

Lecture #17 Second-Order Linear ODEs

I. Singularities

1. Second-order, linear ODEs arise frequently in physics.
2. No universally applicable closed-form solution exists, but methods can be used to form a power-series solution.

A. Singular Points

1. Singularities are important for:
 - a. Classifying ODEs
 - b. Determine feasibility of finding a series solution.

2. Linear, Homogeneous, Second-Order ODE

$$\boxed{y'' + P(x)y' + Q(x)y = 0}$$

3. Classification of Singularities

- a. Ordinary Points: Points x_0 where $P(x)$ and $Q(x)$ are finite.
- b. Regular Singular Point: $P(x)$ or $Q(x)$ diverge as $x \rightarrow x_0$, but $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ remain finite.
- c. Irregular Singular Point (Essential): $P(x)$ or $Q(x)$ diverge as $x \rightarrow x_0$, and $(x-x_0)P(x)$ or $(x-x_0)^2Q(x)$ diverges as $x \rightarrow x_0$.

4. Singularity at $x_0 \rightarrow \infty$:

- a. Set $x = \frac{1}{z}$, substitute into ODE, and examine limit as $z \rightarrow 0$.

b. $w(z) = y(z^{-1})$

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz} \frac{dz}{dx} = \frac{dw(z)}{dz} \left(-\frac{1}{x^2} \right) = -z^2 w' \leftarrow (\text{w.r.t. } z)$$

c. Similarly $y'' = z^4 w'' + 2z^3 w'$

I.A.4. (Continued)

d. Transformation yields

$$w'' + \underbrace{\frac{2z - P(z^{-1})}{z^2}}_{= P'(z)} w' + \underbrace{\frac{Q(z^{-1})}{z^4}}_{= Q'(z)} w = 0 \quad \text{Hawes (2)}$$

e. Ordinary Point at $x_0 = \infty$: IF $P'(z)$ & $Q'(z)$ finite as $z \rightarrow \infty$

f. Regular Singular Point at $x_0 = \infty$: IF $P'(z)$ & $Q'(z)$ diverge as $z \rightarrow \infty$ but $z P'(z)$ and $z^2 Q'(z)$ remain finite

g. Irregular Singular Point at $x_0 = \infty$: IF $P'(z)$ & $Q'(z)$ diverge as $z \rightarrow \infty$ and $z P'(z)$ or $z^2 Q'(z)$ diverge as $z \rightarrow \infty$.

5. Ex: Bessel's Equation: $x^2 y'' + x y' + (x^2 - n^2) y = 0$

a. $y'' + \frac{1}{x} y' + \frac{x^2 - n^2}{x^2} y = 0 \Rightarrow P(x) = \frac{1}{x}, Q(x) = \frac{x^2 - n^2}{x^2}$

b. As $x_0 \rightarrow 0$, $P(x)$ & $Q(x)$ diverge, but $x P(x)$ and $x^2 Q(x)$ are finite.
 $\Rightarrow x_0 = 0$ is a regular singular point

c. For $x_0 \rightarrow \infty$, $z = \frac{1}{x}$ gives $w'' + \left(\frac{2z - z}{z^2}\right) w' + \left(\frac{1 - n^2 z^2}{z^4}\right) w = 0$

As $z \rightarrow 0$, $z P'(z)$ is finite, but $z^2 Q'(z) = \frac{1 - n^2 z^2}{z^2}$ diverges.

$\Rightarrow x_0 = \infty$ is an irregular singular point

6. NOTE: Determination of nature of singularities is important to determine if a series solution is possible (Fuchs Thm).

II. Frobenius' Method for Series Solutions

A. General Procedure:

1. Linear, 2nd-Order, Homogeneous ODE:

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0$$

2. Expand y as power series about point $x = 0$.

3. Ex: Linear Simple Harmonic Oscillator:

$$y'' + \omega^2 y = 0$$

a. Power Series: $y(x) = x^s (a_0 + a_1 x + a_2 x^2 + \dots)$

NOTE: s need not be an integer!

II.A.3. (Continued) $\leftarrow a_0 \neq 0$.

b. $y(x) = \sum_{j=0}^{\infty} a_j x^{s+j} \Rightarrow y''(x) = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2}$ Hence ③

c. Thus $\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \omega^2 \sum_{i=0}^{\infty} a_i x^{s+i} = 0$

d. To collect powers of x , $j=i+2$ has same power of x , so

$\sum_{j=0}^{\infty} f_j = \sum_{i=-2}^{\infty} f_i = f_{-2} + f_{-1} + \sum_{i=0}^{\infty} f_i$, thus

