

Lecture #7 Vector Differentiation and Integration

I. Differential Vector Operators

1. Vector fields $\underline{V}(x, y, z)$ and scalar fields $\phi(x, y, z)$ may be differentiated with respect to spatial dimensions

A. Gradient, ∇

1. Characterizes the change of a scalar quantity with position.

2. In $\mathbb{R}^3$, label coordinates x_1, x_2 , & x_3

Scalar Field $\rightarrow \phi(\underline{r})$ at $\underline{r} = x_1 \underline{\hat{e}}_1 + x_2 \underline{\hat{e}}_2 + x_3 \underline{\hat{e}}_3$

$$3. d\phi = \left(\frac{\partial \phi}{\partial x_1}\right) dx_1 + \left(\frac{\partial \phi}{\partial x_2}\right) dx_2 + \left(\frac{\partial \phi}{\partial x_3}\right) dx_3$$

- a. This can be written $\nabla \phi \cdot d\underline{r}$ where

$$\nabla \phi = \begin{pmatrix} \partial \phi / \partial x_1 \\ \partial \phi / \partial x_2 \\ \partial \phi / \partial x_3 \end{pmatrix} \quad d\underline{r} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \quad \left(\begin{array}{l} \text{column} \\ \text{vector notation} \end{array} \right)$$

or

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}\right) \underline{\hat{e}}_1 + \left(\frac{\partial \phi}{\partial x_2}\right) \underline{\hat{e}}_2 + \left(\frac{\partial \phi}{\partial x_3}\right) \underline{\hat{e}}_3$$

$$d\underline{r} = dx_1 \underline{\hat{e}}_1 + dx_2 \underline{\hat{e}}_2 + dx_3 \underline{\hat{e}}_3$$

4. NOTE: a. $\nabla \phi$ is a vector (no under title)

- b. It can be shown to transform under a rotation $\underline{S}$, $(\nabla \phi)' = \underline{S}(\nabla \phi)$

5. Vector Differential Operator:

$$\nabla = \underline{\hat{e}}_x \frac{\partial}{\partial x} + \underline{\hat{e}}_y \frac{\partial}{\partial y} + \underline{\hat{e}}_z \frac{\partial}{\partial z}$$

- a. Operates on a scalar to produce a vector.

- b. Operates only on what is to the right $\Rightarrow$ order matters! $\nabla \phi$
(Not $\phi \nabla$)

6. Ex: Force expressed as gradient of scalar potential $V(\underline{r})$

$$\underline{F} = -\nabla V(\underline{r})$$

I. A. (Continued)

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7. Ex: $\nabla r = \frac{\partial r}{\partial x} \hat{e}_x + \frac{\partial r}{\partial y} \hat{e}_y + \frac{\partial r}{\partial z} \hat{e}_z$

a. $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2} = \frac{\frac{1}{2} 2x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}$ Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

b. Thus

$$\nabla r = \frac{x}{r} \hat{e}_x + \frac{y}{r} \hat{e}_y + \frac{z}{r} \hat{e}_z = \frac{1}{r} (x \hat{e}_x + y \hat{e}_y + z \hat{e}_z) = \frac{\vec{r}}{r} = \hat{r}$$

c. So $\boxed{\nabla r = \hat{r}}$ Also useful is $\boxed{\hat{r} = \frac{x}{r} \hat{e}_x + \frac{y}{r} \hat{e}_y + \frac{z}{r} \hat{e}_z}$

8. Ex: Central Force $V(r) = \frac{1}{r} \leftarrow$ spherically symmetric potential

a. $\vec{F} = -\nabla V(r) = -\nabla \left(\frac{1}{r} \right)$

b. $\frac{\partial}{\partial r} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{\partial r}{\partial x}$

c. Thus

$$\vec{F} = -\nabla \left(\frac{1}{r} \right) = +\frac{1}{r^2} \frac{\partial r}{\partial x} \hat{e}_x + \frac{1}{r^2} \frac{\partial r}{\partial y} \hat{e}_y + \frac{1}{r^2} \frac{\partial r}{\partial z} \hat{e}_z = +\frac{1}{r^2} \nabla r = +\frac{1}{r^2} \hat{r}$$

d. So $\boxed{\vec{F} = \frac{1}{r^2} \hat{r}} \Rightarrow$ Spherical potential yields radial force!

B. Divergence, $\nabla \cdot$

1. Def: $\boxed{\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$ (Dot Product of ∇ and $\vec{A}$)

2. Ex: $\nabla \cdot \vec{r} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (x \hat{e}_x + y \hat{e}_y + z \hat{e}_z)$

a. NOTE: Cartesian unit vectors are constant, so $\frac{\partial}{\partial x_i} \hat{e}_i = 0!$

b. $= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = \boxed{3}$

3. Physical Significance of Divergence:

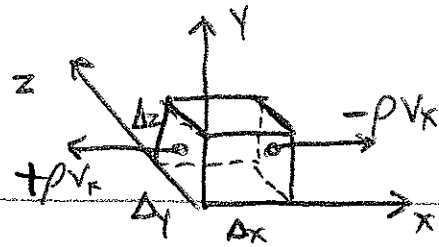
a. Consider the flow $\vec{v}(r)$ of a fluid of mass density $\rho(r)$.

