag{28.5}$$

To convert from `nx` and `ny` to `nakx` and `naky`, here is a conversion table:

nx	nakx	ny	naky
10	7	10	4
15	9	15	5
16	11	16	6
24	15	24	8
32	21	32	11
64	43	64	22

Chapter 29

Performance Issues and Problems

29.1 AstroGK Code Bugs

29.1.1 Outstanding Bugs

1. Adaptive hypercollisionality for any β_i
2. Expand does not work for 256 to 1024 processors (only 1020 files created)
3. For adaptive hypercollisionality, I must use a variable number of species—at the moment it is fixed at 4. (31 OCT 07)
4. There appears a crash for $\beta_i = 100$ run with plasma energy running away. This may be due to centering in time (fexpr=0.5) or some other cause. Adaptive hypercollisionality is not working here.
   It seems that the temporal implicitness parameter may help in this case.
5. Is the minority species crashing because of assumed two species in heating calculations?
6. In EXPAND, the code cannot reduce the number of files created due to a bug that seems to be related to having file_proc both as a global variable and as a subroutine variable. (10 DEC 07)

29.1.2 Potentially Fixed Bugs

1. $\mathbf{E} \times \mathbf{B}$ shear issue with large runs
2. Adaptive hypercollisionality for $\beta_i = 1$
3. Heating diagnostics for more than two species
4. NetCDF filenames for nproc¿10000
5. Expand NetCDF filenames for nproc¿10000
6. Restart in AstroGK used .0000 hardwired for timestep and amplitude NetCDF file. This did not work when proc numbers changed to 5 digits .00000. I believe it has now been fixed. (Nothing seemed to be wrong with Expand).

29.1.3 Fixed Bugs

1. none

29.2 Memory requirements

1. $nmesh = (2 * ntgrid + 1) * 2 * nlambda * negrid * ntheta0 * naky * nspec$

2. nx and ny should have only prime factors 2,3 and 5.

29.3 General Coding

1. Parallel HDF output has been implemented in the code.

2. 3rd order Adams-Bashforth

29.4 Hypercollisionality

1. A wavenumber dependent collisionality coefficient will accomplish resistivity and a perpendicular viscosity (due to FLR terms in collisions).

2. Replace ν with $\nu_0 + (k_\perp \rho_i)^n \nu_n$

29.5 Hyperviscosity

1. Heating began at a nonzero value: This appears to be because an incorrect form of the hyperviscous operator ($\nu_H \nabla_\perp^4 h_s$) was used. Different numbers of stirring modes seems to have no effect on this problem. In decaying runs, heating also begins nonzero. As the timestep is decreased, heating by hyperviscosity increases (slightly).

29.6 Entropy Balance

1. The difference between the QDH heating and collisional heating is simply a scaling factor, but there is no obvious dependence of that scale factor on any parameters.

2. The electron entropy also shows the scale factor, but there is also, in a few cases, a slight phase shift; this could be simply a very small computational effect, however.

3. Is the electron entropy poor because the collisions are much too high (and velocity space resolution too low) to resolve the small electron damping?

4. Does the entropy balance change as the collisionality coefficient is changed?

5. Where is the Landau resonance with respect to the thermal velocity for a given species, and how is this realted to the typical damping rate?

6. Why is the entropy going negative in some places (particularly for electrons)?

7. Consider what the ion and electron heating terms mean compared to the collisional terms. If the collision rate is zero, no entropy can possibly be generated, but there must still be "heating" in the form of structure in velocity space. Is there a time lag between this collisionless "heating" and the actually entropy increase due to collisions?

29.7 MHD Run Stability

1. At high β, the MHD runs tend to be unstable. Bill has seen this strange behavior before.

2. Low β runs seem to be working

3. The problem is related to this. On the RHS of GK equation is dA/dt. When `fexpr`$= 1/2$ and `bakdif`$= 0$, this term vanishes exactly at the Nyquist frequency (when $\lambda = 2\delta x$).

4. Using a Beam-Warming scheme (for parallel space and time) with compact differencing and going from 2nd to 4th order, we can kill off this Nyquist frequency problem.

29.8 Fixed Code Bugs

(a) August 2007: Fixed bug in initial conditions for electromangetic runs. BD and TT.

29.9 Known Code Bugs

(a) In `gs2_diagnostics.f90:init_polar_spectrum`, there is an error in calculating the correction factor to the raw polar spectrum. The correction is calculated to be

$$e_{bincorr} = \frac{\pi k_{\perp bin}}{n_{bin}}, \tag{29.1}$$

but this depends in the scale $k_\perp$, which is incorrect. The correction factor should be scaled to be

$$e_{bincorr} = \frac{\pi k_{\perp bin}}{n_{bin} k_{\perp binmin}}, \tag{29.2}$$

so that the number does not depend on the absolute value of $k_\perp$, only on the relative value compared to the minimum $k_\perp$ in the box.

29.9.1 Tomo's Code Bugs

(a) Can we remove `theta0` and `ntheta0`? — almost done excepet for `agk_dist_io_hdf.f90`.

(b) check the sign of wstar related to background density or temperature gradient.

(c) ei collision frequency included for electron momentum conserving term?

(d) Why at all is `vpac` introduced? It is different from `vpa` when `al` becomes larger than unity which never happens in principle.

