Lecture #24 Random Variables and Binomial and Poisson Distributions

I. Random Variables

A. Basic Concepts

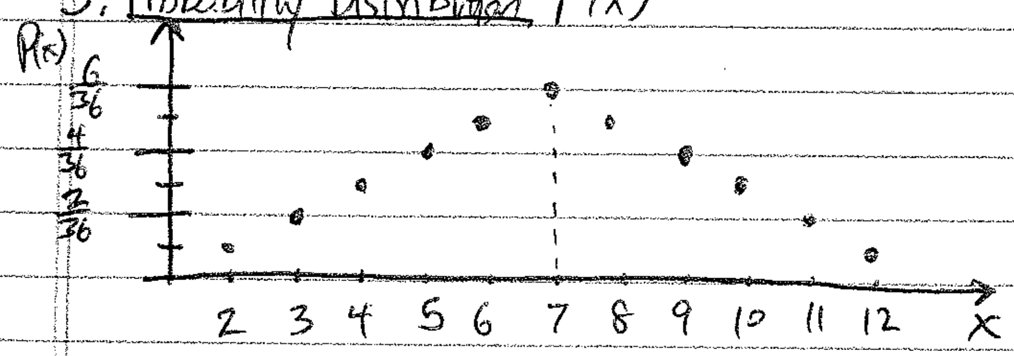
1. Def: A random variable takes on different numerical values with individual probabilities.
2. Quantities to determine are average value and spread of values as well as correlations with other random variables.
3. Random variables may have discrete or continuous values.

B. Probability Distribution for a Discrete Random Variable

1. Rolling Dice: a. Consider rolling one die twice and summing results
   b. Random Variable: X = sum of two rolls $\quad 2 \le X \le 12$ integer
   c. For a single roll, 6 possible mutually exclusive, equally likely events x_i with $P(x_i) = \frac{1}{6}$

2. Compute Probabilities for two rolls:
   a. For $X=2$, one event: $(1,1)$
   Each roll is independent, So $P(1,1) = P(1)P(1) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$
   b. For $X=3$, two events: $(1,2)$ and $(2,1)$
   $P(X=3) = P(1,2) + P(2,1) = P(1)P(2) + P(2)P(1) = 2\left(\frac{1}{6}\right)^2 = \frac{2}{36}$
   c. For $X=4$, three events: $(2,2)$, $(1,3)$, and $(3,1)$
   $P(X=4) = 3\left(\frac{1}{6}\right)^2 = \frac{3}{36}$

3. Probability Distribution $P(x)$



I. B. (Continued)

Haves (2)

4. If X can have n discrete values x_i, where $p_i \equiv P(x_i) \geq 0$,
then $\sum_{i=1}^{n} p_i = 1$. ← Normalization

C. Probability Distribution for a Continuous Random Variable

1. Def: Probability Density $f(x)$ of Continuous random variable X.

a. $\boxed{P(x \leq X \leq x+dx) = f(x)\,dx}$

Probability that X lies in interval $[x, x+dx]$

b. $f(x) \geq 0$

c. Normalization $\int f(x)\,dx = 1$

2. Ex: Hydrogen Atom in Quantum Mechanics

a. 1s electron: Wave function $\psi(x) = \frac{1}{(\pi a^3)^{\frac{1}{2}}} e^{-\frac{r}{a}}$ ← Bohr radius

b. Probability for 1s electron in volume $d^3\underline{r} = |\psi|^2 d^3\underline{r}$

c. Normalization $\int d^3\underline{r}\, |\psi|^2 = 1$ for integration over all space.

D. Computing General Discrete Probability Distributions

1. General Case: a. N independent events

b. Each event has m mutually exclusive outcomes

c. Each outcome $x_1, x_2, \ldots, x_m$ has probability $p_1, p_2, \ldots, p_m$.