$\Rightarrow$ Mass flow rate at any point r is $\boxed{\rho(r) \vec{v}(r)}$

Z. B. 3. (Continued)

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b. Consider rate of change of mass $\frac{\Delta m}{\Delta t}$ due to flow along $\hat{v}_x$



$$\begin{aligned}\frac{\Delta m}{\Delta t} &= + \rho v_x \bigg|_{0, \frac{\Delta y}{2}, \frac{\Delta z}{2}} \Delta y \Delta z - \rho v_x \bigg|_{\Delta x, \frac{\Delta y}{2}, \frac{\Delta z}{2}} \Delta y \Delta z \\ &= - \left(\rho v_x \bigg|_{\Delta x, \frac{\Delta y}{2}, \frac{\Delta z}{2}} - \rho v_x \bigg|_{0, \frac{\Delta y}{2}, \frac{\Delta z}{2}} \right) \Delta x \Delta y \Delta z \\ &\quad \underbrace{\qquad\qquad\qquad}_{\Delta x} \underbrace{\qquad\qquad\qquad}_{\Delta t} \underbrace{\qquad\qquad\qquad}_{\Delta y \Delta z} \\ &\quad \underbrace{\qquad\qquad\qquad}_{\Delta x} \underbrace{\qquad\qquad\qquad}_{\Delta t} \underbrace{\qquad\qquad\qquad}_{\Delta y \Delta z} = \frac{\partial f}{\partial x}\end{aligned}$$

c. NOTE:

$$\lim_{\Delta x \rightarrow 0}$$

$$\frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x} = \frac{\partial f}{\partial x}$$

d. But, we can also have flow in or out along $\hat{e}_y$ or $\hat{e}_z$!

$$\frac{\Delta m}{\Delta t} = - \left[\frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) \right] \Delta x \Delta y \Delta z = - \nabla \cdot (\rho \vec{v}) \Delta x \Delta y \Delta z$$

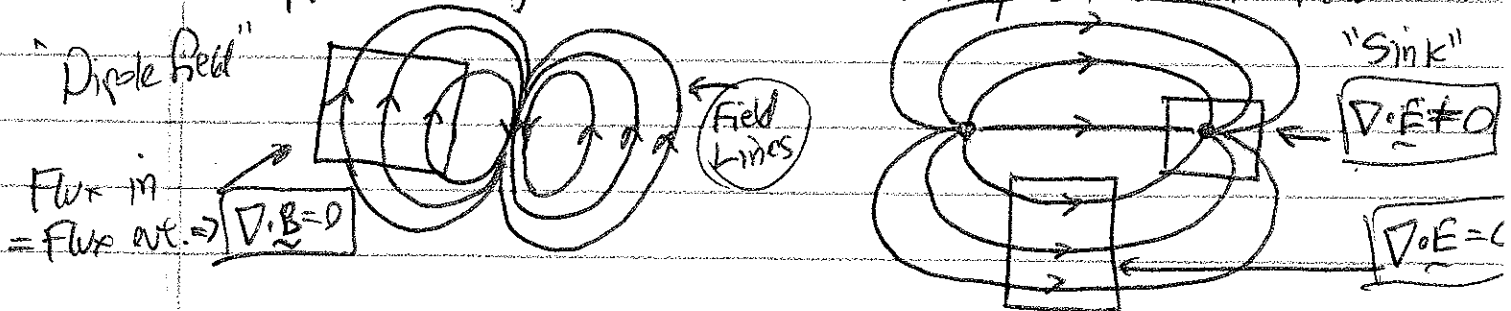
e. So $\frac{\Delta m}{\Delta t} = \nabla \cdot (\rho \vec{v}) \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{v})}$ Equation of Continuity

"Divergence" describes net outflow from a small volume element $\Rightarrow$ reduces mass density.

4. Zero Divergence of a Vector Field

a. Zero Divergence means a steady-state flux within region. "What goes in, comes out"

b. Applies to magnetic & electric fields & incompressible fluid flow.



I. B. Continued

Homes ④

5. Terminology. $\nabla \cdot \underline{B} = 0$ everywhere $\Rightarrow \underline{B}$ is solenoidal.

b. No sources or sinks (No magnetic monopoles).

C. Curl, $\nabla \times$

1. Def:

$$\nabla \times \underline{V} = \hat{e}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{bmatrix}$$

(Cross Product of ∇ and $\underline{V}$)

Top down evaluation "order matters!"

