R2-D

PHYS:4761 Math Methods I

O. Midterm Exam #2

A. 1. Two Page (Two sides of $8\frac{1}{2}" \times 11"$ paper)
   a. Formula Summary Sheet
   b. Ask if you need a formula during exam

2. Calculator OK (you won't need it)

3. I will provide Scratch paper

4. Show your work / explain your reasoning
   a. Partial Credit
   b. No credit if I can't figure out how you got the answer.

B. <u>Covers</u>
   1. Chapters 5–9
   2. HW #6–10
   3. Lectures # 12–21

C. <u>To Study</u>:
   1. Go through HW Solutions: Test problems will be very Similar (ICON)
   2. 7 or 8 problems.
   3. Exam will be difficult

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PHYS 4261 Math Methods I: Midterm #2 Review

I. Chapter 5: Vector Spaces

A. Basic Concepts

1. Vector Space: a. Basis functions, $\psi(x) = \sum_{i=1}^{N} a_i |\phi_i\rangle$

b. Scalar Product: $\langle f(x)|g(x)\rangle = \int_a^b f^*(x) g(x) w(x) dx$

i) Interval: $a \le x \le b$

ii) Weight function: $w(x)$

c. For orthonormal basis, $\langle \phi_i | \phi_j \rangle = \delta_{ij}$

$f = \sum_{i=1}^{n} a_i |\phi_i\rangle$, $a_i = \langle \phi_i | f \rangle$

d. If not normalized, $a_i = \frac{\langle \phi_i | f \rangle}{\langle \phi_i | \phi_i \rangle}$

B. Gram-Schmidt Orthogonalization of Basis χ_i

1. Construct first function, $\phi_0 = \frac{\chi_0}{\langle \chi_0 | \chi_0 \rangle^{1/2}}$

2. a. Form $\psi_1 = \chi_1 - a_{0,1} \phi_0$ where $a_{0,1} = \langle \phi_0 | \chi_1 \rangle$

b. $\phi_1 = \frac{\psi_1}{\langle \psi_1 | \psi_1 \rangle^{1/2}}$

3. Repeat: a. $\psi_n = \chi_n - \sum_{i=1}^{n-1} \langle \phi_i | \chi_n \rangle \phi_i$

b. $\phi_n = \frac{\psi_n}{\langle \psi_n | \psi_n \rangle^{1/2}}$

4. Same procedure for vectors: $|\underset{\sim}{a}_i\rangle$ basis

a. $|\underset{\sim}{b}_n'\rangle = |\underset{\sim}{a}_n\rangle - \sum_{i=1}^{n} \underbrace{\langle \underset{\sim}{b}_i | \underset{\sim}{a}_n \rangle}_{\underset{\sim}{b}_i \cdot \underset{\sim}{a}_n} |\underset{\sim}{b}_i\rangle$ b. $|\underset{\sim}{b}_n\rangle = \frac{|\underset{\sim}{b}_n'\rangle}{\langle \underset{\sim}{b}_n' | \underset{\sim}{b}_n' \rangle^{1/2}}$

(R2-2) I. (Continued)

C. Linear Operators: ← can be differential operator

1. $\mathcal{L}(x)\, y(x) = \lambda\, y(x)$
2. Commutator $[A, B] = AB - BA$
3. Inverse $A^{-1}A = 1$
4. Adjoint $\langle f | A g\rangle = \langle A^{\dagger} f | g\rangle$ ← Use definition of scalar product to compute adjoint
5. Hermitian (self-adjoint) $A = A^{\dagger}$
6. Unitary $U^{\dagger} = U^{-1}$

7. Basis Expansion of Operators:

a. $A\phi_n = \sum_{m,n} a_{mn}\phi_m$ where $a_{mn} = \langle \phi_m | A | \phi_n \rangle$ $(= \langle A^{\dagger}\phi_m | \phi_n\rangle)$

b. $A = \sum_{mn} |\phi_n\rangle a_{nm} \langle \phi_m|$

c. Thus equivalent to matrix equation:

$$b_n = \sum_m a_{nm} c_m \iff \underset{\sim}{b} = \underset{\approx}{A}\,\underset{\sim}{c}$$

d. If Hermitian, $a_{nm} = a_{mn}^{*}$

8. Unitary Operators

a. Transformation between orthonormal bases is unitary

$$\underset{\sim}{c}' = \underset{\approx}{U}\,\underset{\sim}{c} \iff c_n' = \sum_m U_{nm} c_m$$

b. $\underset{\approx}{A}' = \underset{\approx}{U}\,\underset{\approx}{A}\,\underset{\approx}{U}^{-1} = \underset{\approx}{U}\,\underset{\approx}{A}\,\underset{\approx}{U}^{\dagger}$ since $\underset{\approx}{U}^{-1} = \underset{\approx}{U}^{\dagger}$

9. Invariants under unitary transformations

(R2-3)

II. Chapter 6: Eigenvalue Problems

A. Matrix Eigenvalue Problems

1. $\underset{\approx}{H}\,\underset{\sim}{r} = \lambda \underset{\sim}{r}$

Eigenvalues:

2. $\det\left(\underset{\approx}{H} - \lambda \underset{\approx}{1}\right) = 0 \Rightarrow$ Solve to find eigenvalues $= \begin{vmatrix} h_{11}-\lambda & h_{12} \\ h_{21} & h_{22}-\lambda \end{vmatrix}$

Secular / characteristic equation.

