(RF4)

PHYS:4761 Math Methods I: Final Exam Review

I. Math Preliminaries

A. Series Convergence and Expansions

1. Geometric $\sum_{n=0}^{\infty} r^n$ convergent for $|r|<1$

2. Harmonic $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

3. Tests: a. Comparison Test
   b. Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \begin{cases} <1 & \text{conv} \\ >1 & \text{div} \\ =0 & \text{indeterminate} \end{cases}$
   c. Integral $\int_1^{\infty} f(x)\,dx$
   d. Leibniz (Alternating) $\sum_{n=1}^{\infty} (-)^{n+1} a_n$, $\lim_{n\to\infty} a_n = 0 \to$ conv.

4. Taylor Expansion a.: $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$
   b. Know how to expand functions to yield power series.

II. Determinants and Matrices

A. Determinants:

1. $a_1 x_1 + a_2 x_2 + \ldots = 0$
   $b_1 x_1 + b_2 x_2 + \ldots = 0$ $\Rightarrow D = \begin{vmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \\ \vdots & & \end{vmatrix}$
2. If $D \neq 0$, only ~~non~~trivial solution $x_i = 0$ exists
3. Know how to evaluate determinants:
   a. Expansion by minors.
4. Gauss Elimination: Know the procedure.

B. Matrices:

1. Multiplication $\underset{\approx}{A}\,\underset{\approx}{B} = \underset{\approx}{C}$

2. Matrix Inversion: Know the procedure $\begin{pmatrix} a & b & c \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. Know Transpose, Adjoint, Trace, Orthogonal, Unitary, Hermitian.
4. $(\underset{\approx}{A}\,\underset{\approx}{B})^{-1} = \underset{\approx}{B}^{-1}\underset{\approx}{A}^{-1}$ $(T, \dagger)$.

(RF-2)

III. Vector Analysis

1. Dot Product, Cross Product, Scalar & Vector Triple Products.
2. 2D Rotations $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ $\boxed{\text{3D Rot (not on exam)}}$

3. Vector differential operators:
a. Gradient, Divergence, Curl
b. Higher order: Laplacian, $\nabla\times\nabla\phi = 0$, $\nabla\cdot(\nabla\times \underset{\sim}{V}) = 0$, $\nabla\times(\nabla\times\underset{\sim}{V})\ldots$
c. Identities for products $\nabla\cdot(f\underset{\sim}{V})$, $\nabla\times(f\underset{\sim}{V})$, $\nabla(\underset{\sim}{A}\cdot\underset{\sim}{B})$, etc.
$\underset{\sim}{B}\cdot\nabla = B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}$

4. Integral Theorems: a. Gauss' Thm: $\oint_{\partial V} \underset{\sim}{A}\cdot d\underset{\sim}{\sigma} = \int_V \nabla\cdot\underset{\sim}{A}\, d\tau$
b. Stoke's Theorem: $\oint_{\partial S} \underset{\sim}{B}\cdot d\underset{\sim}{r} = \int_S \nabla\times\underset{\sim}{B}\cdot d\underset{\sim}{\sigma}$

5. Potentials: a. $\underset{\sim}{F} = -\nabla\phi$ Scalar
b. $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$ vector
c. Helmholtz Thm: $\underset{\sim}{P} = -\nabla\phi + \nabla\times\underset{\sim}{A}$ (any vector)

6. Curvilinear Coordinates: a. Cylindrical
b. Spherical
c. Know how to use differential operators in these coordinates (NRL Plasma Formulary)

IV. Chap 4 will mostly not be on Exam, except Jacobian:
A. Jacobian: $(x,y)\to(U,V)$ $dx\,dy = J\,dU\,dV$
2. $J = \frac{\partial(x,y)}{\partial(U,V)}$
3. Compute J^{-1} if easier.