(e) Do we want the default value of `chop_side` = `.true.`?

(f) At `MODULE DECLARATIONS` section in `Makefile`, `LINKS` should include `file_utils.f90`: fixed. Also at `DIRECTIVES` section, there are multiple declaration of `file_utils.o` (maybe case dependent?), but is the second line needed in the first declaration? If needed, `file_utils.f90` may also need to be added in dependency.

(g) Inconsistent default values are defined for `fphi`, `fapar` and `faperp` in `run_parameters.f90`.

29.10 Potential Code Bugs

(a) In `collisions.f90`, calculating the array `vnewk` has the conditional `if (spec(is)%type == electron_species) then`. Does this mean that in single species runs, when you are folling the code into thinking it has fluid electrons when really you are using fluid ions, that it is keeping the `zeff` term in the collision operator, which should only be present for electrons and not ions?

29.11 Potential upgrade

I didn't know where to put them so just move whereever you want (TT).

1. ion drag term in ei collision — half done (TT)
2. Legendre pitch-angle collision operator
3. upgrade $\partial/\partial z$ to fourth order compact finite difference
4. upgrade time stepping to Greg Hammetts' suggestion
5. upgrade collision to AB3-BDF2
6. implement isothermal electron fluid eqn — half done (TT)
7. OpenMP (BD)
8. remove ecut for too high value (TT)
9. Is LU-decomposition faster than matrix multiplication? (TT)
10. BLAS library for matrix multiplication? (BD) — Does `aminv` need to be complex? Making it real may enhance the performance. But it may need complex values when `tprim` is finite. Also, store the `transpose`d matrix and calculate by `matmul`. (TT)

Chapter 30

Development

30.1 List of Modifications for Development

1. Code up the entropy and power balances in the heating output
2. Parallel I/O (HDF): This is VERY important for large runs with `nproc` $>$ 4`negaussnegridnspec`.
3. Fourier treatment of the field-aligned coordinate z

30.2 May 20, 2008

AstroGK version r307 known issues:

1. NetCDF output file has 4 or 5 digits.

30.3 April 19, 2007

Here is a list of papers that need to be written.

Computational or Gyrokinetic

1. Nonlinear Spectrum
2. AstroGK Validation
3. The Vegas Rule and minority species heating
4. General heating paper (Bill)
5. Critical Balance in NL simulations
6. Black Hole accretion disk paper
7. Validity of Linear Damping rates

8. Nonlinear interactions (Local or nonlocal)

Analytical

1. Turbulent Cascade (1-D) (almost complete)
2. Slow wind turbulent driving paper
3. Critical Balance with 2-D MHD cascade
4. Full 2-D Kinetic Cascade
5. Kinetic Velocity Shear instability driving paper (Phil)

To Accomplish this week

1. Rework turbulent cascade regarding Alex's comments
2. Wrap up Turbulent cascade paper
3. Try AstroGK nonlinear run (same as swt10a.in) (submitted)
4. Debug Hot Plasma Dispersion relation (WHAMP, Eliot's old version)
5. Begin writeup of SW NL spectrum paper (PRL)
6. Start AstroGK validation paper (JCP)
7. Try isotropic HD cascade numerically
8. Look into Corona turbulence and respond to Ben

Completed tasks:

1. Rework turbulent cascade regarding Alex's comments

30.4 March 29, 2007

Completed tasks:

1. Establish SVN for AstroGK
2. Establish SVN for AstroGK documentation
3. Learn as much as I can about the inner workings of GS2 from Tomo
4. Discuss Alex's Tome
5. Debug Expand and run NL sims
6. Begin writeup of SW NL spectrum paper

30.5 Oct 24, 2006

I would like to do an analytical calculation estimating the power spectra for strong turbulence for all fields in any regime. The following ideas would be incorporated:

1. Linear and nonlinear physics will be decoupled in this model.
2. Linear physics will be derived from gyrokinetic dispersion relation, particularly the equipartition of energy between fields.
3. Nonlinear physics will be treated by a local in wavenumber prescription inspired by the fully nonlinear terms in the equations.
4. Given an input energy at a given scale, the energy in the fields is run until an equilibrium is established.
5. For hydrodynamic Kolmogorov turbluence with Laplacian viscosity, this works nicely. For more complicated systems, determining the interaction between different fields may become somewhat tricky.

30.6 Sept 12, 2006

1. Grand Solar Wind Run: This run will pull together four regime that span from the collisionless MHD regime to the electron Damping regime, covering four decades of scale:

	Regime	$k_\perp \rho_i$ Range
I	Collisionless MHD	0.02 to 0.2
II	Transition	0.2 to 2.0
III	Kinetic Alfven Wave	2.0 to 20
IV	Electron Damping	20 to 200

   We choose as parameters for this run $\beta_i = 1$ and $T_i/T_e = 3$, which puts electron damping scale at $k_\perp \rho_i = 74 k_\perp \rho_e$. Ideally, we can do this run in order of decreasing scale to attempt to match amplitudes as closely as possible.

   Each of these segments can be done on Dawson using $n_x = n_y = 32$. The only outstanding problem at the moment is hypercollisions that operate effectively in the MHD regime. Also, we should be able to reduce computation by reducing velocity phase space in regime I and using adiabatic ions in regime IV.