3. a. Substitute eigenvalue $\underset{\approx}{H}\,\underset{\sim}{r} = \lambda_i \underset{\sim}{r}$

and solve for $\underset{\sim}{r}$ to obtain eigenvector $\underset{\sim}{r}_i$

b. Normalize eigenvector to unity.

4. Degenerate Eigenvalue $\lambda_i = \lambda_j$ for $i \neq j$ (multiple root)
   a. Obtain a 2D manifold
   b. Choose one eigenvector & use Gram-Schmidt orthogonalization to obtain an orthogonal vector

B. Hermitian Eigenvalue Problems

1. H is linear Hermitian operator on a Hilbert Space
   a. λ_i are real
   b. r_i are orthogonal
   c. Eigenfunctions $\underset{\sim}{r}_i$ form a complete set.

2. Eigenvalues are unchanged under unitary transformation.

3. Diagonalization: $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \underset{\approx}{H}' = \underset{\approx}{U}\,\underset{\approx}{H}\,\underset{\approx}{U}^{-1}$

where column vectors of $\underset{\approx}{U}^{-1}$ are normalized eigenvectors of $\underset{\approx}{H}$.

(R2-4) II. B. Continued

4. Hermitian matrices have same eigenvectors if $[\underset{\approx}{A}, \underset{\approx}{B}] = 0$.

5. $\langle H \rangle = \langle \psi | H | \psi \rangle = \sum_n |a_n|^2 \lambda_n$ for $a_n = \langle \phi_n | \psi \rangle$

normalized

C. Normal Matrices

1. Def: $[\underset{\approx}{A}, \underset{\approx}{A}^\dagger] = 0$

2. Eigenvalues: $\underset{\approx}{A} |\underset{\sim}{x}\rangle = \lambda |\underset{\sim}{x}\rangle$, then $\underset{\approx}{A}^\dagger |\underset{\sim}{x}\rangle = \lambda^* |\underset{\sim}{x}\rangle$

3. Eigenvectors are orthogonal.

III. Chap 7: ODEs

A. Linear ODEs — linear differential operator

1. $\mathcal{L}\,\phi(x) = F(x)$

2. Superposition of Solutions

B. First-Order ODEs

1. General form: $y' = \frac{dy}{dx} = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$

2. Separation of Variables: If $\frac{dy}{dx} = -\frac{P(x)}{Q(y)}$

3. Exact Differentials: $\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$

4. Homogeneous in x & y: a. Same combined powers of x & y

b. $y = xv \Rightarrow$ then can be separated.

c. ~~Spec~~ General Case: Isobaric, $y = x^m v$

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III. (Continued)

5. General Form: $\frac{dy}{dx} + p(x)y = q(x)$

a. Integrating factor: $\alpha(x) = \exp\left[\int^x p(x)dx\right]$

b. Homogeneous Solution: $y_1 = \frac{C}{\alpha(x)}$

c. Particular Solution: $y_2 = \frac{1}{\alpha(x)} \int^x \alpha(x) q(x) dx$

C. Constant Coefficients ODEs

1. $y = e^{mx}$, solve for values of m.

2. For n multiple roots, $y_1 = e^{mx}, y_2 = xe^{mx}, \ldots y_n = x^{n-1}e^{mx}$

D. Second-Order ODEs:

1. $y'' + P(x)y' + Q(x)y = 0 \Rightarrow$ Two linearly independent solutions.

2. Singular Points: a) Regular $(x-x_0)P(x), (x-x_0)^2 Q(x)$ finite, otherwise irregular/essential

3. Frobenius' Method: a. $y(x) = \sum_{i=0} a_i x^{s+i}$

b. Indicial Equation: Coefficient of lowest order term in x.

c. Recurrence Relation: Coefficients of x^{s+i} terms.

d. Always yield one solution (Fuch's Thm)

4. Linear independence: Wronskian

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

a. If $W \neq 0$, linearly independent

b. If $W = 0$, linear dependence

(R2-6) III, D, (Continued)

5, <u>Second Independent Solution</u>: $y''+P(x)y'+Q(x)y=0$

a. First Solution $y_1(x)$

b. $$y_2(x) = y_1(x)\int^{x}\left[\frac{\exp\left(-\int^{x_2} P(x_1)\,dx_1\right)}{[y_1(x_2)]^2}\right]dx_2$$ Wronskian Double Integral

c. For $P(x)=0$, $$y_2(x) = y_1(x)\int^{x}\frac{dx_2}{[y_1(x_2)]^2}$$

E. <u>Inhomogeneous 2nd Order ODEs</u>

1. $y''+P(x)y'+Q(x)y = F(x)$ with homogeneous solutions y_1 & y_2.

