(R-0)

PHYS: 4761 Math Methods I

0. Midterm Exam #1

A.

1. Two Page (Two sides of $8\frac{1}{2}$ by 11" paper)
   a. Formula Summary Sheet
   b. Ask if you need a formula.
2. Calculator OK (Won't need it)

3. I will provide scratch paper

4. Show your work / ~~teach~~ explain your reasoning.
   a. Partial Credit
   b. No credit if I can't figure out how you got answer.

B. Covers:
1. Chapters 1-4
2. HW #1-5
3. Lectures #1-11

C. To Study:
1. Go through HW Solutions ∴ Test problems will be very similar. (ICON)
2. a. 11 Problems: Some rather short
3. Exam will be difficult. Aiming for 60 average.
   ⇒ Enables testing of all abilities.

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PHYS;4761 Math Methods I: Midterm # 1 Review

I. Chapter 1: Math Preliminaries

A. Infinite Series and Convergence

1. Geometric Series $\sum_{n=0}^{\infty} r^n$, convergent for $|r|<1$

2. Harmonic Series $\sum_{n=0}^{\infty} \frac{1}{n}$, divergent

3. Tests for Convergence:

a. Comparison Test: $0 \le u_n \le a_n$ convergent
$0 \le b_n \le v_n$ divergent

b. Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \begin{cases} <1 & \text{conv} \\ >1 & \text{div} \\ =1 & \text{indeterminate} \end{cases}$

c. Integral Test $f(n)=a_n \Rightarrow \int_1^{\infty} f(x)\,dx$ converges or diverges.

4. Alternating Series Convergence

a. Leibniz Criterion $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, requires $\lim_{n\to\infty} a_n = 0 \Rightarrow$ conv.

5. Absolute Convergence:

a. If $\sum_{n=1}^{\infty}|u_n|$ converges.

B. For Series of Functions

a. Uniform convergence over interval $x \in [a,b]$

b. Basically, use usual convergence tests, which now may depend on $x \Rightarrow$ range of convergence

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I. (Continued)

B. Taylor Expansion

1. $f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \Rightarrow$ Power Series
2. Know how to expand functions to obtain power series.
3. Power series representation is unique
4. Know some important series $\Rightarrow$ I will provide them also!, $\left[\begin{matrix} e^x \\ \sin(x) \\ \cos x \\ \ln(1+x) \end{matrix}\right\}$ Know!

C. Binomial Thm

1. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$

D. Nothing on complex numbers specifically

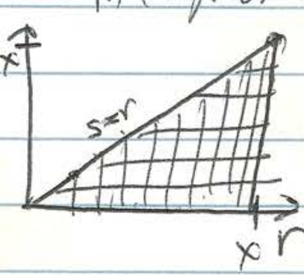
$\Rightarrow$ But you do need to know how to calculate with complex numbers, take complex conjugate, etc.

E. Derivatives and Integrals

1. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$
2. Integration by parts: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$
3. Special Functions: I will provide any you may need
4. Expand function & integrate, eg, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
5. Multiple Integral(s'a) Change order of integration

b) Be careful with limits

c) Jacobian $\Rightarrow$ Chapter 4.



$\int_0^x dr \int_0^r ds$

$\Rightarrow \int_0^x ds \int_s^x dr$

F. Delta Function:

1. $\int_{-\infty}^{\infty} f(x)\,\delta(x-a) = f(a)$
2. Properties: $\delta(ax) = \frac{1}{a}\delta(x)$

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II. Chap 2: Determinants and Matrices

A. Determinants 1. $a_1x_1 + a_2x_2 + \ldots = 0$
$b_1x_1 + b_2x_2 + \ldots = 0$ $\Rightarrow D = \begin{vmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \\ \vdots & \vdots & \end{vmatrix}$

2. If $D \neq 0$, only nontrivial solution $x_i = 0$ exists.

3. $D_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{ijk\ldots} \varepsilon_{ijk\ldots}\, a_{1i} a_{2j} \cdots$

a) $\varepsilon_{ijk} = \begin{cases} +1 & \text{even,} \\ -1 & \text{odd} \\ 0 & \text{repeated} \end{cases}$

4. Evaluating Determinants: a. $\begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$ (Diagonals $\Rightarrow$ R $\rightarrow +1$, L $\rightarrow -1$

b. Expansion by Minors

$\begin{vmatrix} a & a & a & a \\ b & b & b & b \\ c & c & c & c \\ d & d & d & d \end{vmatrix}$ i) Sign $(-1)^{i+j}$

5. Solving Equations: a. Cramer's Rule: $x_2 = \dfrac{1}{D}\begin{vmatrix} a_1 & h_1 & a_3 \\ b_1 & h_2 & b_3 \\ c_1 & h_3 & c_3 \end{vmatrix}$, etc.

