Lecture #22 Green's Functions

I. Green's Functions

A. Basic Concept

1. Used to solve <u>inhomogeneous</u> differential equations, where the inhomogeneous term acts as a <u>source</u>, and you <u>integrate</u> over the source to obtain solution.

2. Ex: Poisson Equation: $-\nabla^2 \Psi(r) = \frac{1}{\epsilon_0} \rho(r)$

   a. <u>Solution</u>: $\Psi(r) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{\rho(r')}{|x - x'|}$

   b. Integral contributes over entire region where $\rho(r') \neq 0$.

   c. Right-hand side is an <u>integral operator</u>
   $\Rightarrow$ Converts charge density $\rho(x)$ to potential $\Psi(r)$.

   d. <u>Green's Function</u>:
   $$G(x, x') = \frac{1}{4\pi\epsilon_0} \frac{1}{|x - x'|}$$

   i) This is the <u>kernel</u> that multiplies the inhomogeneous source term.
   ii) Function of two variables (r, r'): x is constant, x' is integrated.

   e. Therefore $\Psi(x) = \int d^3 x' \, G(x, x') \rho(x')$

3. For more general problems, Green's function may depend on
   a. Boundary Conditions
   b. Form of differential operator.

4. KEY CONCEPT: a. $G(x, x')$ gives contribution to Ψ at point x due to a <u>unit-magnitude source</u> (delta function) at x'
   b. We can determine Ψ at x by summing (integrating) over all points of the source because the differential operator is <u>linear</u>

I. A.4.(continued)

  c. Ex: i. Potential due to charge q at x' $\phi(x) = \frac{1}{4\pi\epsilon_0} \frac{q}{|r - r'|}$

  ii. If I have a collection of charges q_j, then

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \sum_j \frac{q_j}{|x - r_j'|}$$

  iii. The continuum limit yields $\phi(x) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(x')}{|x - x'|}$

     where charge density is $\int d^3r' \, \rho(r) = Q = \sum_i q_j$

                                          total charge

II. Green's Functions in 1D

A. Fundamental Definition

1. Consider a self-adjoint, inhomogeneous ODE $\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) \cdot y = f(x)$

  a. Interval $a \leq x \leq b$

  b. Boundary Conditions: Homogeneous at $x=a$ & $x=b$
    (Meaning if y_1 satisfies BCs, so does $C y_1$)

2. Defining Equation for G: $\boxed{\mathcal{L}\, G(x, x') = \delta(x - x')}$

3. Solution for $y(x)$: $\boxed{y(x) = \int_a^b G(x, x') f(x') dx'}$

  a. Check: $\mathcal{L}y(x) = \int_a^b \underbrace{\mathcal{L}G(x,x')}_{=\delta(x-x')} f(x') dx' = \int_a^b \delta(x-x') f(x') dx' = f(x).$

B. General Properties

Note: → x is constant, but x' is variable that is differentiated.

1. Take $\mathcal{L}(x') G(x, x') = \delta(x - x')$ and integrate $\int_{x-\epsilon}^{x+\epsilon} dx'$

  a. $\int_{x-\epsilon}^{x+\epsilon} \mathcal{L}(x') G(x, x') dx' = \int_{x-\epsilon}^{x+\epsilon} \delta(x-x') dx' = 1$

  b. $\int_{x-\epsilon}^{x+\epsilon} \frac{d}{dx'}\left[p(x') \frac{dG(x,x')}{dx'}\right] dx' + \int_{x-\epsilon}^{x+\epsilon} q(x') G(x,x') dx' = 1$

II. B. (Continued)

2. First integral is exact $\Rightarrow$ $\left. p(x') \frac{dG(x,x')}{dx'} \right|_{x-\varepsilon}^{x+\varepsilon}$

3. Necessary Discontinuity:

$$\left. p(x') \frac{dG(x,x')}{dx'} \right|_{x-\varepsilon}^{x+\varepsilon} + \int_{x-\varepsilon}^{x+\varepsilon} q(x') G(x,x') dx' = 1$$
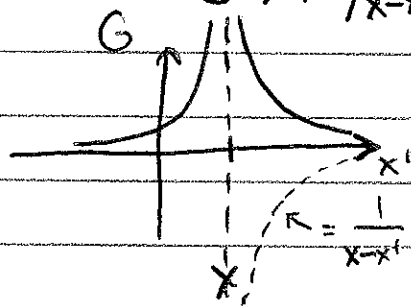
a. In $\lim\limits_{\varepsilon \to 0^+}$, this cannot be satisfied if both

$$\frac{dG(x,x')}{dx'} \text{ and } G(x,x') \text{ are continuous.}$$

b. We require $G(x,x')$ continuous, but allow discontinuity in $\frac{dG(x,x')}{dx'}$

c. In $\lim\limits_{\varepsilon \to 0^+}$, integral vanishes, and we obtain

$$\lim_{\varepsilon \to 0^+} \left[\left. \frac{dG(x,x')}{dx'} \right|_{x'=x+\varepsilon} - \left. \frac{dG(x,x')}{dx'} \right|_{x'=x-\varepsilon} \right] = \frac{1}{p(x)}$$

d. Ex: 1-D $\quad G(x,x') = \frac{1}{|x-x'|}$

$$\frac{dG(x,x')}{dx'} = \begin{cases} \dfrac{1}{(x-x')^2} & x' < x \\[2mm] \dfrac{-1}{(x-x')^2} & x' > x \end{cases}$$