2. Large Transition Regime Run (Black Hole Accretion): This run will test the total damping occuring with large β_i around $k_\perp \rho_i \sim 1$. For this case we need a few supplementary runs to determine that we can achieve total damping of the turbulence onto the ions as $\beta_i \to 1$.

   (a) Main run: $\beta_i = 100$, $T_i/T_e = 100$, and $0.2 \leq k_\perp \rho_i \leq 6.0$ (requiring $n_x = n_y = 96$). Hopefully this run will show 5/3 slope on the MHD portion and 7/3 on the KAW portion.

   (b) Supplementary run: Same as above but increasing β_i to $\beta_i = 300$ or possibly $\beta_i = 1000$ to demonstrate complete ion damping. This can be done with a smaller box $0.2 \leq k_\perp \rho_i \leq 2.0$ (requiring only $n_x = n_y = 32$).

3. Run kaw4 and kaw5 longer to see if things really settle to a nice equilibrium. But KAW is the worst regime because the frequency at the driving scale is slow, while the frequencies at higher modes are much faster. If you need a full turn-around time (or more) at the driving scale to reach equilibrium, then you prefer a much faster frequency at the driving scale. (Perhaps KAW would be much better done at $\beta_i \sim 1$ and $T_i/T_e \sim 3$ because then you don't have to run for so long.

4. Estimate total requirements (size and number of timesteps as well as memory and CPU hours) for Seaborg sized runs.

5. Run code with driving in the middle of the box to look for any possible inverse cascade of magnetic field. If B-field energy piles up at large scale, we have evidence for dynamo action.

6. Try to figure out how low you can go with velocity phase space for MHD runs. (I think $(8,4)$ should definitely work, but can we go to $(5,3)$?)

   (a) Test $(5,3)$ first with same parameters as other MHD run. If unstable, then increase to $(8,4)$. If stable, try $(4,2)$ (why not get greedy?)

30.7 Code Modifications

30.7.1 Diagnostics

1. (GGH) Put fluctuating energy diagnostic into heating.

2. (GGH) Fix energy and entropy balance for single species runs

3. (GGH) Determine correction procedure for $E(k_\perp)$ calculations.

4. Moving GS diagnostic for $k_\parallel$ out of code.

5. Energy diagnostic $E(|k_\perp|)$.

6. Energy diagnostic $E(|k_\parallel|)$.

7. Energy Contours on $k_\parallel$-$k_\perp$ plane for both weak and strong turbulence

8. Determine structure of k by k nonlinear energy transfer.

   (a) Use reduced MHD as a test bed for this energy transfer diagnostic.

   (b) Possible of velocity-space dependence of nonlinear effects and energy transfer. Do not integrate over velocity and look at $h^2(v)$ at a given k to see if cascading energy has any dependence on v. REAL PHYSICS!

   (c) Look for velocity space tails in distribution function driving by turbulence. $E_\parallel$ is never very big, and you must overcome the collisions with a Driecer field.

30.7.2 Dynamics

1. (BD) Antenna amplitudes in continuation files for restart.
2. (BD) Check for restart bug.
3. (BD) Fix BD norm for heating calculations.

30.7.3 Eye Candy

1. Field-line following with tracer particles.
2. Tracer particle evolution.
3. Flux tube in a black hole accretion disk to zoom into flux tube and looking at small-scale turbulence.

30.8 Analytical

30.8.1 Short Term

1. Determine hyperdiffusivity terms in reduced MHD limit.
   (a) Compare with dispersion relation for reduced MHD.
   (b) Look at relative amplitude of $\phi, A_\parallel$ as a function of ν, η.
2. Linear dispersion relation using density and momentum conserving Krook collision operator. Comparison to reduced MHD slow wave dispersion relation.
3. Model energy cascade in turbulence numerically to estimate effect of changing damping rate on cascade.
4. Derive nonlinear tranfer terms analytically. Nonlinear transfer terms dependence on particle energy or pitch angle may be illuminated by this.
5. Conserved Quantities
   (a) Clean up Energy
   (b) Helicity is $\int A_\parallel B_\parallel$ plus helicity breaking terms.
   (c) Enstrophy, Helicities, etc.
   (d) Magnetic Moment, $\mu = \frac{mv_\perp^2}{2B}$ is a conserved quantity in gyrokinetics. Integrated over the box it should still be conserved, so perhaps we can use that as a clue to find another quantity. HINT: Look at Antonsen and Lane in their formulation of GK which uses coordinates (E, μ). Perhaps finding energy in their simulation will be similar.
6. Clean decomposition of Alfven and Slow mode energy (Entropy mode).
7. Determine Poynting Flux and Local version of conserved energy.
8. Determine Reynolds and Maxwell Stress

30.8.2 Long Term

1. Turbulent cascade model using GK as base and critical balance as criterion for nonlinear energy transfer.

30.9 Documentation

30.9.1 Short Term

1. Treatment of Fourier series and complex terms.

30.9.2 Long Term

1.