2. What is probability that n_1 events have outcome x_1, n_2 events have outcomes x_2, etc.? Here $n_1 + n_2 + \ldots + n_m = N$

a. $P(n_1, n_2, \ldots, n_m) = \underbrace{B(n_1, n_2, \ldots, n_m)}(p_1)^{n_1}(p_2)^{n_2}\ldots(p_m)^{n_m}$

Number of ways n_1 events have x_1, n_2 events have x_2, etc.

Multinomial coefficient

Z. D. 2. (Continued)

b. Multinomial coefficient $B(n_1, n_2, \ldots, n_m) = \frac{N!}{n_1! n_2! \ldots n_m!}$

c. Thus $\boxed{P(n_1, n_2, \ldots, n_m) = \frac{N!}{n_1! n_2! \ldots n_m!} (p_1)^{n_1} (p_2)^{n_2} \ldots (p_m)^{n_m}}$

E. Mean and Variance

1. Consider taking the average of n measurements of a variable x, giving values x_j, such that

$$\underbrace{\overline{x}}_{\text{average}} = \underbrace{\frac{1}{n}}_{\text{Probability of each measurement (equally likely).}} \sum_{j=1}^{n} x_j$$

2. Def: Mean Value, $\langle x \rangle$

a. Discrete $\boxed{\langle x \rangle \equiv \sum_i x_i p_i}$

b. Continuous $\boxed{\langle x \rangle \equiv \int x f(x) dx}$

3. Why do we chose to use the arithmetic mean?

a. Consider minimizing the sum of squares of deviations, $\sum_{i=1}^{n} (x - x_i)^2$

b. $\frac{d}{dx}\left[\sum_{i=1}^{n} (x - x_i)^2\right] = \sum_{i=1}^{n} 2(x - x_i) = 0$

c. Solve for x: $\sum_{i=1}^{n} x - \sum_{i=1}^{n} x_i = 0 \Rightarrow nx = \sum_{i=1}^{n} x_i \Rightarrow \underbrace{x = \frac{1}{n}\sum_{i=1}^{n} x_i}_{\text{Arithmetic mean}}$

d. This method of lease squares is due to Gauss and others.

4. Def: Standard Deviation, σ

a. $\boxed{\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \langle x \rangle)^2}}$

5. <u>Useful Formula</u> $\boxed{\sigma^2 = \langle x^2\rangle - \langle x\rangle^2}$ (Discrete case)

a. Square $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \langle x\rangle)^2$

b. $n\sigma^2 = \sum_{i=1}^{n} x_i^2 - 2\langle x\rangle \underbrace{\sum_{i=1}^{n} x_i}_{=n\langle x\rangle} + n\langle x\rangle^2 = n\langle x^2\rangle - 2n\langle x\rangle^2 + n\langle x\rangle^2$

$= n(\langle x^2\rangle - \langle x\rangle^2)$.

6. Def: <u>Variance</u>, σ^2

a. Discrete $\boxed{\sigma^2 = \sum_j (x_j - \langle x\rangle)^2 p_j}$

b. Continuous: $\boxed{\sigma^2 = \int_{-\infty}^{\infty}(x - \langle x\rangle)^2 f(x)\,dx}$

7. <u>Properties of Random Variables</u>; X, Y

a. <u>Variance</u>: $\boxed{\sigma^2 = \langle X^2\rangle - \langle X\rangle^2}$ (discrete or continuous)

b. If two random variables are related by $Y = aX + b$, then

i. $\boxed{\langle Y\rangle = a\langle X\rangle + b}$

ii. $\boxed{\sigma^2(Y) = a^2\sigma^2(X)}$

c. <u>Chebyshev Inequality</u> $\boxed{P(|x - \langle x\rangle| \geq k\sigma) \leq \frac{1}{k^2}}$

i. Note, for $k=3$, $P(|x - \langle x\rangle| \leq 3\sigma) \leq \frac{1}{9}$, standard 3σ estimate.