2. Ex: Curl of a central force field: $\underline{F} = f(r) \hat{r}$

a. $\nabla \times [f(r) \hat{r}]$ where $\hat{r} = \frac{x}{r} \hat{e}_x + \frac{y}{r} \hat{e}_y + \frac{z}{r} \hat{e}_z$

b. x-component: $\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} = \frac{\partial}{\partial y} \left(f(r) \frac{z}{r} \right) - \frac{\partial}{\partial z} \left(f(r) \frac{y}{r} \right)$

Since $\frac{\partial z}{\partial y} = 0$

$$= z \frac{\partial}{\partial y} \left(\frac{f(r)}{r} \right) - y \frac{\partial}{\partial z} \left(\frac{f(r)}{r} \right) = z \frac{\partial r}{\partial y} \frac{\partial}{\partial r} \left(\frac{f(r)}{r} \right) - y \frac{\partial r}{\partial z} \frac{\partial}{\partial r} \left(\frac{f(r)}{r} \right)$$

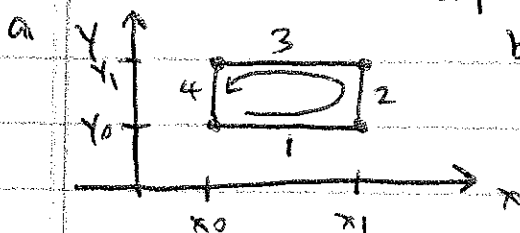
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \frac{\partial r}{\partial y}, \text{ etc.}$$

c. Finally, note $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$, so

$$= \frac{zy}{r} \frac{\partial}{\partial r} \left(\frac{f(r)}{r} \right) - \frac{yz}{r} \frac{\partial}{\partial r} \left(\frac{f(r)}{r} \right) = 0!$$

d. All other components yield 0 $\Rightarrow \boxed{\nabla \times [f(r) \hat{r}] = 0}$

3. Geometric Interpretation:



b. $\oint \underline{B} \cdot d\underline{s} = \int_{x_0}^{x_1} B_x dx + \int_{y_0}^{y_1} B_y dy - \int_{x_1}^{x_0} B_x dx$

c. If contributions do not cancel, $-\int_{y_1}^{y_0} B_y dy$, then $\nabla \times \underline{B} \neq 0$.

L.C. (Continued)

4. Terminology: a Circulation = $\frac{\oint \underline{B} \cdot d\underline{s}}{\oint da}$ (per unit area) Hawes 5

$$= (\nabla \times \underline{B}) \cdot \hat{n} \leftarrow \text{Normal to loop } d\underline{s}$$

b. Fluid dynamics: Vorticity $\underline{\omega} \equiv \nabla \times \underline{v}$

c. $\nabla \times \underline{B} = 0 \Rightarrow \underline{B}$ is irrotational.
everywhere

D. Higher Order Derivatives:

1. Laplacian:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2. Irrational and Solenoidal Vector Fields:

a. $\nabla \times \nabla \phi = 0$ Any gradient has a vanishing curl
 $\Rightarrow$ irrotational!

b. $\nabla \cdot (\nabla \times \underline{v}) = 0$ Any curl has a vanishing divergence
 $\Rightarrow$ Solenoidal!

3. Vector Laplacian, $\nabla^2 \underline{v}$ (Laplacian of a vector).

$$a. \nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \cdot \underline{v}) - \nabla^2 \underline{v}$$

4. Ex: Maxwell's Equations (SI Units)

a. ① $\nabla \cdot \underline{B} = 0$

② $\nabla \cdot \underline{E} = \rho / \epsilon_0$

③ $\nabla \times \underline{B} = \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t} + \mu_0 \underline{J}$

④ $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

Same as: Charge density $\rho(x)$

Current density $\underline{J}(x)$

where $\frac{1}{c^2} = \mu_0 \epsilon_0$
magnetic permeability $\nwarrow$ electric permittivity $\searrow$

b. Electromagnetic Waves in Vacuum ($\rho=0, \underline{J}=0$)

i. $\frac{\partial}{\partial t} \textcircled{3} \Rightarrow \frac{\partial}{\partial t} (\nabla \times \underline{B}) = \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2}$

ii. $\nabla \times \textcircled{4} \Rightarrow \nabla \times (\nabla \times \underline{E}) = -\frac{\partial}{\partial t} (\nabla \times \underline{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2}$

iii. Use $\nabla \times (\nabla \times \underline{E}) = \nabla (\nabla \cdot \underline{E}) - \nabla^2 \underline{E}$ and $\epsilon_0 \mu_0 = \frac{1}{c^2}$
 $\nabla \cdot \underline{E} = 0$ for $\rho=0$!

iv. $\boxed{\nabla^2 \underline{E} = \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2}}$

Wave equation for propagation of light waves at speed c .