30.10 Code Runs

30.10.1 Short Term

1. DONE: Locate and implement some run experiement/project management software to coordinate our runs.
2. Determining window of validity of collision frequency (Plataeu). How is minimum collisionality dependent on velocity space resolutions? How is max collisionality related to desired damping rate and does this depend on parameters β. etc?
3. Determine amplitude of driving for weak/strong turbulence
4. Run $\beta = 100$ case over $k_\perp \rho_i$ from 0.1 to 3.
5. Testing hyperviscosity, especially convergence test in nonlinear runs while scaling hyper term with box size keeping all else fixed (gridnorm=T).
6. What is difference in resolution in ngauss vs. negrid? How does it affect collisionality, etc? Bill believes only one is important (ngauss more important than negrid?)?
7. Determine minimal velocity-space resolution for MHD runs. Would like 256^3 grid in space.
8. Perform HUGE MHD run as soon as possible.
9. Entropy balance convergence with space and time resolution. Could be that energy balance calculations need implicit time derivative, not explicit.
10. Test higher $k_\parallel$ modes in nonlinear run to verify that code is recovering the linear theory result adequately.
    (a) Extract a single mode and run it to see if it agrees with linear theory.
    (b) Issue with timestep $\omega\Delta t$: For the maximum $k_\perp v_\perp$ on the grid, the timestep satisfies $k_\perp v_\perp \Delta t < F_{CFL}$. More precisely, $[\chi,] \sim k_\perp v_\parallel A_\parallel$. Because there are a few high $v_\parallel$'s in code, so although $\phi > A_\parallel$, this $v_\parallel A_\parallel$ may dominate the Courant condition.

11. Run Slow waves in Alfven turbulence, and see that $k_\parallel$ of slow mode does not increase.

30.10.2 Intermediate Term

1. Heating scan of parameters
   (a) Define a standard acceptable box size and resolution for transition regime simulations $64 \times 64 \times 128$ (k range of 20) running from $k_\perp \rho_i$ from 0.2 to 4.0.
   (b) Run β scan at fixed T_i/T_e.
   (c) Run T_i/T_e scan at fixed β.
   (d) Compare to Quataert-Gruzinov predictions about heating vs. parameters. Put points on there plot to check their prediction.
2. Orszag-Tang Decaying Turbulence problem

30.10.3 Long Term

1.

30.11 Future Projects

1. Demonstrate or test locality of nonlinear interactions in k-space.
2. Show the transition from weak to strong turbulence and determine the threshold and its parameter dependence. Study the transition.
3. Spectral features of density fluctuations in turbulence and see if slope depends on beta (other than 5/3 would be exciting).
4. MHD turbulence heating at very low β for solar coronal heating. Include non-hydrogenic particle species to look at Z-dependence of minority heating.
5. Oft-cited papers:
   (a) More thorough investigation of heating with more beta and temperature ratios.
   (b) Low beta coronal heating problem.
   (c) Weak to strong transition and overdriving. Do we believe Goldreich-Sridhar?
6. Reconnection
   (a) Linear Dispersion relation for reconnection problem. (Shooting code, global vs. local boundary layer solution).
   (b) Heating out of reconnection.
7. Generalize gyrokinetic theory for Bimaxwellian equilibrium distribution function $F(E,\mu) \propto \exp -[(E-\mu B)/T_\parallel] \exp -[(\mu B)/T_\perp]$.

8. Heating due to microturbulence in tokamaks.
9. Study of velocity-space anisotropy in perturbed distribution due to plasma heating in turbulence. Document existence or non-existence of non-thermal features.

30.12 Astrophysical Topics/Questions for Study

1. Reconnection in the strong guide field limit
2. Collisionless shocks in the limits of a strong field parallel to the shock front (perpendicular to the normal). This will only work if the gradient is large scale with respect to the ion Larmor radius (thus the need for a strong field).
3. Work out collisional gyrokinetic dispersion relation and follow the evolution of the slow wave through the mean free path scale.
4. Add diagnostics for density perturbations
5. Plot gamma vs. nu for the whole range of varying nu (Expect poor unconverged behavior too low nu, and then collisionless damping to gradually decrease as collisionality is increased.)
6. What is the relation of the pitch angle scattering collision operator to real collisions and resisitivity? How does the behavior limit as collision rate increases, does it asymptote to the viscous and resistive behavior of an MHD fluid (non-ideal)?
7. AND, in what limit does resistive MHD pertain to reality?

30.13 Reading

1. Morrison papers: Full nonlinear stability criterion for a Vlasov Plasma.
2. Hallatschek
3. Krommes
4. Kadomtsev: Plasma Turbulence (80 pages)
5. Brizard and Hahm (2006)

30.14 May 9, 2006

1. Hypercollisionality: Implement and test
2. eMHD cascade slope (see Bale)
3. Particle and momentum conservation in collisions (and hypercollisions)
4. Density conserving Krook operator in dispersion relation

5. Determine the dependence of the collisionless damping rate plataeu in collision frequnecy depending on velocity space resolution.

6. Determine limits of weak/strong turbulence depending on amplitude of driving. Determine the linear saturation amplitude and how it relates to driving amplitude.

