

PHYS: 4761 Mathematical Methods of Physics I

Lecture #1: Infinite Series, Series of Functions, Binomial TheoremI. Infinite SeriesA. Fundamentals:

1. Infinite series are an extremely powerful way to represent functions, enabling solution by relatively straightforward means.
 - a. Determine whether a series is convergent is often critical.

2. Infinite Series $u_1 + u_2 + u_3 + \dots$

3. Def: Partial Sum $S_i \equiv \sum_{n=1}^i u_n$

4. Def: Convergence $\lim_{i \rightarrow \infty} S_i = S$

then the infinite series converges $\sum_{n=1}^{\infty} u_n \equiv S$

a. Necessary condition $\lim_{n \rightarrow \infty} u_n = 0$ (but not sufficient)
for convergence

5. Def: Divergence $\lim_{i \rightarrow \infty} S_i = \pm \infty$

6. Def: Oscillatory: Ex: $\sum_{n=1}^{\infty} u_n = 1 - 1 + 1 - 1 + \dots = (-1)^n + \dots$
often classed with divergent. (not convergent)

B. Two Important Series:

1. Geometric Series a. $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$

NOTE: Lower limit $n=0$.

b. Partial Sum $S_n = \frac{1-r^{n+1}}{1-r}$

c. Sum: $\lim_{n \rightarrow \infty} S_n = \boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$, convergent for $|r| < 1$
 divergent for $r \geq 1$, $r < -1$
 oscillatory for $r = -1$

2. Harmonic Series a. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

b. Although $\lim_{n \rightarrow \infty} \frac{1}{n} \Rightarrow 0$, it is not sufficient for convergence

c. The Harmonic Series diverges $\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \infty$

Both of these Series are valuable for the Comparison Test.

C. The Comparison Test for Convergence

1. Convergence: If $0 \leq u_n \leq a_n$ for all n , and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is convergent.

2. Divergence: If $0 \leq b_n \leq v_n$ for all n , and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} v_n$ diverges.

D. D'Alembert Ratio Test (Easy Convergence Test to Apply)

1. If $\frac{a_{n+1}}{a_n} \leq r < 1$ for $n > N$ and r is independent of n , $\sum_{n=1}^{\infty} a_n$ is convergent

2. D. (Continued)

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2. If $\frac{a_{n+1}}{a_n} \geq 1$ for $n > N$, then $\sum_{n=1}^{\infty} a_n$ is divergent

3. Simple Limiting Form $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \begin{cases} < 1 & \text{convergence} \\ > 1 & \text{divergence} \\ = 1 & \text{indeterminate} \end{cases}$

4. If indeterminate, a more sensitive test is necessary:

For example: a) Cauchy Root Test

b) Kummer's Theorem

c) Gauss's Test $\leftarrow$ Good choice when D'Alembert Ratio Test is indeterminate.

5. Exs Harmonic Series (Failure of D'Alembert Ratio Test)

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1$$

BUT cannot choose an $r < 1$ independent of n , since $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$,
 $\Rightarrow$ Test indeterminate!

E. Cauchy Integral Test:

1. Let $f(x)$ be a continuous, monotonically decreasing function in which $f(n) = a_n$.

Then $\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges if } \int_1^{\infty} f(x) dx \text{ is finite.} \\ \text{diverges if } \int_1^{\infty} f(x) dx \text{ is infinite.} \end{cases}$

F. Alternating Series

1. Generally convergence is more rapid due to cancellation.

2. Leibniz Criterion (Strict sign alternation)

For $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with $a_n > 0$, if a_n is monotonically decreasing and $\lim_{n \rightarrow \infty} a_n = 0$, the series converges.

3. Absolute Convergence

a. A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |u_n|$ converges.

b. Otherwise, the series is termed conditionally convergent.

c. Ex: Alternating Harmonic Series: i) $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1} \Rightarrow$ convergent by Leibniz Criterion
 ii) But $\sum_{n=1}^{\infty} |(-1)^{n-1} n^{-1}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

iii) Thus, this series is conditionally convergent.

4. Conditions for Operating on Series

a. An absolutely convergent series may be reordered

- i) Series sum is independent of order in which terms are added.
- ii) You may add, subtract, or multiply term wise two absolutely convergent series; the resulting series is absolutely convergent.
- iii) You may multiply whole series: the limit of the product is simply the product of the limits.

b. Riemann's Theorem: By a rearrangement of terms, a conditionally convergent series may be made to converge to any desired value or to diverge.

BOTTOM LINE: Be careful with conditionally convergent series.

G. Improvement of Convergence

- 1. The rate of convergence may be improved by forming a linear combination of a slowly converging series with a known series.
- 2. Important for efficient numerical evaluation of series.

II. Series of Functions

A. Basics: 1. Consider each term is a function, $U_n = U_n(x)$

2. Partial Sum $S_n(x) = U_1(x) + U_2(x) + \dots + U_n(x)$

3. Series sum: $\sum_{n=1}^{\infty} U_n(x) = S(x) = \lim_{n \rightarrow \infty} S_n(x)$

B. Uniform Convergence:

1. IF for any small $\epsilon > 0$, there exists a number N , independent of x over interval $[a, b]$ (that is, $a \leq x \leq b$) such that

$$|S(x) - S_n(x)| < \epsilon \text{ for all } n > N,$$

then the series is uniformly convergent in $[a, b]$.