6. Gauss Elimination:
a. Know the procedure (methodical)

B. Matrices:
1. Multiplication: $\underset{\approx}{A}\underset{\approx}{B} = \underset{\approx}{C}$ $\quad C_{ij} = \sum_k a_{ik} b_{kj}$

2. Commutator ~~AB~~ $[\underset{\approx}{A}, \underset{\approx}{B}] = \underset{\approx}{A}\underset{\approx}{B} - \underset{\approx}{B}\underset{\approx}{A}$

3. ~~Identity~~ Matrix Inversion: Know the Procedure $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \underset{\approx}{A} \Rightarrow \underset{\approx}{A}^{-1}$

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II. B. (Continued)

4. $\det(\underset{\approx}{A}\underset{\approx}{B}) = \det(\underset{\approx}{A})\det(\underset{\approx}{B})$

5. a. Transpose, $\underset{\approx}{A}^T$, $a_{ij} \to a_{ji}$

b. Adjoint: $\underset{\approx}{A}^\dagger$, $a_{ij} \to a_{ji}^*$

c. $\mathrm{Trace}(\underset{\approx}{A}) = \sum_{i=1}^{n} a_{ii}$

6. $(\underset{\approx}{A}\underset{\approx}{B})^{-1} = \underset{\approx}{B}^{-1}\underset{\approx}{A}^{-1}$ (same for T, †).

7. a. Orthogonal $\boxed{\underset{\approx}{S}^{-1} = \underset{\approx}{S}^T}$

b. Unitary $\underset{\approx}{U}^\dagger = \underset{\approx}{U}^{-1}$

c. Hermitian (Self-adjoint) $\underset{\approx}{H} = \underset{\approx}{H}^\dagger$

III. <u>Chap 3: Vector Analysis</u>

A. <u>Vector Properties</u>

1. $\underset{\sim}{A}\cdot\underset{\sim}{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$

2. a. $\underset{\sim}{A}\times\underset{\sim}{B} = AB\sin\theta\,\underset{\sim}{\hat{e}}_c$ (RH Rule) $(\underset{\sim}{\hat{e}}_x \times \underset{\sim}{\hat{e}}_y = \underset{\sim}{\hat{e}}_z)$

b. $C_i = \sum_{jk} \varepsilon_{ijk} A_j B_k$

c. $\underset{\sim}{A}\times\underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{\hat{e}}_x & \underset{\sim}{\hat{e}}_y & \underset{\sim}{\hat{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

3. $\underset{\sim}{A}\cdot(\underset{\sim}{B}\times\underset{\sim}{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$, $\underset{\sim}{A}\cdot(\underset{\sim}{B}\times\underset{\sim}{C}) = (\underset{\sim}{A}\times\underset{\sim}{B})\cdot\underset{\sim}{C} = (\underset{\sim}{B}\times\underset{\sim}{C})\cdot\underset{\sim}{A}$

4. $\underset{\sim}{A}\times(\underset{\sim}{B}\times\underset{\sim}{C}) = \underset{\sim}{B}(\underset{\sim}{A}\cdot\underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A}\cdot\underset{\sim}{B})$ "BAC − CAB"

B. 1. <u>2D Rotations</u>: 1. $\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}}_{\underset{\approx}{S} \to \text{orthogonal matrix } (\underset{\approx}{S}^{-1} = \underset{\approx}{S}^T)} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$

2. <u>3D Rotations will not be on the exam.</u>

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III (Continued)

C. <u>Vector Differential Operators</u>

1. Gradient: $\nabla = \hat{\underset{\sim}{e}}_x \frac{\partial}{\partial x} + \hat{\underset{\sim}{e}}_y \frac{\partial}{\partial y} + \hat{\underset{\sim}{e}}_z \frac{\partial}{\partial z}$

2. Divergence: $\nabla \cdot \underset{\sim}{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

3. Curl:
$$\nabla \times \underset{\sim}{V} = \begin{vmatrix} \hat{\underset{\sim}{e}}_x & \hat{\underset{\sim}{e}}_y & \hat{\underset{\sim}{e}}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

