Lecture #16 Ordinary Differential Equations

I. Introduction

A. Basics

1. Physics is often formulated in terms of differential equations.
   a. Space (x, y, z) and time (t) are independent variables
   b. Functions being differentiated are dependent variables.

2. Partial Differential Equations (PDEs) involve more than one independent variable

3. Ordinary Differential Equations (ODEs) have a single independent variable.

B. Linear Operators

1. Taking a derivative is a linear operation, $\mathcal{L}=\frac{d}{dx}$. $\quad \mathcal{L}[a\phi(x)+b\psi(x)] = a\frac{d\phi}{dx}+b\frac{d\psi}{dx}$

2 Higher order derivatives are also linear operators $\frac{d^2}{dx^2}[a\phi(x)+b\psi(x)] = a\frac{d^2\phi}{dx^2}+b\frac{d^2\psi}{dx^2}$

3. NOTE: Linearity refers to the operator $\mathcal{L}$, not the functions $\phi(x)$, $\psi(x)$.
   a. Ex: $\mathcal{L}\equiv p(x)\frac{d}{dx}+q(x)$

$$\mathcal{L}[a\phi(x)+b\psi(x)] = a\left(p(x)\frac{d\phi}{dx}+q(x)\phi\right)+b\left(p(x)\frac{d\psi}{dx}+q(x)\psi\right) = a\mathcal{L}\phi + b\mathcal{L}\psi$$

   b. In general, linear differential operators have form $\boxed{\mathcal{L} = \sum_{i=0}^{n} p_i(x)\left(\frac{d^i}{dx^i}\right)}$

C. Homogeneous and Inhomogeneous ODEs

1a. Def: Homogeneous ODE: Dependent variable occurs to same power in all terms.

b. Otherwise, ODE is inhomogeneous.

I.C. (Continued)

2. A linear ODE can be written in the form

$$\mathcal{L}\,\phi(x) = F(x) \leftarrow \text{algebraic, not differential, function of } x$$

a. **NOTE:** This equation is inhomogeneous since $F(x)$ has $\overset{o}{\phi}(x)$.

b. A linear, homogeneous equation has form $\boxed{\mathcal{L}\,\phi(x) = 0.}$

3. Superposition Principle

a. For a homogeneous, linear ODE, any multiple of a solution is also a solution (not unique).

b. It is important to identify solutions that are linearly independent.

c. In general, a linear combination of solutions is a solution,

$$\boxed{\text{If } \mathcal{L}\phi = 0 \text{ and } \mathcal{L}\psi = 0, \text{ then } \mathcal{L}(a\phi + b\psi) = 0.}$$

d. Examples: i. 1D Schrödinger Equation: $\mathcal{H}\psi - E\psi = 0$

ii. Electrodynamics, optics, etc.

D. Notation for ODEs

1. Independent variable, x Dependent variable $y(x)$
2. Thus general linear ODE $\mathcal{L}y = F(x)$
3. $y' \equiv \frac{dy}{dx}$ (prime notation)

E. Nonlinear ODEs

1. Fluid mechanics, plasma physics, chaos theory often involve nonlinear differential equations,

a. Ex: $\boxed{y' = p(x)y + q(x)y^n}$ $n \neq 0,1$

b. Can not be written in terms of a linear operator on y.

II. Solving First-Order ODEs

1. General Form:

$$\boxed{y' = \frac{dy}{dx} = f(x,y) = -\frac{P(x,y)}{Q(x,y)}}$$

II. (Continued)

A. Separation of Variables

1. For equations of special form $\frac{dy}{dx} = -\frac{P(x)}{Q(y)}$, we may write

a. $P(x)\,dx = -Q(y)\,dy$

b. Integrate $\boxed{\int_{x_0}^{x} P(x)\,dx = -\int_{y_0}^{y} Q(y)\,dy}$

c. This does <u>not require</u> the ODE to be linear!