30.15 Feb 16, 2006

1. Determine all possible quantities to output
   (a) Figure out all pieces of Energy/Entropy Balances
2. Figure out hyperresisitivity implementation for $\delta B_\parallel$
3. Verify entropy and power balances for hyperresistivity
4. Test convergence of entropy balance with increasing velocity-space resolution
5. Return response to referee
6. Sort out definition of entropy, enstrophy, etc. and their equivalents in gyrokinetics
   (a) What are the conserved quantities in gyrokinetics?
   (b) In 2-D fluid turbulence, energy $E \equiv |\nabla_\perp \phi|^2 = v^2$ and enstrophy $|\nabla \phi|^2$ are conserved quantities. (these may be wrong.)
   (c) Can we define similar quantities in gyrokinetics?
   (d) In reduced MHD, magnetic and cross helicity are conserved. They must also be conserved in the long wavelength limit of gyrokinetics. What form do they take?
   (e) The fluctuating energy $\tilde{E}$ is a conserved quantity, which includes $(\delta f)^2$ and $|\delta \mathbf{B}|^2$. But is there more included in here? What is form of the fluctuating energy in the long wavelength limit?
   (f) Reference: Hallatschek derives the Gibbs Free Energy (fluctuating energy).
   (g) Hyper terms may be accomplished by using a collision operator with a factor of k^2 in it. A Hyperresistivity may be accomplished by using the same collision operator but with a k dependent coefficient, such as $\nu C(h) + \nu_H k^2 C(h)$.
   (h) Current collision operator is second order in pitch angle, $\partial^2/\partial\psi^2$. We could operate twice, effectively giving a hyper operator that goes like $\partial^4/\partial\psi^4$.
7. Hyperdiffusivity
   (a) Hyper terms may be accomplished by using a collision operator with a factor of k^2 in it. A Hyperresistivity may be accomplished by using the same collision operator but with a k dependent coefficient, such as $\nu C(h) + \nu_H k^2 C(h)$.

(b) Current collision operator is second order in pitch angle, $\partial^2/\partial\psi^2$. We could operate twice, effectively giving a hyper operator that goes like $\partial^4/\partial\psi^4$.

(c) If you have no resistivity and only viscosity then you may end up with a high Prandtl number plasma and accodringly weird behavior.

(d) Is a hyperviscous term in the electron equation equivalent to a hyperresistivity? One needs to work this out to see what you get in the reduced MHD limit.

8. Think about the transfer of energy from k to k and how to define that in terms of diagnostics

   (a) $\int d^3\mathbf{r} \int d^3\mathbf{v} f[f,g] = 0$ because of integration, but each term in Fourier series is not zero, only the sum.

   (b) Perhaps we can calculate the typical nonlinear transfer term and see how the size depends on mass ratio; then perhaps we can put a mass-ratio dependence into the hyperviscosity coefficient in order to even out the amount of heating going into different species.

   (c) We want the hyperviscosity to mimic a tranfer of energy to smaller scales. So we don't want an excessive amount of energy poured into the electrons (as seems to be the case now.)

   (d) Energy balance with both hyoerdiffusive terms seems to be dominated by the hyperviscosity.

9. Linear Collisional Gyrokinetic Dispersion relation

   (a) Reference: Phil Snyder's Thesis

   (b) Electron collisions are important for shear Alfven wave, so one must ensure that the collision operator is momentum conserving.

10. Linear Hyperviscous Gyrokinetic Dispersion relation

   (a) Why is it ill-behaved? Can I get a nice comparison that will work?

Chapter 31

Visualization Tools

31.1 Plotting Notes in Gnuplot

1. To plot, type
   `plot 'swpb1t1k12.w' u 2:4, 'swpb1t1k12.w' u 2:(A*g*(($2-w)**2 + g**2)**(-0.5))`

   This will plot the data against a Lorentzian of the form

$$A(\omega) = \frac{A_0\gamma}{\sqrt{(\omega-\omega_0)^2+\gamma^2}}$$

2. Surface plots:
   `splot 'iaw1a.dist' u 1:2:3`
   This plots a surface plot of variable in column 3 on the plane defined by variables 1 and 2. To set the viewing angle, use
   `set view 30,60,1,1`
   Other good possible choices are $(0, 30, 1, 1)$ or $(90, 90, 1, 1)$.
   `set cntrp levels 50` sets number of contour levels.
   `(un)set con` sets contours on bottom.
   `(un)set surf` sets surface plot.

3. Shortcuts:
   `set da s l` means `set data style lines`.
   `set au` to go back to automatic ranging.

31.2 Graphics and Plotting Packages

1. SCI Run (downloadable from the web

2. Vis5d

31.3 Plotting Magic with SuperMongo

31.3.1 Colors

The default colors in are: white, black, red, green, blue, cyan, magenta, yellow. Here are a few commands for defining some additional colors:

```
   #DEFINE NEW COLORS
# Brown
CTYPE = CTYPE() concat 139 + 256*(69 + 256*19)
CTYPE = CTYPE(STRING) concat 'brown'
# Orange
CTYPE = CTYPE() concat 255 + 256*(165 + 256*0)
CTYPE = CTYPE(STRING) concat 'orange'
# Purple
CTYPE = CTYPE() concat 160 + 256*(32 + 256*240)
CTYPE = CTYPE(STRING) concat 'purple'
# Pink
CTYPE = CTYPE() concat 255 + 256*(20 + 256*147)
CTYPE = CTYPE(STRING) concat 'pink'
```

Chapter 32

Proposed GS2 Runs

32.1 Grand Solar Wind Runs

Here we want to model the solar wind from the Collisionless MHD Regime ($k_\perp \rho_i \ll 1$) through the Transition Regime ($k_\perp \rho_i \sim 1$) through the Kinetic Alfven Wave Regime ($k_\perp \rho_i \gg 1$ and $k_\perp \rho_e \ll 1$) to the Electron Dissipation Regime ($k_\perp \rho_e \sim 1$). We choose $\beta_i = 1$ and $T_i/T_e = 3$.