$\underbrace{a_0 s(s-1)}_{=0} x^{s-2} + \underbrace{a_1 (s+1)s}_{=0} x^{s-1} + \sum_{i=0}^{\infty} \underbrace{[a_{i+2}(s+i+2)(s+i+1) + \omega^2 a_i]}_{=0} x^{s+i} = 0$

e. By uniqueness of power series solution, each coefficient must vanish!

x^{s-2} f. Indicial Equation: $a_0 s(s-1) = 0$ where $a_0 \neq 0$ by definition
 Solutions: $\boxed{s=0, s=1} \Rightarrow$ Series solution series with x^0 or x^1 term!

x^{s-1} g. $a_1 (s+1)s = 0$ i. If $s=0$, a_1 is unconstrained
 ii. If $s=1$, $a_1 = 0$.

x^{s+i} Terms h. $a_{i+2} (s+i+2)(s+i+1) + \omega^2 a_i = 0$
 i. $\boxed{a_{i+2} = \frac{-\omega^2}{(s+i+2)(s+i+1)} a_i}$ Recurrence Relation
 ii. Gives a_2, a_4, a_6, \dots in terms of a_0 and a_3, a_5, a_7, \dots in terms of a_1 . $a_0 \neq 0$
 Let us take $a_1 = 0!$

i. $\boxed{s=0}$ solution $a_{i+2} = \frac{-\omega^2}{(i+2)(i+1)} a_i$
 $\Rightarrow a_2 = \frac{-\omega^2}{2!} a_0$ $a_4 = \frac{-\omega^2}{3 \cdot 4} a_0 = \frac{-\omega^2}{3 \cdot 4} \left(\frac{-\omega^2}{2!} a_0 \right) = \frac{+\omega^4}{4!} a_0$
 \Rightarrow In general, $\boxed{a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0}$

Solution: $y(x) = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \dots \right] = \boxed{a_0 \cos(\omega x) = y(x)}$

II. A. 3. (Continued)

Hines (4)

J. S=1 solution

$$a_{i+2} = \frac{-\omega^2}{(i+3)(i+2)} a_i$$

$$\Rightarrow a_2 = -\frac{\omega^2}{3!} a_0 \Rightarrow a_4 = \frac{\omega^4}{5!} a_0 \Rightarrow a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

Solution $y(x) = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \dots \right] = \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \dots \right]$

$$\boxed{y(x) = \frac{a_0}{\omega} \sin(\omega x)}$$

K. Thus, general solution:

$$\boxed{y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)}$$

4. Cautionary Note about Series Solutions using Frobenius' Method:

- Always substitute solutions back into original ODE as a check!
- Need to check that series converges over region of interest.

B. Additional Considerations

- Expanding about a point $x_0 \neq 0$, $y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{s+j}$, $a_0 \neq 0$.
 - Legendre, Chebyshev, and Hypergeometric eqs. use $x_0 = 1$.
 - x_0 should not be an irregular singularity!

C. Ex: Bessel's Equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$

1. Let $y(x) = \sum_{j=0}^{\infty} a_j x^{s+j}$ $a_0 \neq 0$

2. $\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j} + \sum_{i=0}^{\infty} a_i x^{s+i+2} - \sum_{j=0}^{\infty} a_j n^2 x^{s+j} = 0$

x^s

3. Indicial Equation $j=0$ term:

$$a_0 s(s-1) + a_0 s - a_0 n^2 = a_0 [s^2 - s + s - n^2] = a_0 (s^2 - n^2) = 0 \quad a_0 \neq 0!$$

a. Solutions $\boxed{s = +n, s = -n}$

x^{s+1}

4. a. $a_1 (s+1)s + a_1 (s+1) - a_1 n^2 = a_1 (s+1-n)(s+1+n) = 0$

II Cont (Continued)

Homework 5

b. For either $S=+n$ or $S=-n$, solution requires $a_1=0$.

$x^{S+1/2}$ 5. Recurrence Relation: ($j=i+2$):

$$a_j [(S+j)(S+j-1) + S^2 - n^2] + a_{j-2} = 0$$

6. For $S=n$, we obtain:
$$a_{j+2} = \frac{-1}{(j+2)(2n+j+2)} a_j$$

7. We can obtain the general form

$$a_{2p} = (-1)^p \frac{a_0 n!}{2^{2p} p! (n+p)!}, \text{ yielding } y(x) = a_0 2^n n! \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{x}{2}\right)^{n+j}$$

$$y(x) = J_n(x) \text{ for } a_0 = \frac{1}{2^n n!}$$

8. For $S=-n$,
$$a_{j+2} = \frac{-1}{(j+2)(-2n+j+2)} a_j$$

a. But, when $-2n+j+2=0$, or $j=2(n-1)$, coefficient diverges!

b. For integral n , Frobenius' method fails to find a second series solution!