4. <u>Higher Order</u>:

a. Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

b. $\nabla \times \nabla \phi = 0$, $\quad \nabla \cdot (\nabla \times \underset{\sim}{V}) = 0$

$\nabla \phi$ irrotational $\quad \nabla \times \underset{\sim}{V}$ solenoidal

c. $\nabla \times (\nabla \times \underset{\sim}{V}) = \nabla(\nabla \cdot \underset{\sim}{V}) - \nabla^2 \underset{\sim}{V}$

5. <u>Other Identities</u>

a. $\nabla \cdot (f\underset{\sim}{V}) = \nabla f \cdot \underset{\sim}{V} + f \nabla \cdot \underset{\sim}{V}$

b. $\nabla \times (f\underset{\sim}{V}) = f(\nabla \times \underset{\sim}{V}) + (\nabla f) \times \underset{\sim}{V}$

c. $\nabla(\underset{\sim}{A} \cdot \underset{\sim}{B}) = (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{A} + (\underset{\sim}{A} \cdot \nabla)\underset{\sim}{B} + \underset{\sim}{B} \times (\nabla \times \underset{\sim}{A}) + \underset{\sim}{A} \times (\nabla \times \underset{\sim}{B})$

<u>NOTE</u>: $\underset{\sim}{B} \cdot \nabla = B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$

D. <u>Integrals and Integral Theorems</u>

1. <u>Line</u> $\int_C \phi \, d\underset{\sim}{r}$ or $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

2. <u>Surface</u>: $\int_S \underset{\sim}{V} \cdot d\underset{\sim}{\sigma}$

3. <u>Volume</u>: $\int_V \underset{\sim}{V} \, d\tau \quad d\tau = dx\,dy\,dz$

4. Gauss' Thm: $\boxed{\oint_{\partial V} \underset{\sim}{A} \cdot d\underset{\sim}{\sigma} = \int_V \nabla \cdot \underset{\sim}{A} \, d\tau}$

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III D. (Continued)

5. Stokes Theorem: $\oint_{\partial S} \underset{\sim}{B} \cdot d\underset{\sim}{r} = \int_S \nabla \times \underset{\sim}{B} \cdot d\underset{\sim}{\sigma}$

6. Use Arbitrary vector to compute alternative versions.

a. Ex: $\underset{\sim}{B}(x,y,z) = B(x,y,z)\ \underset{\sim}{a}$ ← constant, arbitrary vector

b. Gauss's Thm: $\int_{\partial V} \underset{\sim}{B} \cdot d\underset{\sim}{\sigma} = \int_V \nabla \cdot \underset{\sim}{B}\, d\tau$

$\Rightarrow \int_{\partial V} B\underset{\sim}{a} \cdot d\underset{\sim}{\sigma} = \int_V \nabla \cdot (B\underset{\sim}{a})\, d\tau$ → factor out $\underset{\sim}{a}$

$\Rightarrow \boxed{\oint_{\partial V} B\, d\underset{\sim}{\sigma} = \int \nabla B\, d\tau}$

E. Potential Theory

1. $\underset{\sim}{F} = -\nabla \phi$ Scalar potential
2. $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ Vector potential
3. Helmholtz Theorem: Any vector $\underset{\sim}{P} = -\nabla \phi + \nabla \times \underset{\sim}{A}$

F. Curvilinear Coordinates (ρ, ϕ, z)

1. Cylindrical: a. $x = \rho\cos\phi \quad y = \rho\sin\phi \quad z = z$

b. $\underset{\sim}{r} = \rho\, \hat{\underset{\sim}{e}}_\rho + z\, \hat{\underset{\sim}{e}}_z$

c. $\hat{\underset{\sim}{e}}_\rho = \hat{\underset{\sim}{e}}_x \cos\phi + \hat{\underset{\sim}{e}}_y \sin\phi$

$\hat{\underset{\sim}{e}}_\phi = -\hat{\underset{\sim}{e}}_x \sin\phi + \hat{\underset{\sim}{e}}_y \cos\phi$

$\hat{\underset{\sim}{e}}_z = \hat{\underset{\sim}{e}}_z$

d. $\underset{\sim}{V} = V_\rho\, \hat{\underset{\sim}{e}}_\rho + V_\phi\, \hat{\underset{\sim}{e}}_\phi + V_z\, \hat{\underset{\sim}{e}}_z$

e. Differential Operators: $\nabla\phi,\ \nabla \cdot \underset{\sim}{V},\ \nabla \times \underset{\sim}{V},\ \nabla^2 \phi$

f. Know how to use these!