1. Collsionless MHD Regime $0.08 \leq k_\perp \rho_i \leq 1.04$
2. Transition Regime $0.4 \leq k_\perp \rho_i \leq 4.0$
3. Kinetic Alfven Wave Regime $4.0 \leq k_\perp \rho_i \leq 40$
4. Electron Dissipation Regime $40 \leq k_\perp \rho_i \leq 400$
5. Collisional Transition Regime $0.4 \leq k_\perp \rho_i \leq 5.2$

32.2 Black Hole Accretion/Transition Regime

Here we want to study high β_i, two temperature $T_i/T_e = 100$ turbulence to understand how much of the turbulent energy is deposited in ions as opposed to electrons as a possible explanation for underluminous super-massive black hole accretion disks.

1. Electron Energy vs. Beta: Plot δ_e vs. β_i to see how the ratio P_i/P_e changes with parameters.

32.3 Effect of Driving Different Modes

1. How different are the dynamics and energetics when modes are driven in both directions as opposed to just one direction?
2. How is this related to imbalanced cascades (lithwick and Goldreich)?

32.4 Driving Amplitude Scan

1. KAW regime.
2. Investigate the transition from weak to strong turbulence in the various regimes
3. Develop a scaling law for the transition from weak to strong turbulence
4. What happens when you overdrive?

32.5 Driving at High Wavenumbers

1. Is there any evidence of an inverse cascade in either collisionless MHD, transition, or KAW regimes? Which fields might show this?
2. Drive at high $k_{\|}$.
3. Drive the box weakly to observe effects.
4. Look at the difference when drving with waves going in one or both directions.

32.6 Minor Ion Heating

Add in a third species with $n = 10^{-5}$ of the other species with a small q/m and look at its heating rates. Is this due primarily to nonlinear phase mixing?

1. Do minority ion species with low q/m get preferentially heated through nonlinear phase mixing?
2. Third species: $q = 2$, $m = 12$ and $n = 10^{-5}$.
3. Step 1: Do a linear run with and without the minority species. If you are not near $k_\perp \rho_m \sim 1$, then the minority ion m should be heated very little, the majority of heating going to protons or electrons.
4. Step 2: Perform a nonlinear run and look at the heating of the minority. The heavier ion, with a much larger $\rho_m > \rho_i$, should be strongly heated by the process of nonlinear phase mixing (responsible for the entropy cascade).

32.7 Nonlinear Transfer of Energy

Is there any evidence that energy can be transferred nonlocally in wavenumber space, particularly when ρ_i is a special scale in the problem?

1. Can we calculate the energy that is transferred from one wavenumber to another? Since interactions have at least three waves, can you say from which mode energy came and to which it went?

32.8 Slow Mode Damping

1. Put by hand a slow mode at longest wavelength in the box
2. Then put in into a fully turbulent box

32.9 Alfvén Wave Packet Collisions

1. Watch them not interact until they hit each other, then they scatter off of each other.
2. Can we do the same thing in the KAW regime?

Appendix A

Useful Mathematics

A.1 Gamma Function Definitions

test

Integrations over $v_\perp$ involve pairs of Bessel functions and can be written as modified Bessel functions. Three such integrals arise in the calculation of the linear gyrokinetic dispersion relation; we label them $\Gamma_0(\alpha)$, $\Gamma_1(\alpha)$, and $\Gamma_2(\alpha)$. These integrals are

$$\begin{aligned}
\Gamma_0(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-v_\perp^2/v_{th}^2} = I_0(\alpha)e^{-\alpha}, \\
\Gamma_1(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} \frac{2v_\perp^2}{v_{th}^2} \frac{J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}} e^{-v_\perp^2/v_{th}^2} = [I_0(\alpha) - I_1(\alpha)]e^{-\alpha}, \\
\Gamma_2(\alpha) &= \int_0^\infty \frac{2v_\perp dv_\perp}{v_{th}^2} \frac{4v_\perp^4}{v_{th}^4} \left[\frac{J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)}{\frac{k_\perp v_\perp}{\Omega}}\right]^2 e^{-v_\perp^2/v_{th}^2} = 2\Gamma_1(\alpha),
\end{aligned} \tag{A.1}$$

where I_0 and I_1 are the modified Bessel functions, the argument is $\alpha = \frac{k_\perp^2 \rho^2}{2}$.

In the large-scale limit $k_\perp^2 \rho_i^2 \ll 1$, or $\alpha_i \ll 1$, we can expand the functions $\Gamma_0(\alpha_s)$, $\Gamma_1(\alpha_s)$, and $\Gamma_2(\alpha_s)$ as follows: $\Gamma_0(\alpha_s) \simeq 1 - \alpha_s$, $\Gamma_1(\alpha_s) \simeq 1 - 3\alpha_s/2$, and $\Gamma_2(\alpha_s) \simeq 2 - 3\alpha_s$.