$\Rightarrow$ Discontinuous slope.

4. NOTE: a. The key to understanding Green's functions is to be very careful handling its two variables

b. One variable is always a constant, while the other varies (and is the one over which you integrate).

5. Eigenfunction expansions in two variables:

a. We are going to expand $G(x,x')$ in the orthonormal eigenfunctions of the Hermitian operator $\mathcal{L}$, ϕ_i

b. NOTE: Both x and x' are separately expanded in the ϕ_i

c. To expand a function $f(x,x')$ in ϕ_i

  i. First, expand in $\phi_i(x)$ with $x' = $ constant

$$f(x,x') = \sum_n a_n(x')\,\phi_n(x) \text{, where } a_n(x') = \langle \phi_n(x) | f(x,x') \rangle$$

Here $\langle \phi_n(x) | f(x,x') \rangle = \int_a^b \phi_n^*(x)\, f(x,x')\,dx = a_n(x')$ with $x' = $ const.

  ii. Second, expand $a_n(x')$ in $\phi_m(x')$ with $x = $ constant

$$a_n(x') = \sum_m b_m\, \phi_m(x') \quad \text{where } b_m = \langle \phi_m(x') | a_n(x') \rangle$$

  iii. Thus, $b_m = \langle \phi_m(x') | a_n(x') \rangle = \int_a^b \phi_m^*(x')\, a_n(x')\,dx'$

$$\boxed{= \int_a^b \phi_m^*(x') \left[\int_a^b \phi_n^*(x)\, f(x,x')\,dx \right] dx' \equiv C_{nm}}$$

  iv. Finally $\boxed{f(x,x') = \sum_{n,m} C_{nm}\, \phi_n(x)\, \phi_m(x')}$

6. Now, Let's expand $G(x,x')$ in terms of eigenfunctions of $\mathcal{L}$

  a. First, expand $\delta(x-x')$ in terms of $\phi_i(x)$

  i. $\delta(x-x') = \sum_n c_n(x')\, \phi_n(x)$

  ii. $c_n(x') = \langle \phi_n(x) | \delta(x-x') \rangle = \int_a^b \phi_n^*(x)\, \delta(x-x')\,dx = \phi_n^*(x')$

  iii. Thus $\boxed{\delta(x-x') = \sum_n \phi_n(x)\, \phi_n^*(x')}$

  b. Now, $G(x,x') = \sum_{nm} g_{nm}\, \phi_n(x)\, \phi_m^*(x')$ ← Take $\phi_m^*(x')$, instead of $\phi_m(x')$, to match $\delta(x-x')$.

  c. Use defining eq. for G: $\mathcal{L}(x) G(x,x') = \delta(x-x')$ and substitute.

  i. NOTE: Here $\mathcal{L}(x)$ is operating on x, not x' !

  ii. $\mathcal{L}(x) \sum_{nm} g_{nm}\, \phi_n(x)\, \phi_m^*(x') = \sum_m \phi_m(x)\, \phi_m^*(x')$

  iii. $\sum_{nm} g_{nm}\, \lambda_n\, \phi_n(x)\, \phi_n^*(x') = \sum_m \phi_m(x)\, \phi_m^*(x')$

d. Take scalar product with $\phi_{m'}^*(x')$

$$\sum_{nm} g_{nm} \lambda_n \phi_n(x) \underbrace{\langle \phi_m^*(x') | \phi_{m'}^*(x') \rangle}_{\delta_{mm'}} = \sum_m \phi_m(x) \underbrace{\langle \phi_m^*(x') | \phi_{m'}^*(x') \rangle}_{= \delta_{mm'}}$$

e. Take scalar product with $\phi_{n'}(x)$

$$\sum_n g_{nm'} \lambda_n \underbrace{\langle \phi_{n'}(x) | \phi_n(x) \rangle}_{\delta_{n'n}} = \underbrace{\langle \phi_{n'}(x) | \phi_{m'}(x) \rangle}_{\delta_{n'm'}}$$

f. Thus, we obtain $\quad g_{n'm'} \lambda_{n'} = \delta_{n'm'} \quad$ or $\quad \boxed{g_{nm} = \dfrac{\delta_{nm}}{\lambda_n}}$

g. Plugging back into expansion for $G(x,x')$: $\quad \boxed{G(x,x') = \sum_n \dfrac{\phi_n(x)\, \phi_n^*(x')}{\lambda_n}}$

7. Symmetry of $G(x,x')$: $\quad \boxed{G(x,x') = G^*(x',x)}$

   a. This key property will enable us to establish a procedure for computing Green's Functions.