2. <u>Ex: Parachute velocity vs time</u>

a. $\underset{\text{acceleration}}{m\dot{v}} = \underset{\text{gravity (positive down)}}{mg} - \underset{\text{air drag}}{bv^2}$ Initial Conditions $v=0$ at $t=0$.

b. As $t \to \infty$, parachute reaches terminal velocity, so $\dot{v} \to 0. \Rightarrow mg = bv_0^2$
where we define terminal velocity $v_0 \equiv \sqrt{\frac{mg}{b}}$

c. Rewriting equation $\boxed{\frac{m}{b}\dot{v} = v_0^2 - v^2}$ where $\dot{v} = \frac{dv}{dt}$

d. Separate variables: $\int \frac{dv}{v_0^2 - v^2} = \int \frac{b}{m}\,dt$

e. Use partial fractions to simplify LHS: $\frac{1}{v_0^2 - v^2} = \frac{c}{v_0 + v} + \frac{d}{v_0 - v} \Rightarrow 1 = c(v_0 - v) + d(v_0 + v)$

i) In powers of v: $\begin{aligned} 1 &= (c+d)\,v_0 \xrightarrow{\qquad\qquad} c = \frac{1}{2v_0} \\ 0 &= (d-c)\,v_0 \rightarrow c = d \end{aligned}$

ii) Thus $\int \frac{1}{2v_0}\left[\frac{1}{v_0+v} + \frac{1}{v_0-v}\right]dv = \frac{1}{2v_0}\left[\ln(v_0+v) - \ln(v_0-v)\right] + C = \frac{1}{2v_0}\ln\left(\frac{v_0+v}{v_0-v}\right) + C$

f. Therefore, $\frac{1}{2v_0}\ln\left(\frac{v_0+v}{v_0-v}\right) + C = \frac{b}{m}t$

i) Applying initial conditions: $v=0$ at $t=0 \Rightarrow C=0$.

$$\ln\left(\frac{v_0+v}{v_0-v}\right) = \frac{2v_0 bt}{m}$$

g. After manipulation, $v = v_0 \frac{e^{-\frac{t}{\tau}} - e^{+\frac{t}{\tau}}}{e^{-\frac{t}{\tau}} + e^{+\frac{t}{\tau}}} = v_0 \frac{\sinh(t/\tau)}{\cosh(t/\tau)} \Rightarrow \boxed{v = v_0 \tanh\left(\frac{t}{\tau}\right) \text{ where } \tau \equiv \sqrt{\frac{gb}{m}}}$

II. A2 (Continued)

h. Always check solution! i. $\frac{dv}{dt} = \frac{d}{dt}\left[v_0 \tanh\left(\frac{t}{\tau}\right)\right] = \frac{v_0}{\tau}\operatorname{sech}^2\left(\frac{t}{\tau}\right)$

ii. So $\frac{m\dot{v}}{b} = \frac{m v_0}{\underbrace{b\tau}_{=v_0^2}}\operatorname{sech}^2\left(\frac{t}{\tau}\right) = v_0^2 - \left[v_0 \tanh\left(\frac{t}{\tau}\right)\right]^2 = v_0^2\left[1-\tanh^2\left(\frac{t}{\tau}\right)\right] = v_0^2 \operatorname{sech}^2\frac{t}{\tau}$ ✓

B. Exact Differentials

1. Rewriting general form as $P(x,y)\,dx + Q(x,y)\,dy = 0$, the equation is an exact differential if it matches

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \quad \text{where} \Rightarrow \quad \frac{\partial\phi}{\partial x} = P(x,y), \quad \frac{\partial\phi}{\partial y} = Q(x,y)$$

2. We can check if such a function ϕ exists by calculating cross derivative $\frac{\partial^2\phi}{\partial x\partial y} \Longrightarrow \boxed{\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}}$

3. If so, solution is $\phi(x,y) = \text{constant}$, or

$$\boxed{\phi(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy = \text{constant}}$$

4. NOTE! All separable ODEs are exact, but not all exact ODEs are separable.

5. Ex: $y' + \left(1 + \frac{y}{x}\right) = 0$

a. Multiply by $x\,dx$ to obtain $(x+y)\,dx + x\,dy = 0$ ← not separable.