2. Note that absolute and uniform convergence are different concepts.

3. Weierstrass M (Majorant) Test (for uniform convergence)

Text is a little confusing about how to test for uniform convergence!

a. A series $\sum_{n=1}^{\infty} U_n(x)$ will be uniformly convergent in $[a, b]$

if we can construct a convergent series $\sum_{n=1}^{\infty} M_n$ where $M_n \geq |U_n(x)|$ for all x in $[a, b]$.

b. NOTE: This test requires absolute convergence to establish uniform convergence.

4. Key Point: Use a standard convergence test, which is now a function of x .
 $\Rightarrow$ Conditions on x for convergence gives range of uniform convergence in x .

C. Properties of Uniformly Convergent Series

1. IF $\sum_{n=1}^{\infty} U_n(x)$ is uniformly convergent in $[a, b]$ and $U_n(x)$ are continuous!

Almost always satisfied

a. The Sum $S(x) = \sum_{n=1}^{\infty} U_n(x)$ is continuous

b. Sum of integrals is equal to integral of sum! $\int_a^b S(x) dx = \sum_{n=1}^{\infty} \int_a^b U_n(x) dx$

More restrictive

c. Sum of derivatives is equal to derivative of sum:

$\frac{d}{dx} S(x) = \sum_{n=1}^{\infty} \frac{d}{dx} U_n(x)$ if $\frac{dU_n(x)}{dx}$ is continuous in $[a, b]$; $\sum_{n=1}^{\infty} \frac{dU_n(x)}{dx}$ is uniformly convergent.

II. Continued

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D. Taylor's Expansion

1. Perhaps one of the physicist's most widely used tools is the Taylor Expansion of a function into a power series.
- 2a. Assuming a function $f(x)$ has n continuous derivatives in $[a, b]$, one may integrate $f^{(n)}(x)$ n times to obtain:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \underbrace{R_n}_{\text{Remainder}}$$

- b. This is an exact expression with remainder

$$R_n = \int_a^x dx_n \dots \int_a^{x_2} dx_1 f^{(n)}(x_1)$$

- c. The mean value theorem, $\int_a^x g(x) dx = (x-a)g(\xi)$ for $a \leq \xi \leq x$, can be used to estimate R_n .

$$\Rightarrow R_n = \frac{(x-a)^n}{n!} f^{(n)}(\xi)$$

- d. NOTE: In this form, the series is not infinite, and therefore converges. The only question is the magnitude of R_n .

3. Taylor's Series: When $\lim_{n \rightarrow \infty} R_n = 0$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)$$

where $0! \equiv 1$.

This is the value at point x in terms of value and derivatives at reference point a .

4. Maclaurin Series: Set $a=0$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

II.D. (Continued)

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5. Ex: Power Series representation of $f(x) = e^x$

a. Note: $f'(x) = e^x$, so $f^{(n)}(0) = 1$.

b. Thus, using Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

c. Check convergence by D'Alembert Ratio Test

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} x = \lim_{n \rightarrow \infty} \frac{x}{n+1} = 0! \text{ Convergent for } -\infty < x < \infty!$$

d. Absolute value of series also converges $\Rightarrow$ Absolutely convergent!

E. Properties of Power Series

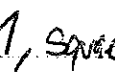
1. General Form $f(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$

2. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = R^{-1}$, series converges for $-R < x < R$ with a Radius of Convergence R .

3. a. Power series are uniformly and absolutely convergent for any interior interval $-S \leq x \leq S$ where $S < R$.
[Can be proven by Weierstrass M test].

b. Since $u_n(x) = a_n x^n$ are all continuous & $f(x)$ is uniformly convergent, $f(x)$ must be continuous in $-S \leq x \leq S$.

4. Thus, infinite power series can only represent continuous functions!

b. Discontinuous functions (sawtooth wave , square wave ) are often expressed as infinite series of trigonometric functions (sine, cosine $\Rightarrow$ Fourier Transform).

II E. (Continued)

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5. Uniqueness Theorem: Any power series representation is unique.

a. Proof: i) $\sum_{n=0}^{\infty} a_n x^n \stackrel{?}{=} \sum_{n=0}^{\infty} b_n x^n$ ii) Set $x=0 \Rightarrow a_0 = b_0$

iii) Differentiate & set $x=0 \Rightarrow a_1 = b_1$, etc... $a_n = b_n$!

b. This property is extremely valuable in solving physics problems.

6. L'Hôpital's Rule: If the ratio of two differentiable functions $f(x)$ and $g(x)$ becomes indeterminate ($\frac{0}{0}$) at $x=x_0$, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

a. Ex: Use power series to evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

ii) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} = 1$

III Binomial Theorem:

A. Binomial Expansion

1. Applying the Maclaurin expansion to $(1+x)^m$ where m need not be positive nor integral.

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

2. Binomial Coefficients: a. In general, $\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}$,

so $(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$

b. For an integer $m > 0$, $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ "m choose n"
Number of ways to choose n out of m objects.

3. Ex: Relativistic Energy: $E = mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

a. $E = \underbrace{mc^2}_{\text{Rest Energy}} + \underbrace{\frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \dots}_{\text{Classical limit of kinetic energy for } v \ll c}$