A.2 Useful Manipulations and Properties

1. Ring Average Defintions

   (a) Ring Average at constant guiding center $\mathbf{R}$

$$\langle a(\mathbf{r}, \mathbf{v}, t)\rangle_{\mathbf{R}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{R} - \frac{\mathbf{v} \times \hat{\mathbf{z}}}{\Omega}, \mathbf{v}, t) \tag{A.2}$$

(b) Ring Average at fixed position $\mathbf{r}$

$$\langle A(\mathbf{R}, \mathbf{v}, t)\rangle_{\mathbf{r}} = \frac{1}{2\pi} \oint d\theta A(\mathbf{r} + \frac{\mathbf{v} \times \hat{\mathbf{z}}}{\Omega}, \mathbf{v}, t) \tag{A.3}$$

2. Demonstrate the property

$$\int d^3\mathbf{r} \langle A\rangle_{\mathbf{r}} a = \int d^3\mathbf{R} A \langle a\rangle_{\mathbf{R}} \tag{A.4}$$

Here we take a function of position in space $a(\mathbf{r}, \mathbf{v}, t)$and a function of guiding center position $A(\mathbf{R}, \mathbf{v}, t)$. The Fourier decompositions of these functions are

$$a(\mathbf{r}, \mathbf{v}, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{A.5}$$

and

$$A(\mathbf{R}, \mathbf{v}, t) = \sum_{\mathbf{k}} A_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{R}} \tag{A.6}$$

Performing ring averages on each of these to convert them to functions of the alternative spatial coordinates, we have

$$\langle a(\mathbf{r}, \mathbf{v}, t)\rangle_{\mathbf{R}} = \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) a_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{R}} \tag{A.7}$$

$$\langle A(\mathbf{R}, \mathbf{v}, t)\rangle_{\mathbf{r}} = \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{A.8}$$

The left-hand side of (A.4) becomes

$$\begin{aligned} &\int d^3\mathbf{r} \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \sum_{\mathbf{k}'} a_{\mathbf{k}'}(\mathbf{v}, t) e^{i\mathbf{k}'\cdot\mathbf{r}} = \sum_{\mathbf{k}}\sum_{\mathbf{k}'} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}}(\mathbf{v}, t) a_{\mathbf{k}'}(\mathbf{v}, t) \int d^3\mathbf{r} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \\ &= \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}}(\mathbf{v}, t) a_{-\mathbf{k}}(\mathbf{v}, t) = \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}}(\mathbf{v}, t) a^*_{\mathbf{k}}(\mathbf{v}, t) \end{aligned} \tag{A.9}$$

where we have used the properties

$$\int d^3\mathbf{r} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}} = \delta(\mathbf{k} + \mathbf{k}'), \tag{A.10}$$

and the reality condition

$$a_{-\mathbf{k}} = a^*_{\mathbf{k}}. \tag{A.11}$$

The right hand side can be handled in the same way to yield

$$\begin{aligned} &\int d^3\mathbf{r} \sum_{\mathbf{k}'} A_{\mathbf{k}'}(\mathbf{v}, t) e^{i\mathbf{k}'\cdot\mathbf{R}} \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) a_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}}\sum_{\mathbf{k}'} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{\mathbf{k}'}(\mathbf{v}, t) a_{\mathbf{k}}(\mathbf{v}, t) \int d^3\mathbf{r} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}} \\ &= \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A_{-\mathbf{k}}(\mathbf{v}, t) a_{\mathbf{k}}(\mathbf{v}, t) = \sum_{\mathbf{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) A^*_{\mathbf{k}}(\mathbf{v}, t) a_{\mathbf{k}}(\mathbf{v}, t) \end{aligned} \tag{A.12}$$

Since the reality condition means that quadratic functions must be real, we note that $A_{\mathbf{k}}(\mathbf{v},t)a^*_{\mathbf{k}}(\mathbf{v},t) = A^*_{\mathbf{k}}(\mathbf{v},t)a_{\mathbf{k}}(\mathbf{v},t)$. Thus, we have proven (A.4).

A.3 General Math

1. Vector integration by parts

$$\int d^3\mathbf{r}\mathbf{A}\cdot\frac{\partial f}{\partial \mathbf{r}} = A_x f|_{x_i}^{x_f} + A_y f|_{y_i}^{y_f} + A_z f|_{z_i}^{z_f} - \int d^3\mathbf{r} f \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{A} \tag{A.13}$$

2. Vector Differential

$$\frac{\partial}{\partial \mathbf{r}} = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} = \nabla \tag{A.14}$$

A.4 Bessel Functions

1. One definition of the Bessel Function J_0 (9.1.18) is

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{iz\cos\theta} \tag{A.15}$$

2. Derivatives: (9.1.28)

$$\frac{dJ_0(z)}{dz} = -J_1(z) \tag{A.16}$$

3. Higher Order Forms: (9.1.21)

$$J_n(z) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} d\theta\cos(n\theta)e^{iz\cos\theta} \tag{A.17}$$

4. The form of an Ascending Series expansion for the Bessel functions (9.1.10) is given by

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)} \tag{A.18}$$

Thus for the zeroth Bessel Function

$$J_0(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \cdots \tag{A.19}$$

This can be used to find the small argument expansion for the Bessel Function.