C. General Solution for $G(x,x')$

   1. Consider a Hermitian operator $\mathcal{L}$ with homogeneous BCs on (a,b).

   $$\boxed{\mathcal{L}y = \frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] + q(x)y = f(x)} \quad \xleftarrow{} \text{Source in inhomogeneous ODE.}$$
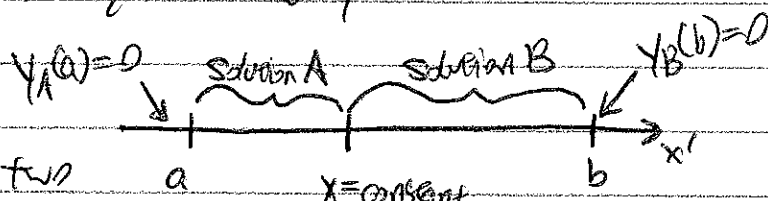
   2. We will determine $G(x,x')$ in terms of solutions $\boxed{y_1(x),\ y_2(x)}$ of the homogeneous equation $\mathcal{L}y = 0$.

   3. Strategy:

   a. Split range $a \le x' \le b$ into two sections, $a \le x' < x$ and $x < x' \le b$, where x is constant.

   b. In each half, satisfy B.C.'s: $y_A(a) = 0, \quad y_B(b) = 0$!

II. C. (Continued)

4. The general homogeneous solution $\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$

5. Interval: $a \le x' < x$

   a. Chose C_1 & C_2 such that $y_A(a) = C_1 y_1(a) + C_2 y_2(a) = 0$

   b. The most general form may have a constant $C_A(x)$ ← can be function of x.

   c. Thus $G(x,x') = C_A(x) y_A(x')$, $a \le x' < x$

6. Following same procedure, $G(x,x') = C_B(x) y_B(x')$, $x < x' \le b$

7. Thus
$$G(x,x') = \begin{cases} C_A(x) \, y_A(x') & a \le x' < x \\ C_B(x) \, y_B(x') & x < x' \le b \end{cases}$$

8. Apply Symmetry Condition $G(x,x') = G^*(x',x)$

   a. Solution is consistent only if $C_A^*(x) = A y_B(x)$ and $C_B^*(x) = A y_A(x)$, where A is a constant

   b. If we assume y_A and y_B can be chosen to be real, we begin
$$G(x,x') = \begin{cases} A \, y_A(x') \, y_B(x) & a \le x' < x \\ A \, y_B(x') \, y_A(x) & x < x' \le b \end{cases}$$
$\boxed{\text{Solution for } G(x,x')}$

   c. But, we still need to determine A.

9. Apply $\lim\limits_{\varepsilon \to 0^+} \left[\dfrac{dG(x,x')}{dx'} \Big|_{\substack{x' \le x+\varepsilon \\ x' > x}} - \dfrac{dG(x,x')}{dx'} \Big|_{\substack{x' = x-\varepsilon \\ x' < x}} \right] = \dfrac{1}{p(x)}$

   a. $A\left[y_B'(x) y_A(x) - y_A'(x) y_B(x) \right] = \dfrac{1}{p(x)}$

   b. Thus $\boxed{A = \left\{ p(x) \left[y_B'(x) y_A(x) - y_A'(x) y_B(x) \right] \right\}^{-1}}$ $\boxed{\text{Solution for } A}$

   c. NOTE: i. A is actually a constant → does not depend on x!

   ii. Term in brackets is Wronskian, $[\cdots] = W = \dfrac{C}{p(x)}$ ← can be shown.

II. (Continued)

D. Example 1:

1. Consider $-y'' = f(x)$, where $y(0) = y(1) = 0$.

2. General Solution to Homogeneous equation: $y'' = 0$

$$y = C_0 + C_1 x$$

3. Determine Solutions y_0 and y_1 that satisfy B.C.'s:

   a. $y(0) = 0 = C_0 + C_1(0) \Rightarrow C_0 = 0 \Rightarrow y_0(x) = x$

   b. $y(1) = 0 = C_0 + C_1(1) \Rightarrow C_1 = -C_0 \Rightarrow y_1(x) = 1-x$

4. Compute coefficient A: $p(x) = -1$, $y_0'(x) = 1$, $y_1'(x) = -1$

$$A = \left\{ p(x)\left[y_1'(x)y_0(x) - y_0'(x)y_1(x) \right] \right\}^{-1} = \left\{ (-1)\left[(-1)(x) - (1)(1-x) \right] \right\}^{-1} = 1$$