RF-3

V. Vector Spaces

1. a. $\psi(x) = \sum_{i=1}^{N} a_i |\phi_i\rangle \rightarrow a_i = \frac{\langle\phi_i|\psi\rangle}{\langle\phi_i|\phi_i\rangle}$

   b. $\langle f(x)|g(x)\rangle = \int_a^b f^*(x) g(x)\, w(x)\, dx$ $(a, b, w(x))$

   c. Orthonormal $\langle\phi_i|\phi_j\rangle = \delta_{ij} \rightarrow a_i = \langle\phi_i|\psi\rangle$

   ~~d. Decomposi~~

2. Gram-Schmidt Orthogonalization: Know the procedure.
3. Linear Operators: $\mathcal{L}y = \lambda y$
   a. Commutator $[A,B] = AB - BA$, Inverse $A^{-1}A = 1$
   b. Adjoint $\langle f|Ag\rangle = \langle A^\dagger f|g\rangle$
   c. Self-Adjoint $A = A^\dagger$, Unitary $U^\dagger = U^{-1}$
   d. Basis Expansion: i. $A = \sum_{mn} |\phi_n\rangle a_{nm} \langle\phi_m|$, $a_{mn} = \langle\phi_m|A\phi_n\rangle$

   ii. Hermitian: $a_{nm} = a^*_{mn}$

4. Unitary Transformations: a. $\underset{\sim}{b} = \underset{\approx}{A}\,\underset{\sim}{c}$ $\underset{\sim}{c}' = \underset{\approx}{U}\,\underset{\sim}{c}$ $\underset{\approx}{A}' = \underset{\approx}{U}\,\underset{\approx}{A}\,\underset{\approx}{U}^{-1}$
   b. Invariances under Unitary Trans.

VI. Eigenvalue Problems

1. a. $\underset{\approx}{H}\,\underset{\sim}{r} = \lambda \underset{\sim}{r}$, $\det(\underset{\approx}{H} - \lambda \underset{\approx}{1}) = 0$ → Know how to Solve for eigenvalues and eigenvectors.

   b. If degenerate $\lambda_i = \lambda_j$, use Gram-Schmidt Orthog.

2. Hermitian Eigenvalue Problems: a. λ_i real, $\underset{\sim}{r}_i$ orthogonal, complete set.
   b. Diagonalization $\underset{\approx}{U}\,\underset{\approx}{H}\,\underset{\approx}{U}^{-1}$, where columns of $\underset{\approx}{U}^{\dagger}$ are normalized eigenvectors.
3. If $[\underset{\approx}{A}, \underset{\approx}{B}] = 0$, Hermitian Matrices have same ~~eigenvalues~~, eigenvectors.

RF-4

VI. (Continued)

4. Expectation values $\langle H \rangle = \langle \Psi | H | \Psi \rangle = \sum_n |a_n|^2 \lambda_n \quad a_n = \langle \phi_n | \Psi \rangle$ normalized

VII ODEs

1. First Order: a. $y' = -\dfrac{P(x,y)}{Q(x,y)}$

b. Separation of Variables: $\dfrac{dy}{dx} = -\dfrac{P(x)}{Q(y)}$

c. Exact Diff: $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$

d. Homogeneous in x & y: $y = xv$

ii. Isobaric $y = x^m v$

e. General: $\dfrac{dy}{dx} + p(x) y = q(x)$

i. Integrating Factor: $\alpha(x) = \exp\left[\int^x p(x)\, dx \right]$

ii. Homogeneous Sol.: $y_1 = \dfrac{C}{\alpha(x)}$

iii. Particular: $y_2 = \dfrac{1}{\alpha(x)} \int^x \alpha(x)\, q(x)\, dx$

2. Constant Coefficients: $y = e^{mx}$, solve for m.

3. Second-Order ODE's: $y'' + P(x) y' + Q(x) y = 0$

a. Two Solutions

4. Frobenius Method (know how to do this): $y(x) = \sum_{j=0} a_j x^{s+j}$

a. Indicial eq: (lowest order in x)

b. Recurrence Relation

5. Linear Independence: Wronskian $W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} \begin{cases} = 0 & \text{lin. dependent} \\ \neq 0 & \text{indep.} \end{cases}$