F. <u>Moments of Probability Distributions</u>

1. Def: <u>k^{th} Moment</u> $\langle (X - \langle x\rangle)^k\rangle$

a. Discrete $\boxed{\langle (X - \langle x\rangle)^k\rangle = \sum_j (x_j - \langle x\rangle)^k p_j}$

b. Continuous $\boxed{\langle (X - \langle x\rangle)^k\rangle = \int_{-\infty}^{\infty}(x - \langle X\rangle)^k f(x)\,dx}$

F. (Continued)

2. Moment Generating Function: $\langle e^{tX} \rangle$

a. $$\langle e^{tX} \rangle = \int e^{tx} f(x)dx = \int \left[1 + tx + \frac{t^2}{2!}x^2 + \frac{t^3}{3!}x^3 + \dots\right] f(x)dx$$
$$= 1 + t\langle X \rangle + \frac{t^2}{2!}\langle X^2 \rangle + \frac{t^3}{3!}\langle X^3 \rangle + \dots$$

b. $$\frac{d}{dt}\langle e^{tX} \rangle\Big|_{t=0} = \langle X \rangle + \cancel{\frac{t}{1}}^{0}\langle X^2 \rangle + \cancel{\frac{t^2}{2!}}^{0}\langle X^3 \rangle + \cancel{\dots}^{0} = \langle X \rangle$$

c. Similarly, $$\langle X^2 \rangle = \frac{d^2}{dt^2}\langle e^{tX} \rangle\Big|_{t=0}, \dots \quad \langle X^n \rangle = \frac{d^n}{dt^n}\langle e^{tX} \rangle\Big|_{t=0}$$

G. Multiple Random Variables:

1. For X and Y a. $$\langle X \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy, \quad \langle Y \rangle = \iint_{-\infty}^{\infty} y f(x,y)\,dx\,dy$$

b. $$\sigma^2(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \langle X \rangle)^2 f(x,y)\,dx\,dy, \text{ etc.}$$

H. Covariance and Correlation

1. These measure how two random variables are related.
   a. Independent if $f(x,y) = g(x)\,h(y)$

2. Def: Covariance, $\mathrm{cov}(X,Y)$

a. $$\boxed{\mathrm{Cov}(X,Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle}$$

b. For X and Y independent, $\mathrm{cov}(X,Y) = 0$.

3. Def: Correlation (Normalized Covariance)

a. $$\boxed{\frac{\mathrm{cov}(X,Y)}{\sigma(X)\,\sigma(Y)}}$$

b. Values fall in interval $[-1, 1]$.

4a. Theorem: $P(Y = aX + b) = 1$ if and only if the correlation $\frac{\mathrm{cov}(X,Y)}{\sigma(X)\sigma(Y)} = \pm 1$.

b. Implies ~~a~~ linear relationship between X and Y.

I. Marginal Probability Distributions

1. For a two-variable probability distribution, we can integrate out one of the random variables.

Marginal Probability Distributions → $F(x) = \int f(x,y)\,dy$ or $G(y) = \int f(x,y)\,dx$

2. Marginal probability distributions remain properly normalized,
$$\int F(x)\,dx = 1.$$

II. Binomial Distribution

A. Basic Properties

1. Occurs typically in repeated, independent trials of random events.

2. Ex: Rolling a Six

a. Probability to roll a Six: $a = P(6) = \frac{1}{6}$ $\Rightarrow a + b = 1$.
to roll anything else: $b = \frac{5}{6}$

b. If you roll 4 times, let random variable $S = s$ be number of sixes,
$0 \le S \le 4$

c. $$P(S=s) = \underbrace{\frac{4!}{s!(4-s)!}}_{\text{(Number of ways to obtain S sixes)}} \underbrace{a^s b^{4-s}}_{\text{(Probability of S sixes)}}$$

II, A2 (Continued)

d. Note: $\sum_{s=0}^{4} \binom{4}{s} a^s b^{4-s} = (a+b)^4 = \left(\frac{1}{6} + \frac{5}{6}\right)^4 = 1.$

e. Here: $f(0) = b^4$, $f(1) = 4ab^3$, $f(2) = 6a^2b^2$, $f(3) = 4a^3b$, $f(4) = a^4$