9. Thus Frobenius' method for series solution does not always yield two linearly independent solutions.

D. Regular vs. Irregular Singularities

1. Consider two equations: (A) $y'' - \frac{6}{x^2} y = 0$

(B) $y'' - \frac{6}{x^3} y = 0$

Substitute:

$$y = \sum_{j=0}^{\infty} a_j x^{S+j}$$

2. For (A), indicial equation is $S(S-1) - 6 = 0 \Rightarrow (S-3)(S+2) = 0$

a. Solutions $S=3, S=-2$

b. All terms have same power in x (ODE is homogeneous in x & y), so there is no recurrence relation. \Rightarrow Solutions: $y_1 = c_1 x^3, y_2 = c_2 x^{-2}$

II. D. (Continued)

3. For (B), $\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} - 6 \sum_{j=0}^{\infty} a_j x^{s+j-3} = 0$

Hawes (6)

a. Indicial equation: Lowest order x^{s-3} : $-6a_0 = 0$, but $a_0 \neq 0!$
 \Rightarrow No solution!

b. Expansion about irregular singular point at $x=0$ fails!

Fuchs Theorem: We can always obtain at least one power-series solution if we expand about an ordinary point or a regular singular point.

III. Obtaining a Second, Linearly Independent Solution

A. Determining Linear Independence of Solutions: Wronskian

1. Consider a set of n functions ϕ_i

a. The set is linearly dependent if one can solve
$$\sum_{i=1}^n k_i \phi_i(x) = 0 \quad (\text{with not all } k_i = 0.)$$

b. If only solution is $k_i = 0$, functions are linearly independent.

2. Mutually orthogonal functions are linearly independent.

a. $S = \langle \sum_i k_i \phi_i \mid \sum_j k_j \phi_j \rangle = \sum_{ij} k_i^* k_j \underbrace{\langle \phi_i \mid \phi_j \rangle}_{=\delta_{ij}} = \sum_i |k_i|^2 > 0$ unless $k_i = 0!$
 \Rightarrow linearly independent!

3. Wronskian:

a. Assuming differentiable functions ϕ_i ,
$$\sum_i k_i \phi_i'(x) = 0$$

$$\sum_i k_i \phi_i''(x) = 0, \text{ etc.}$$

b. Do this until you generate n equations.

III A. 3. (Continued)

Howes 7

c. This set of n homogeneous equations has a nontrivial solution (i.e. $k_i \neq 0$) only if determinant vanishes!

d. Wronskian $W \equiv \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix} = 0$

- e. 1) If Wronskian $W \neq 0$, functions ϕ_i are linearly independent.
2) Only if $W = 0$ over entire range of x are ϕ_i linearly dependent.

4. Ex: Linear Oscillator: $\phi_1 = \sin \omega t$, $\phi_2 = \cos \omega t$

a. $W = \begin{vmatrix} \sin \omega x & \cos \omega x \\ \omega \cos \omega x & -\omega \sin \omega x \end{vmatrix} = -\omega(\sin^2 \omega x + \cos^2 \omega x) = -\omega \neq 0 \Rightarrow$ linearly independent.

B. Number of Solutions

1. Theorem: A linear, 2nd-order homogeneous ODE has at most two linearly independent solutions. An n -th order, linear, homogeneous ODE has at most n linearly independent solutions.
Proof in text.

C. Constructing a Second, Linearly Independent Solution

1. For $y'' + P(x)y' + Q(x)y = 0$ with solution $y = y_1(x)$.

2. A second solution can be constructed by $y_2(x) = y_1(x) \int \frac{\exp(-\int^{x_2} P(x_1) dx_1)}{[y_1(x_2)]^2} dx_2$

3. A common special case has $P(x) = 0$, so

$$y_2(x) = y_1(x) \int \frac{dx_2}{[y_1(x_2)]^2}$$

III. C. (Continued)

Hines (8)

4. NOTE: Any linear, 2nd-order homogeneous ODE can be transformed into a form with $P(x) = 0$.

5. Ex: Simple Harmonic Oscillator. $y'' + y = 0$

a. One solution: $y_1(x) = \sin x$

$0 \rightarrow$ only adds $C y_1$

b. $y_2(x) = \sin x \int \frac{dx_2}{\sin^2 x_2} = \sin x [-\cot x + C] = \sin x \frac{-\cos x}{\sin x} = -\cos x$

c. Thus $y_2(x) = \cos x$

D. Series Form of Second Solution

1. We can expand $P(x) = \sum_{i=-1}^{\infty} p_i x^i$ and $Q(x) = \sum_{j=2}^{\infty} q_j x^j$

to compute second solution by integrating the form for $y_2(x)$ [above in III.C.2] term by term.

2. Series Form of Second Solution

$$y_2(x) = y_1(x) \ln|x| + \sum_{j=-n}^{\infty} d_j x^{j+\alpha}$$

where n is an integer, where power series starts with x^{-n} .