PHYS:4731 Homework #10

Due at the beginning of class, Thursday, December 8, 2022.

1. Resistive MHD Dispersion relation:
   Calculate the dispersion relation for Resistive MHD equations below:

$$\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{U}$$

$$\rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{U}$$

   Assume the equilibrium plasma conditions are constant in time and homogeneous with a mean equilibrium magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. Take the wavevector to be of the form $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$.

   (a) Find the linearized equations for Resistive MHD. State clearly any assumptions you have made and please box the final form of each equation.

   (b) Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, yielding algebraic equations for the Fourier coefficients of the variables $\hat{\rho}_1(\mathbf{k})$, $\hat{\mathbf{U}}_1(\mathbf{k})$, $\hat{\mathbf{B}}_1(\mathbf{k})$, and $\hat{p}_1(\mathbf{k})$. Be sure to box the final form of each equation.

   (c) Eliminate the Fourier coefficients for all of the variables except for $\hat{\mathbf{U}}_1$, writing the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \end{pmatrix} = 0$$

   in terms of ω, η, μ_0, $k_\parallel$, $c_s = \sqrt{\gamma p_0/\rho_0}$, and $v_A = B_0/\sqrt{\mu_0 \rho_0}$.

   (d) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$.

   (e) Solve for the frequency ω of the Alfvén wave mode.

2. Eigenfunctions for Resistive MHD with $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$

   (a) Use your results from the problem above to find the complete eigenfunction (determined by the relationships between the Fourier coefficients of the different field components) for a mode that has a Fourier coefficient for the perturbed fluid velocity $\hat{\mathbf{U}}_1(\mathbf{k}) = \hat{U}_0 \hat{\mathbf{z}}$ (thus with $\hat{U}_x = 0$ and $\hat{U}_y = 0$). Your answer should provide the solutions for $\hat{\rho}_1$, $\hat{p}_1$, and $\hat{\mathbf{B}}_1$ in terms of $\hat{U}_0$, B_0, γ, p_0, ρ_0, η, μ_0, $k_\parallel$, $c_s = \sqrt{\gamma p_0/\rho_0}$, and $v_A = B_0/\sqrt{\mu_0 \rho_0}$.

   (b) If $c_s < v_A$, does this mode correspond to the fast mode or the slow mode?

3. Shallow Water Equations

The Shallow Water Equations are a two-dimensional system of equations describing the dynamics of motions with wavelengths greater than the depth of the water H in a rotating frame of reference, valid for wavenumbers $kH \ll 1$. They are useful in describing ocean dynamics. The equations are

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{U}) = 0$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + f\hat{\mathbf{z}} \times \mathbf{U} = -g\nabla\eta$$

$$h = H + \eta$$

where the fluid velocity $\mathbf{U}(x, y, t) = U_x(x, y, t)\hat{\mathbf{x}} + U_y(x, y, t)\hat{\mathbf{y}}$ and the gradient $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}$ are two-dimensional. The term $f\hat{\mathbf{z}} \times \mathbf{U}$ represents the Coriolis force due to the rotation of the earth, with the frequency f representing the effective frequency of this rotation at a given latitude; we assume $f = f_0 =$ constant. The height of the mean sea level above the ocean bottom is given by $H(x, y)$ and does not depend on time. The fluctuation of the sea surface due to waves from mean sea level is $\eta(x, y, t)$, so that $h(x, y, t)$ represents to total depth of water at any given position and time. We may assume $H \gg |\eta|$ and $\nabla H = 0$.

(a) Find the linearized Shallow Water Equations. State clearly any assumptions you have made and please box the final form of each equation.

(b) Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Be sure to box the final form of each equation.

(c) Put the system of equations into matrix form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{U}_x \\ \hat{U}_y \end{pmatrix} = 0.$$

(d) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$, using $c = \sqrt{gH}$ as the surface gravity wave velocity.

(e) For the non-trivial root, determine simplified dispersion relation in the short- and long-wavelength limits. Be sure to define each limit. (The short-wavelength solution corresponds to surface gravity waves and the long-wavelength solution corresponds to inertial waves.)

(f) Which of the above limits (short or long) gives non-dispersive wave behavior?

4. Reflection from a Plasma

Light waves in vacuum (left) are incident on a slab of cold, unmagnetized plasma (right) at an angle θ as shown below. The fully-ionized plasma of protons and electrons has a uniform density. The index of refraction for a plasma is given by $n = ck/\omega$. Using Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, find the critical angle θ_c for total internal reflection (within the vacuum free space region) as a function of the light wave frequency ω and the plasma frequency ω_p.