$$\underbrace{}_{-x-1+x}$$

5. Thus
$$G(x,x') = \begin{cases} x'(1-x), & 0 \le x' < x \\ (1-x')x, & x < x' \le 1 \end{cases}$$

6. Thus, Solution $y(x)$ is $y(x) = \int_0^1 G(x,x')f(x')\,dx'$ for any $f(x)$.

7. a. For example, $f(x) = \sin(\pi x)$ yields

  b. $y(x) = \int_0^1 G(x,x')\sin\pi x'\,dx' = \int_0^x x'(1-x)\sin\pi x'\,dx' + \int_x^1 (1-x')x\sin\pi x'\,dx'$

  c. Using $\int_a^b x'\sin\pi x'\,dx' = \left. \frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi} \right|_a^b$, we obtain

$$= (1-x)\left[\frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi} \right]_0^x + x\left[\frac{-\cos\pi x'}{\pi} \right]_x^1 - x\left[\frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi} \right]_x^1$$

$$= (1-x)\left[\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi} \right] + x\left[\frac{-(1)}{\pi} + \frac{\cos\pi x}{\pi} \right] - x\left[\frac{-(1)(-1)}{\pi} - \frac{\sin\pi x}{\pi^2} + \frac{x\cos\pi x}{\pi} \right]$$

$$= \frac{\sin\pi x}{\pi^2} - \frac{\cancel{x\cos\pi x}}{\pi} - x\left(\cancel{\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}} \right) + \frac{\cancel{x\cos\pi x}}{\pi} + x\left(\cancel{\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}} \right)$$

$$\boxed{y(x) = \frac{\sin\pi x}{\pi^2}}$$

8. NOTE! With Green's functions, if you change $f(x)$, you need only repeat the integration of the Green's function kernel with $f(x)$.

II. (Continued)

E. Other Forms of Boundary Conditions

1. Non-homogeneous boundary conditions: $y(a) = C_1$ $y(b) = C_2$

a. Simply change y to $U = y - \dfrac{C_1(b-x) + C_2(x-a)}{b-a}$

b. In terms of U, $U(a) = 0$, $U(b) = 0$.

2. Initial Value Problems: $\mathcal{L} y(t) = f(t)$ $f(0) = 0$, $f'(0) = 0$.

a. Can construct $G(t, t')$ by invoking continuity of G at $t = t'$!

b. BUT, $G(t, t')$ is no longer symmetric.

3. Ex: Initial Value Problem: $\mathcal{L} y = \dfrac{d^2 y}{dt^2} + y = f(t)$, $y(0) = 0$, $y'(0) = 0$

a. Homogeneous Solution: $\mathcal{L} y = 0 \Rightarrow y_1 = \sin t$, $y_2 = \cos t$

b. Key point: No linear combination $y_0 = C_1 y_1 + C_2 y_2$ satisfies BC's at $t = 0$, except for $C_1 = C_2 = 0$.

   i. Thus, for $t' < t$, we have $G(t, t') = 0$.

c. Second key point: At $t' > t$, we have no constraining BC's, so

   i. For $t' > t$, $G(t, t') = C_1(t) \sin t' + C_2(t) \cos t'$

d. Thus $G(t, t') = \begin{cases} 0 & t' < t \\ C_1(t) \sin t' + C_2(t) \cos t' & t' > t \end{cases}$

e. Enforce continuity requirements:

   i. $\lim\limits_{\varepsilon \to 0^+} \left[\underset{t' > t}{G(t, t+\varepsilon)} - \underset{t' < t}{G(t, t-\varepsilon)} \right] = 0 \Rightarrow \boxed{C_1(t) \sin t + C_2(t) \cos t - 0 = 0}$

   ii. $\lim\limits_{\varepsilon \to 0^+} \left[\left. \dfrac{dG(t,t')}{dt'} \right|_{\underset{t' > t}{t' = t+\varepsilon}} - \left. \dfrac{dG(t,t')}{dt'} \right|_{\underset{t' < t}{t' = t-\varepsilon}} \right] = \dfrac{1}{p(t)}$ where $p(t) = 1$ for $\mathcal{L}$.

   iii. $\boxed{C_1(t) \cos t + C_2(t)(-\sin t) - 0 = 1}$

   iv. Solving these equations yields $C_1(t) = \cos t$, $C_2(t) = \sin(t)$.

f. Thus $\cos t \sin t' - \sin t \cos t' = \sin(t' - t) \Rightarrow \boxed{G(t, t') = \begin{cases} 0 & t' < t \\ \sin(t' - t) & t' > t \end{cases}}$