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III. F. (Continued) (r, θ, ϕ)

2. Spherical a. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

b. $\underset{\sim}{r} = r\hat{\underset{\sim}{e}}_r$

c. $\underset{\sim}{V} = V_r \hat{\underset{\sim}{e}}_r + V_\phi \hat{\underset{\sim}{e}}_\phi + V_z \hat{\underset{\sim}{e}}_z$

d. $\hat{\underset{\sim}{e}}_r = \hat{\underset{\sim}{e}}_x \sin\theta\cos\phi + \hat{\underset{\sim}{e}}_y \sin\theta\sin\phi + \hat{\underset{\sim}{e}}_z \cos\theta$

$\hat{\underset{\sim}{e}}_\theta = \hat{\underset{\sim}{e}}_x \cos\theta\cos\phi + \hat{\underset{\sim}{e}}_y \cos\theta\sin\phi - \hat{\underset{\sim}{e}}_z \sin\theta$

$\hat{\underset{\sim}{e}}_\phi = -\hat{\underset{\sim}{e}}_x \sin\phi + \hat{\underset{\sim}{e}}_y \cos\phi$

e. Differential Operators: $\nabla\psi$, $\nabla\cdot\underset{\sim}{V}$, $\nabla\times\underset{\sim}{V}$, $\nabla^2\psi$

f. Know how to use them.

IV. Chapter 4: Tensors, Jacobians, Differential Forms

A. Tensors:

1. Scalars (0), Vectors (1), Tensors (2+)

2. Contravariant $\quad (A')^i = \frac{\partial (x')^i}{\partial x^j} A^j \quad$ (Summation)

covariant $\quad (A')_i = \frac{\partial x^j}{\partial (x')^i} A_j$

3. Quotient Rules

$$K_{kl} C^{kl}_{ij} = A_{ij}$$

a. If holds under coordinate transformations, $\underset{\sim}{K}$ is tensor of indicated rank and co/contra-variant nature.

4. General Coordinates:

a. Covariant Basis $\quad \underset{\sim}{\varepsilon}_i = \frac{\partial x}{\partial q_i}\hat{\underset{\sim}{e}}_x + \frac{\partial y}{\partial q_i}\hat{\underset{\sim}{e}}_y + \frac{\partial z}{\partial q_i}\hat{\underset{\sim}{e}}_z$

b. Covariant Metric Tensor $\quad g_{ij} = \underset{\sim}{\varepsilon}_i \cdot \underset{\sim}{\varepsilon}_j$

c. Contra: $\underset{\sim}{\varepsilon}^i = \frac{\partial q^i}{\partial x}\hat{\underset{\sim}{e}}_x + \frac{\partial q^i}{\partial y}\hat{\underset{\sim}{e}}_y + \ldots \quad g^{ij} = \underset{\sim}{\varepsilon}^i \cdot \underset{\sim}{\varepsilon}^j$

5. Will not cover Covariant Derivatives on exam.

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IV. (Continued)

B. Jacobians 1. $(x,y) \rightarrow (u,v) \Rightarrow dxdy = J\,du\,dv$

where $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \equiv \frac{\partial(x,y)}{\partial(u,v)}$

2. Can compute J^{-1} if easier.

C. Differential Forms & Exterior Algebra

1. 1-form $\omega_1 = A\,dx + B\,dy$, 2-form: $\omega_2 = C\,dx \wedge dy + D\,dy \wedge dz$

2. a. $dx_i \wedge dx_j = -dx_j \wedge dx_i$ (Permutational anti-symmetry)

b. $a(\omega_1 \wedge \omega_2) = (a\omega_1) \wedge \omega_2$

c. $dx_i \wedge dx_i = 0$

3. Hodge Star Operator: $*\omega$

a. Sign: (indices of ω) $\rightarrow$ (indices of ω')

b. $(-1)^{\mu}$ for μ-1 diagonal elements in metric tensor. $\rightarrow$ Not on exam

4. Exterior derivative: $d\omega$

a. $d(f\omega) = df \wedge \omega + f\,d\omega$

b. $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$

p-form p'-form

c. $d(d\omega) = 0$

5. Stoke's Theorem $\boxed{\int_R d\omega = \int_{\partial R} \omega}$ Not on exam