6. Second Solution: $y_2(x) = y_1(x) \int^x \left\{ \dfrac{\exp\left[-\int^{x_2} P(x_1)\, dx_1 \right]}{[y_1(x_2)]^2} \right\} dx_2$

(RF-5)

VII. ODEs (Continued)

7. Inhomogeneous 2nd order ODEs:

$$y'' + P(x)y' + Q(x)y = F(x)$$

a. $y_p = u_1 y_1 + u_2 y_2$ $\quad y_1, y_2$ homogeneous sol.

b. Solve $\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= F(x) \end{aligned}$ for u_1', u_2'

c. $y = C_1 y_1 + C_2 y_2 + y_p$

VIII. Sturm-Liouville Theory

1. $\mathcal{L}\,\psi(x) = \lambda\,\psi(x)$

a. Apply BC's to get discrete λ_i solutions.

2. Hermitian Op: $\mathcal{L} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$

a. Self-adjoint if $p_0'(x) = p_1(x)$

b. Make self adjoint $w(x) = p_0^{-1} \exp\left[\int \frac{p_1(x)}{p_0(x)}dx\right]$

IX. PDE's

1. First-order PDEs: $\mathcal{L}\phi = a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial y}$

a. Characteristics: $t = bx - ay$ = constant. $(s = ax + by)$

b. $\Rightarrow \mathcal{L}\phi = (a^2+b^2)\frac{\partial\phi}{\partial s} = 0$

2. Second-order PDE's

a. Hyperbolic: $a^2\frac{\partial^2\phi}{\partial x^2} - b^2\frac{\partial^2\phi}{\partial y^2} = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)$

Two Characteristics $bx \pm ay$

3.

RF-6

PDEs
IX (Continued)

3. Separation of Variables: a. $\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

b. Introduce constants of separation.

c. BC's determine allowable values of constants.

4. Cylindrical Coordinates: a. $\psi(\rho,\phi,z) = P(\rho)\,\Phi(\phi)\,Z(z)$

b. Yield's Bessel's Eq.

5. Spherical Coordinates: a. $\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$

b. Assoc. Legendre equation for $x = \cos\theta$

c. $\cancel{R(r) = \frac{Z(kr)}{(kr)^{\frac{1}{2}}}}$ ~~⇒ Bessel's eq. of order~~ $\cancel{l+\frac{1}{2}}$

~~⇒ Spherical Bessel fn~~

6. Specific Types of PDEs:

a. Elliptic: Laplace's & Poisson's Eq.

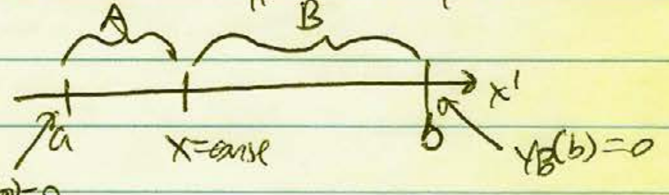
b. Hyperbolic: Wave Equation

c. Parabolic: Diffusion equation, $\xi = \frac{x}{\sqrt{t}} \Rightarrow \text{PDE} \rightarrow \text{ODE in } \xi$.

RF-7

X. Green's Functions (Only 1D Green's Functions will be tested)

1. For a self-adjoint, inhomogeneous ODE
   a. $\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = f(x)$
   b. Defined on an interval $a \le x \le b$
   c. Homogeneous BCs at $x=a$ & $x=b$.
2. Def: $\mathcal{L}G(x,x') = \delta(x-x')$
3. $y(x) = \int_a^b G(x,x')f(x')\,dx'$
4. NOTE: Symmetry $G(x,x') = G^*(x',x)$
5. How to solve a Problem:
   a. Find homogeneous solutions $y_1(x)$ and $y_2(x)$ to $\mathcal{L}y=0$
   b. Split Interval



   c. i. Use BC at $x=a$ to determine $y_A(x)$
      ii. Use BC at $x=b$ to determine $y_B(x)$
   d.
$$G(x,x') = \begin{cases} A\, y_A(x')\, y_B(x) & a \le x' \le x \\ A\, y_A(x)\, y_B(x') & x \le x' \le b \end{cases}$$
   e. $A = \left\{ p(x)\left[y_B'(x)\, y_A(x) - y_A'(x)\, y_B(x) \right] \right\}^{-1}$