b. Check for exact differential:

$\frac{\partial P(x,y)}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1 \qquad \frac{\partial Q(x,y)}{\partial x} = \frac{\partial(x)}{\partial x} = 1$ ✓ exact!

c. $\phi(x,y) = \int_{x_0}^{x}(x+y)\,dx + \int_{y_0}^{y} x_0\,dy = \left(\frac{x^2}{2} + xy - \frac{x_0^2}{2} - \cancel{x_0 y}\right) + \cancel{x_0 y} - x_0 y_0$

$$= \frac{x^2}{2} + xy \underbrace{- \frac{x_0^2}{2} - x_0 y_0}_{\text{constant}} = \text{constant}.$$

d. Thus $\boxed{\frac{x^2}{2} + xy = C}$ solution.

e. Solve for y and check solution satisfies equation!

II. (Continued)

C. Homogeneous ODE of order n in x and y

1. Def: ODE in Homogeneous in x and y if combined powers of x and y in each term add to n.
   a. NOTE: Different meaning from $\mathcal{L}\phi(x)=0$. Here applies to combined powers of x and y!

2. If homogeneous in x & y, can be solved by substitution $\boxed{y = xV}$ where $dy = x\,dV + V\,dx$
   a. All terms with dV are order x^{n+1}
   b All terms with dx are order x^n

   $\Big\}$ ~~then x & V can be separated!~~

3. Ex: Homogeneous ODE in x and y
   a. $(2x+y)\,dx + x\,dy = 0$
   b. Substitute $y = xV$, $dy = x\,dV + V\,dx$

   $[2x+(xV)]dx + x[x\,dV + V\,dx] = (2x+2xV)dx + x^2 dV = 0$

   c. Divide by x to obtain $2(1+V)dx + x\,dV = 0$
   d. Solve by separation $\int \frac{dV}{2(1+V)} = -\int \frac{dx}{x} \Rightarrow \frac{1}{2}\ln(1+V) = -\ln x + C'$
   e. Can be manipulated to $x^2(1+V) = C$ (where $C = e^{2C'}$)
   f. $y = xV \Rightarrow x^2 + x^2V = x^2 + xy = C \Rightarrow \boxed{y = \frac{C}{x} - x}$

D. Isobaric Equations

1. Generalizing the approach for ODEs homogeneous in x and y, assign different weights to x and y.
   a. Let x have weight 1, y have weight m.
   b. NOTE: dx and dy must have corresponding weights 1 and m.
   c. Substitution by $\boxed{y = x^m V}$ will make equation separable.

II. D. (Continued)

2. Ex: Isobaric ODE $(x^2-y)dx+xdy=0$

a. weight: 3 $\quad 1+m \quad 1+m$

b. Set $3=1+m \Rightarrow \boxed{m=2}$ substitute $y=x^2v$

c. $(x^2-x^2v)dx+x(2xvdx+x^2dv)=(x^2+x^2v)dx+x^3dv=0$

d. Divide by x^2: $(1+v)dx=-xdv$

e. Separate $\int\frac{dv}{1+v}=-\int\frac{dx}{x} \Rightarrow \ln(1+v)=-\ln x+\overset{\text{Constant}}{\ln C}$

$\Rightarrow \boxed{1+v=\frac{+C}{x}}$

f. Solve $v=\frac{y}{x^2} \Rightarrow 1+\frac{y}{x^2}=\frac{C}{x} \Rightarrow \boxed{y=Cx-x^2}$

E. General Strategy for Solving Linear, First-Order ODEs

1. General form $\boxed{\frac{dy}{dx}+p(x)y=q(x)}$

2. If equation is not exact, it can be made so by an integrating factor $\alpha(x)$.

$$\alpha(x)\frac{dy}{dx}+\underbrace{\alpha(x)p(x)}_{=\frac{d\alpha}{dx}}y=\frac{d}{dx}\left[\alpha(x)y\right]=\alpha(x)q(x)$$