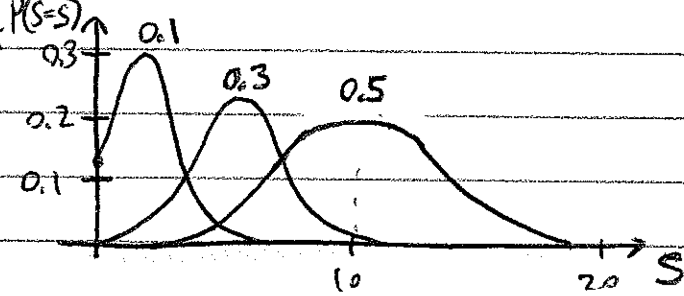
3. Common Special Case:

a. n repeated trials with two possible outcomes
Probability p for a hit, probability $q = 1-p$ for a miss.

b. For $S = s$ successful hits

$$\boxed{P(S=s) = \frac{n!}{s!(n-s)!} p^s q^{n-s} = \binom{n}{s} p^s q^{n-s}}$$ Binomial Probability Distribution

c. Plot



$n = 20$, $p = 0.1, 0.3, 0.5$

III. Poisson Distribution

A. Basics

1. Describes situations where events occurs repeatedly at constant probability.
Ex: Radioactive decay, when slow enough that depletion is negligible

2. Assumptions:

a. Small event rate, so in a measurable interval at most one event occurs.

b. Useful to model events by discrete probability distributions.

B. Derivation of Poisson Distribution

1. Define $P_n(t)$ as probability that n events occur in time t.

2. For a short interval dt, $P_1(dt) = \mu dt$ where μ = constant and $\mu dt \ll 1$.

3. Thus $P_n(t+dt) = P_n(t)\, P_0(dt) + P_{n-1}(t)\, P_1(dt)$

mutually exclusive events

III. B. (Continued)

4. NOTE: A short enough interval dt is chosen so that only 0 or 1 events occur. Thus $P_0(dt)+P_1(dt)=1$.

5. Substituting $P_1(dt)=\mu dt$, after some manipulation, we obtain

$$\boxed{\frac{dP_n(t)}{dt} = \frac{P_n(t+dt)-P_n(t)}{dt} = \mu P_{n-1}(t) - \mu P_n(t)}$$ Recursion Relation.

6. For $n=0$, $P_{n-1}(t)=0$ (not possible to have -1 event), (0 events in 0 time)

$$\frac{dP_0(t)}{dt} = -\mu P_0(t) \Rightarrow P_0(t) = e^{-\mu t}$$ using initial condition $P_0(0)=1$

7. Solving recursively, $$\boxed{P_n(t) = \frac{(\mu t)^n}{n!} e^{-\mu t}}$$ Poisson Distribution

8. a. For a discrete random variable X, standard form is

$$\boxed{p(n) = \frac{\mu^n}{n!} e^{-\mu}}$$ where $X = n = 0, 1, 2, \ldots$

b. Normalization $$\sum_{n=0}^{\infty} p(n) = \underbrace{\left(\sum_{n=0}^{\infty} \frac{\mu^n}{n!}\right)}_{=e^{+\mu}} e^{-\mu} = \left(e^{+\mu}\right)e^{-\mu} = 1.$$

9. Mean & Variance

a. $$\langle x \rangle = \sum_{n=0}^{\infty} (n) \frac{\mu^n}{n!} e^{-\mu} = e^{-\mu} \sum_{n=1} \frac{\mu^n}{(n-1)!} = e^{-\mu} \mu \underbrace{\left(\sum_{p=0}^{\infty} \frac{\mu^p}{p!}\right)}_{=e^{+\mu}} = \mu$$

b. $\langle x^2 \rangle = \mu^2 + \mu$

c. $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \mu$

10. Relation to Binomial Distribution

Theorem: For $n \to \infty$ and $p \to 0$ such that $np \to \mu$ remains finite, the binomial distribution becomes a Poisson Distribution.