PHYS:4731 Homework #9

Due at the beginning of class, Thursday, December 1, 2022.

1. The Conservation of Energy in Ideal MHD
   In this problem we are going to derive the equation for the evolution of the energy in Ideal MHD.

   (a) First, take the dot product of the momentum equation with $\mathbf{U}$ and derive the relation

   $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2\right) + \nabla \cdot \left(\frac{1}{2}\rho U^2 \mathbf{U}\right) = -(\mathbf{U}\cdot\nabla)p + \frac{1}{\mu_0}\mathbf{U}\cdot[(\nabla\times\mathbf{B})\times\mathbf{B}]$$

   Hint: Use the continuity equation.

   (b) Use the pressure equation (see HW#7 problem 1) to derive the relation

   $$(\mathbf{U}\cdot\nabla)p = \frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) + \nabla\cdot\left(\frac{\gamma p}{\gamma-1}\mathbf{U}\right)$$

   (c) Show that the second term on the right-hand side of the answer to part (a) may be written

   $$\frac{1}{\mu_0}\mathbf{U}\cdot[(\nabla\times\mathbf{B})\times\mathbf{B}] = -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) - \nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)$$

   Hint: Don't forget about Ohm's Law and Faraday's Law.

   (d) Put everything together to derive the final form of the equation for the evolution of the energy in Ideal MHD.

   $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\frac{1}{2}\rho U^2\mathbf{U} + \frac{\gamma p}{\gamma-1}\mathbf{U} + \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right) = 0$$

2. Distinguishing Fast waves from Slow waves
   The characteristic eigenfunctions can be used to distinguish fast wave fluctuations from slow wave fluctuations. Consider the specialized case (as discussed in class) with $k_\parallel = k_\perp = k_0$ (this is $\theta = 45°$) and $c_s = v_A$.

   (a) Calculate the ratio of the density fluctuation ρ_1 to the z-component of the magnetic field fluctuation B_z for the fast wave.

   (b) Do the same for the slow wave.

   (c) Satellite measurements of fluctuations in the solar wind plasma can measure the density and magnetic field fluctuations vs. time. How would one distinguish observationally between a fast wave fluctuation and a slow wave fluctuation?

3. Group velocity
   The (vector) group velocity is given by

   $$\mathbf{v}_g = \frac{\partial\omega}{\partial\mathbf{k}}$$

   where the shorthand notation $\partial/\partial\mathbf{k} = \hat{\mathbf{x}}\partial/\partial k_x + \hat{\mathbf{y}}\partial/\partial k_y + \hat{\mathbf{z}}\partial/\partial k_z$ is used. Consider below a plasma with $c_s = v_A$. You may assume $\mathbf{k} = k_\perp\hat{\mathbf{x}} + k_\parallel\hat{\mathbf{z}}$.

   (a) Calculate the (vector) group velocity for the Alfvén wave.

   (b) Calculate the (vector) group velocity for the Fast wave.
   Hint: You'll want to write the frequency in terms of the components $k_\parallel = k_z$ and $k_\perp = k_x$ rather than the angle θ.

4. Theta Pinch
(Note that for cylindrical coordinates defined as (r, ϕ, z), rather than (r, θ, z), this should be called the "Phi" Pinch.)

In class we have investigated the properties of cylindrical MHD equilibria with magnetic fields of the form $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} + B_z(r)\hat{\mathbf{z}}$. A "Theta" Pinch may be used to confine a hot plasma within a cylindrical column using only a radially varying axial magnetic field, $\mathbf{B} = B_z(r)\hat{\mathbf{z}}$.

(a) For a Theta pinch with

$$B_z(r) = B_0 \left(1 - e^{-r/a}\right),$$

find the thermal pressure profile $p(r)$ given by the equilibrium force balance.

(b) Find the current $\mathbf{j}$ required to maintain this magnetic field.