(RF-8)

XI Probability and Statistics

A. Probabilities

1. Exp: $P(x_i) = \frac{\text{Number of } x_i}{\text{Number of trials}}$ Theor: $P(x_i) = \frac{\text{Number of outcomes } x_i}{\text{Number of all events}}$

2. Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3. Conditional Probability: a. $P(B|A) = \frac{P(A, B)}{P(A)}$

   b. If A & B independent, $P(B|A) = P(B)$

4. Permutations: a. Order of n distinct objects: $n!$

   b. n objects in K spaces: $\frac{n!}{(n-k)!}$ $(n>k)$

5. Combinations: a. Choose k of n distinct objects: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$
   (Binomial Coefficient)

6. Classical vs. Quantum Statistics: (n particles, k states)

   Distinguishable → a. Maxwell-Boltzmann: k^n

   Stat

   Indistinguishable
   - b. Bose-Einstein: $\binom{n+k-1}{n}$
   - c. Fermi-Dirac: $\binom{k}{n}$

B. Random Variables

1. Discrete Case: N events, m mut. exclus. outcomes, $(x_1, p_1), (x_2, p_2), \ldots (x_m, p_m)$, $(\sum_i n_i = N)$

2. $P(n_1, n_2, \ldots n_m) = \frac{N!}{n_1!\, n_2! \ldots n_m!} (p_1)^{n_1} (p_2)^{n_2} \ldots (p_m)^{n_m}$

3. Mean: $\langle x \rangle = \sum_i x_i p_i$

   $\langle x \rangle = \int x f(x)\, dx$

RF-9

XI B. (Continued)

4. <u>Variances</u> $\sigma^2 = \sum_j (x_j - \langle x \rangle)^2 p_j$ $\quad\sigma$ = Standard Deviation

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 f(x)\, dx$$

5. $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

6. For $Y = aX + b$, $\langle Y \rangle = a\langle x \rangle + b$, $\sigma^2(Y) = a^2 \sigma^2(X)$

7. <u>Multiple Variables:</u> $f(x,y)$
   a. $\langle X \rangle = \iint (x)\, f(x,y)\, dx\, dy$, etc.
   $$\sigma^2(X) = \iint (x - \langle x \rangle)^2 f(x,y)\, dx\, dy$$
   b. If independent $f(x,y) = f_1(x) f_2(y)$
   c. Marginal Prob: $f(x) = \int f(x,y)\, dy$

8. <u>Covariance:</u> $cov(X,Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle$

9. correlation $\quad \dfrac{cov(X,Y)}{\sigma(X)\sigma(Y)}$

<u>C. Distributions</u>

1. Binomial: $P(S=s) = \dfrac{n!}{s!(n-s)!} p^s q^{n-s}$ $\quad p+q=1$

2. Poisson: $p(n) = \dfrac{\mu^n}{n!} e^{-\mu}$
   a. $\langle x \rangle = \mu$
   b. $\sigma^2 = \mu$

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3. Gaussian: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

RF-10

X D. Transformations

1. $y = y(x) \Rightarrow g(y) = f[x(y)]\frac{dx}{dy}$

2. $(X,Y) \rightarrow (U,V)$ $\quad g(U,V) = f[x(U,V), y(U,V)] \cdot |J|$

$$J = \frac{\partial(x,y)}{\partial(U,V)}$$

E. Statistics:

1. Error Propagation:

$$f(\bar{x} \pm \sigma_x, \bar{y} \pm \sigma_y) = f(\bar{x},\bar{y}) \pm \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2}$$

2. Repeated Measurements: $\sigma^2(\bar{x}) = \frac{\sigma^2}{n}$

3. Sample Standard Deviation: $S = \sqrt{\frac{\sum_j (x_j - \bar{x})^2}{n-1}}$

F. Curve Fitting, χ^2 distribution, Student t Distribution, and Confidence intervals will not be on exam.