5. The large argument expansion for the Bessel Function (9.2.1) is

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{\cos(z - \nu\pi/2 - \pi/4) + e^{|\mathcal{J}_z|}\mathcal{O}(|z|^{-1})\right\} \tag{A.20}$$

This tells us that for large z

$$J_0(z) \sim \mathcal{O}(|z|^{-1}) \tag{A.21}$$

6. The modified Bessel function $I_n(z)$ obeys the derivative relation (9.6.27)

$$\frac{dI_0(z)}{dz} = I_1(z) \tag{A.22}$$

7. The recurrence relation for modified Bessel functions (9.6.26) is

$$\frac{dI_\nu(z)}{dz} = I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z) \tag{A.23}$$

8. Ascending series form for modified Bessel functions (9.6.10) is

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)} \tag{A.24}$$

9. From Watson, Basic Integrals of Bessel Functions,

$$\int_0^\infty e^{-a^2x^2} J_n(px) J_n(qx) x dx = \frac{1}{2a^2} e^{-(p^2+q^2)/(4a^2)} I_n\left(\frac{pq}{2a^2}\right) \tag{A.25}$$

A.5 Plasma Dispersion Function

1. Definition of the Plasma Dispersion Function Z

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dt \frac{e^{-t^2}}{t-\xi} \tag{A.26}$$

2. Another oft-used formula

$$1 + \xi Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dt \frac{-te^{-t^2}}{\xi - t} \tag{A.27}$$

3. And another handy formula,

$$\xi[1 + \xi Z(\xi)] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dt \frac{-t^2 e^{-t^2}}{\xi - t} \tag{A.28}$$

4. Limits of the Plasma Dispersion Function (found on page 30 of the NRL Plasma Formulary)

   (a) A power series representation of the plasma dispersion function for a small argument $|\xi| \ll 1$ is

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - 2\xi\left(1 - \frac{2\xi^2}{3} + \frac{4\xi^4}{15} - \frac{8\xi^6}{105} + \cdots\right) \tag{A.29}$$

(b) An asymptotic series representation of the plasma dispersion function for a large argument $|\xi| \gg 1$ is

$$Z(\xi) = i\sqrt{\pi}\sigma e^{-\xi^2} - \frac{1}{\xi}\left(1 + \frac{1}{2\xi^2} + \frac{3}{4\xi^4} + \frac{15}{8\xi^6} + \cdots\right) \tag{A.30}$$

where

$$\sigma = \begin{cases} 0 & y > |x|^{-1} \\ 1 & |y| < |x|^{-1} \\ 2 & y < |x|^{-1} \end{cases} \tag{A.31}$$

and $\xi = x + iy$.

5. Limits for $\xi Z(\xi)$

(a) $\xi \gg 1$

$$\xi Z(\xi) \simeq i\sqrt{\pi}\xi\sigma e^{-\xi^2} - 1 + \frac{1}{2\xi^2} \tag{A.32}$$

(b) $\xi \ll 1$

$$\xi Z(\xi) \simeq i\sqrt{\pi}\xi - 2\xi^2 + \frac{4\xi^4}{3} \tag{A.33}$$

A.6 Poisson Bracket Math

1. Definitions

$$[u, v] = \left(\frac{\partial u}{\partial \mathbf{R}} \times \hat{\mathbf{z}}\right) \cdot \frac{\partial v}{\partial \mathbf{R}} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}. \tag{A.34}$$

2.

$$[u, u] = 0 \tag{A.35}$$

3.

$$u[u, v] = \left[\frac{u^2}{2}, v\right] \tag{A.36}$$

4. For periodic boundary conditions,

$$\int d^3\mathbf{r}[u, v] = 0 \tag{A.37}$$

A.7 Useful Integrals

These formulas have been extracted from a number of references for mathematics [Abramowitz and Stegun, 1972, Dwight, 1961, Gradshteyn and Ryzhik, 1965, Prudnikov et al., 1986a, Prudnikov et al., 1986b, Spiegel, 1968].

Useful references for mathematical functions and integrals are [], 1.

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{A.38}$$

2.

$$\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}} \tag{A.39}$$

where $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(1/2) = \sqrt{\pi}$.

Appendix B

References

B.1 References for GS2

1. Bill Dorland's Thesis
2. Mike Beer's Ph.D. Thesis (online)
   (a) Energy transfer between k's
   (b) Found a conserved quantity
   (c) Calculates sources, sinks, and transfers at each k
   (d) In a tokamak, all cascade is in $k_{\|}$, none in $k_{\perp}$.
   (e) Near the end of his thesis
   (f) Energy functional for GS2 may be h^2?
3. Phil Snyder's Thesis
   (a) At www.pppl.gov/ hammett/papers/ (may be w3.pppl.gov etc.)
   (b) Nicholson, Plasma Theory (thin blue book). Great Landau Damping discussion.
4. John P. Boyd, Chebyshev and Fourier Spectral Methods, 2nd edition, Dover, New York 2001, 688 pp.
   (a) THE book for Spectral Methods
   (b) Free online from the author's homepage.
5. Dale R. Durran, Numerical Methods for Wave Equations in Geophysical Fluid Dynamics, Springer, Berlin, 1998. $79.95
   (a) A great book with some advanced algorithms, better than the standard schemes.

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