2. a. $y_p = u_1 y_1 + u_2 y_2$

b. Solve $y_1 u_1' + y_2 u_2' = 0$
$y_1' u_1' + y_2' u_2' = F(x)$ for u_1' & u_2'

c. Integrate to obtain u_1 & u_2.
$\cancel{\text{d.Sol}}$ $y = c_1 y_1 + c_2 y_2 + y_p$

IV. <u>Chapter 8: Sturm-Liouville Theory</u>

A. <u>Basic Concepts</u> $\mathcal{L}(x)\psi(x)=\lambda\psi(x)$ - <u>ODE Eigenvalue Problems</u>

1. Applying BCs yields discrete λ_i Solutions.
2. Hermitian operators $\mathcal{L}(x) = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$
   a. Self-adjoint if $\boxed{p_0'(x) = p_1(x)}$
3. Make self-adjoint using $w(x)\mathcal{L}(x)\psi(x) = w(x)\lambda\psi(x)$
   where $w(x) = p_0^{-1}\exp\left[\int \frac{p_1(x)}{p_0(x)}dx\right]$

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V. PDEs

A. First-Order PDEs

1. Characteristics: $\mathcal{L}\phi = a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} = 0$

   a. $(x,y) \Rightarrow (s,t)$

   b. Characteristic: $t = bx - ay = \text{constant}$

   c. Orthogonal coordinate $s = ax + by$

$\boxed{\text{PDE} \Rightarrow \text{ODE}}$ d. $\mathcal{L}\phi = (a^2+b^2)\frac{\partial \hat{\phi}}{\partial s} = 0 \qquad \hat{\phi}(s,t)$

   e. Solution: $\phi(x,y) = f(t) = f(bx-ay)$

Skip 2. For General case, $\mathcal{L}\phi = a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} + q(x,y)\phi = F(x,y)$

   a. Use same characteristics to convert to ODE.

   b. Solve ODE using Separation of Variables

B. Second-Order PDEs

1. Characteristics: Hyperbolic PDE: $a^2\frac{\partial^2 \phi}{\partial x^2} - b^2\frac{\partial^2 \phi}{\partial y^2} = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)\phi = 0.$

   a. Two characteristics:

   $bx - ay \quad , \quad bx + ay$

   b. Thus $\phi(x,y) = f(bx-ay) + g(bx+ay)$

C. Separation of Variables

1. $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2\psi = 0$

2. a. $\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

   b. Substitute, convert $\frac{\partial}{\partial x} \to \frac{d}{dx}$, and divide by XYZ.

R2-8 V C. (Continued)

3. $\frac{1}{X}\frac{d^2X}{dx^2} = -\ell^2 = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2}$

constant of separation.

4. Boundary conditions determine allowable values of ℓ^2.

5. In cylindrical coordinates ⇒ Bessel's Equation.
 a. $\psi(\rho,\phi,z) = P(\rho)\Phi(\phi)Z(z)$

6. Spherical Coordinates: a. $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$
 b. Associated Legendre Equation for $x = \cos\theta$
 c. Transform $R(r) = \frac{Z(kr)}{(kr)^{\frac{1}{2}}}$ → Bessel's Eq of order $\ell + \frac{1}{2}$
 d. Solutions of $R(r)$: Spherical Bessel functions.

D. Specific Types of PDE's:
1. Elliptic: Laplace's & Poisson's Eqs.
 a. Unique solutions, no extrema in domain for Laplace's Eq.

2. Hyperbolic: Wave Equation
 a. Characteristics yield traveling waves
 b. Sum of traveling waves → standing wave
 c. d'Alembert's Solution:

$$\psi(x,t) = \frac{1}{2}\left[\psi(x+ct,0) + \psi(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \left.\frac{\partial\psi(x,t)}{\partial t}\right|_{t=0} dx$$

(R2-9) IV D. (Continued)

3. Parabolic : Diffusion Eq.

a. Form linear combinations of solutions to satisfy BC's.

$$\psi(x,t) = \sum_n \left[A_n X_1(x) + B_n X_2(x)\right] e^{-\alpha^2 t}$$

b. Substitute: $\xi = \frac{x}{\sqrt{t}}$ $\Rightarrow$ $\psi(x,t) = U(\xi)$

i. PDE $\Rightarrow$ ODE in ξ.

ii. Solve by separation of variables

c. Spherical symmetry: $\psi(r,t) = \frac{f(r,t)}{r}$

a. Convert to form similar to Cartesian diff. eq.