3. Thus, we must solve $\frac{d\alpha}{dx}=\alpha(x)p(x)$ ← this is separable.

a. $\int^{\alpha}\frac{d\alpha}{\alpha}=\int^{x}p(x)dx \Rightarrow \underset{\text{Integrating Factor}}{\boxed{\alpha(x)=\exp\left[\int^{x}p(x)dx\right]}}$

4. Thus, integrating full equation gives

$$y(x)=\underbrace{\frac{1}{\alpha(x)}\int^{x}\alpha(x)q(x)dx}_{=y_2(x)}\underbrace{+\frac{C}{\alpha(x)}}_{=y_1(x)}$$

5. Parts of Solutions:

a. $y_1(x)=\frac{C}{\alpha(x)}$ is the homogeneous solution (Solution with $q(x)=0$)

i. $\frac{dy_1}{dx}+p(x)y_1=0 \Rightarrow \int\frac{dy_1}{y_1}=-\int^{x}p(x)dx+C$

II. E. 5. a. (Continued)

ii. Thus $\ln y_1 = -\ln\alpha + \ln C \Rightarrow \boxed{y_1 = \frac{C}{\alpha(x)}}$ Homogeneous Solution

b. Particular Solution: Set $C=0$ (remove homogeneous solution by canceling $p(x)y_1$ term)

$$y_2 = \frac{1}{\alpha(x)}\int^x \alpha(x)q(x)\,dx$$

No arbitrary constant in particular solution.

6. Theorem 1:

$\boxed{\text{The solution of an inhomogeneous first-order linear ODE is unique except for an arbitrary multiple of homogeneous solution.}}$

7. Theorem 2:

$\boxed{\text{A first-order, linear homogeneous ODE has only one linearly independent solution.}}$

8. Ex: RL Circuit $L\frac{dI(t)}{dt} + RI(t) = V(t)$

a. In general form $\frac{dI}{dt} + \frac{R}{L}I = \frac{V(t)}{L}$ So $p(t) = \frac{R}{L}$

$q(t) = \frac{V(t)}{L}$

b. Integrating factor: $\alpha(t) = \exp\left[\int^t p(t)dt\right] = \exp\left[\int^t \frac{R}{L}dt\right] = e^{Rt/L}$

c. Thus $I(t) = \frac{1}{\alpha(t)}\left[\int^t \alpha(t)q(t)dt + C\right] = e^{-Rt/L}\left[\int^t \frac{V(t)}{L}e^{Rt/L}dt + C\right]$

d. For the special case $V(t) = V_0 =$ constant and $I=0$ at $t=0$,

$$I(t) = e^{-Rt/L}\left[\frac{V_0}{\cancel{L}}\frac{\cancel{L}}{R}e^{Rt/L} + C\right] = \frac{V_0}{R} + Ce^{-Rt/L}$$

e. Applying initial conditions, $C = \frac{-V_0}{R}$, so $\boxed{I(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)}$

III. ODEs with Constant Coefficients

A. Special Case

1. Any order linear ODE with constant coefficients on homogeneous terms,

$$\boxed{\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0 y = f(x)}$$

2. Homogeneous equation has solutions of form $y = e^{mx}$, where m is solution of $m^n + a_{n-1}m^{n-1} + \ldots a_1 m + a_0 = 0$

3a. If m has a multiple root (degeneracy of α), you will not obtain n linearly independent solutions.

b. In this case, solutions are $e^{mx}, x^1 e^{mx}, x^2 e^{mx}, \ldots x^{\alpha-1} e^{mx}$.

4. Ex: Hooke's Law Spring $M\frac{d^2y}{dt^2} = -ky$

a. General form $y'' + \frac{k}{M}y = 0$

b. Assume $y = e^{mt} \Rightarrow m^2 + \frac{k}{M} = 0 \Rightarrow m = \pm i\sqrt{\frac{k}{M}} \equiv \pm i\omega$

c. General solution $\boxed{y = C_1 e^{+i\omega t} + C_2 e^{-i\omega t}}$

d. Fit to initial conditions $y(0)$ and $y'(0)$ to solve for C_1, C_2.