5. Other Vector Identities:

a. $\nabla \cdot (f \underline{V}) = (\nabla f) \cdot \underline{V} + f \nabla \cdot \underline{V}$

b. $\nabla \times (f \underline{V}) = f (\nabla \times \underline{V}) + (\nabla f) \times \underline{V}$

c. $\nabla (\underline{A} \cdot \underline{B}) = (\underline{B} \cdot \nabla) \underline{A} + (\underline{A} \cdot \nabla) \underline{B} + \underline{B} \times (\nabla \times \underline{A}) + \underline{A} \times (\nabla \times \underline{B})$

NOTE: $\underline{B} \cdot \nabla = B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$

II. Vector Integration

A. Line Integrals

1. Forms:

$$\int_C \phi d\mathbf{r}$$

$$\int_C \underline{E} \cdot d\mathbf{r}$$

$$\int_C \underline{V} \times d\mathbf{r}$$

2. Using $d\mathbf{r} = dx \hat{e}_x + dy \hat{e}_y + dz \hat{e}_z$, [noting $\phi = \phi(x, y, z)$]

NOTE: ϕ is a scalar! $\rightarrow \int_C \phi d\mathbf{r} = \hat{e}_x \int_C \phi dx + \hat{e}_y \int_C \phi dy + \hat{e}_z \int_C \phi dz \leftarrow$ sum of scalar integrals!

II.A. (Continued)

Hawes 7

3. NOTE! The path must be specified!

a. $\int_C \phi(x, y, z) dx$ where we need to know $y(x)$ and $z(x)$.
Equation for path.

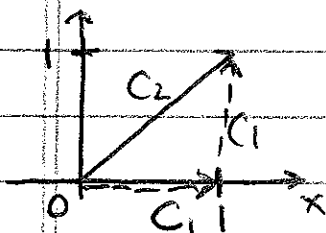
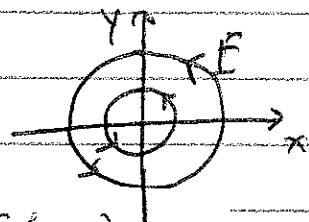
4. Work: $W = \int_C \underline{F} \cdot d\underline{r}$

NOTE! Answer is

a scalar!

$\rightarrow = \int_C F_x dx + \int_C F_y dy + \int_C F_z dz$

5. Path Dependence: a. $\underline{F} = -y \hat{e}_x + x \hat{e}_y$



b. $\int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^1 dx F_x(x, 0) + \int_0^1 dy F_y(1, y)$
 $= \int_0^1 (0) dx + \int_0^1 (1) dy = \boxed{1}$

$C_2 \Rightarrow x=y$

c. $\int_{C_2} \underline{F} \cdot d\underline{r} = \int_0^1 F_x(x, x) dx + \int_0^1 F_y(y, y) dy = \int_0^1 (-x) dx + \int_0^1 y dy = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$

Value of line integral depends on path!

Unless integrand has special properties, line integrals depend on path!

B. Surface Integrals

1. Forms:

$\int \phi d\sigma$

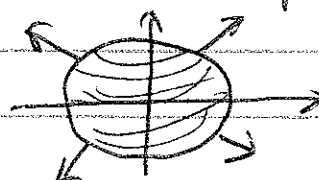
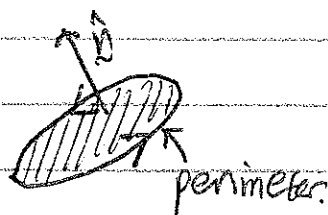
$\int \underline{V} \cdot d\underline{\sigma}$

$\int \underline{V} \times d\underline{\sigma}$

2. a. Often $d\underline{\sigma} = \hat{n} dA$, where $\hat{n}$ is normal to area dA .

b. Which direction normal? i. Closed Surface: Outward normal is positive!

ii. Open Surface: Right-hand Rule



II. B. (Continued)

Howes (8)

3. Most Common Form: $\int \underline{V} \cdot d\underline{\sigma} \leftarrow$ Flow or Flux through a surface.

C. Volume Integrals:

1.

$$\int \underline{V} d\tau = \hat{e}_x \int V_x d\tau + \hat{e}_y \int V_y d\tau + \hat{e}_z \int V_z d\tau$$

(Vec sum of scalar integrals)

2. For terms which vanish at infinity,

$$\int f(r) \nabla \cdot \underline{A}(r) d\tau = - \int \underline{A}(r) \cdot \nabla f(r) d\tau$$

$$\int \underline{C}(r) \cdot [\nabla \times \underline{A}(r)] d\tau = \int \underline{A}(r) \cdot [\nabla \times \underline{C}(r)] d\tau$$

Can be proven using integration by parts