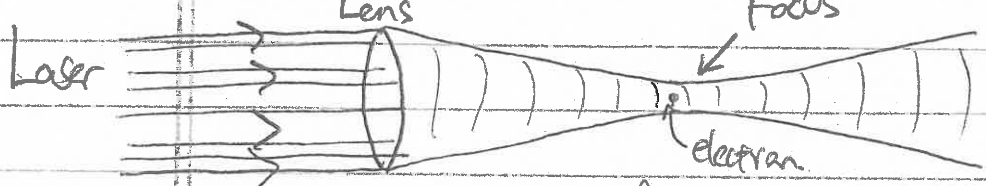
Lecture #10 The Ponderomotive Force

I. Particle Motion in High Frequency Electromagnetic Wave

A. Laser Plasma Interaction:



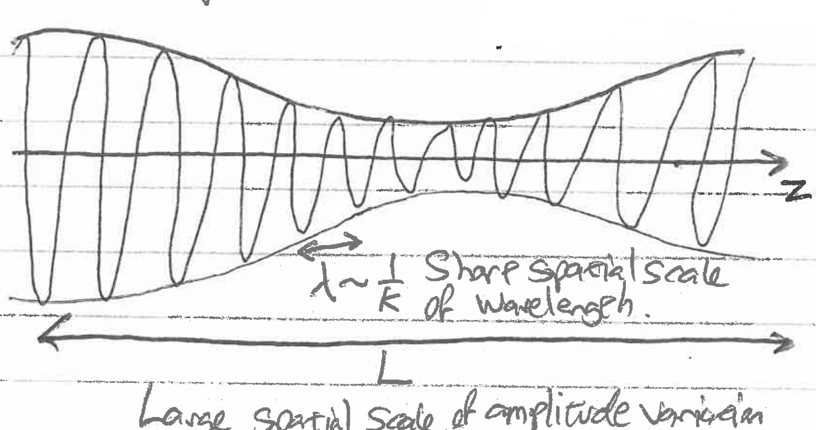
1. What is the motion of an electron in a high-frequency electromagnetic wave with variation in wave amplitude over space, ie., near the focal point of a laser?

2. In this case, the plasma is unmagnetized. Only $\underline{E}$ and $\underline{B}$ from wave are present.
3. We'll use multiple timescale analysis to determine the lowest order nonlinear effect.

B. Multiple Timescale Analysis:
1. Consider an electromagnetic wave of high frequency ω whose amplitude may vary on a long timescale and large spatial scale.

$$\underline{E}(\underline{x},t) = \underline{E}_0(x,\tau)\cos(\omega t - \underline{k}\cdot\underline{x})$$

a. Two spatial scales:



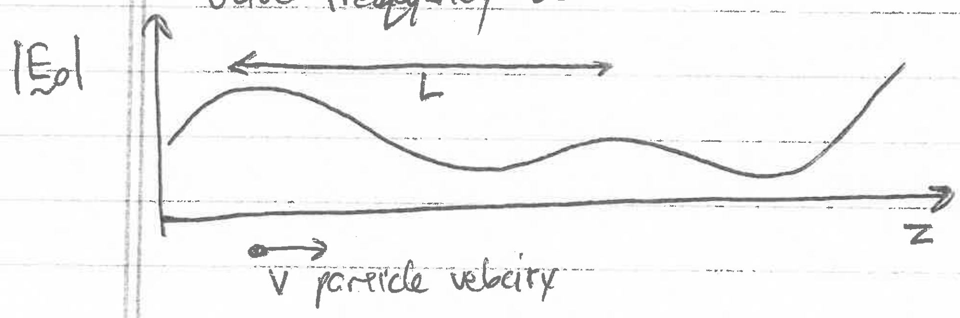
Lecture #10 (Continued) Howes ②

I. B. (Continued)

2. We'll show in HW#5 that $\underset{\sim}{E}(\underset{\sim}{x},\tau,t)$ yields

$$\underset{\sim}{B}(\underset{\sim}{x},\tau,t) = -\frac{1}{\omega}\left\{ \nabla\times \underset{\sim}{E}_0(\underset{\sim}{x},\tau)\sin(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x}) - \underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{x},\tau)\cos(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})\right\}$$

3. We want a slow variation of EM wave magnitude due to motion of the particle through space compared to the wave frequency ω!



a. To particle, amplitude varies in time due to motion $\underset{\sim}{v}\cdot\nabla \underset{\sim}{E}_0$

$\nabla \sim \frac{1}{L}$ large spatial scale $\quad |\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0| \sim \frac{v}{L}E_0$

b. Frequency of EM wave gives $|\omega \underset{\sim}{E}_0| \sim \omega E_0$

c. We want $\underbrace{|\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0|}_{\text{Slow timescale}} \ll \underbrace{|\omega\underset{\sim}{E}_0|}_{\text{Fast timescale}} \Rightarrow \frac{v}{L} \ll \omega$ or $\boxed{\frac{v}{L\omega} \ll 1}$

d. This will be our ordering parameter

$$\boxed{\epsilon \sim \frac{v}{L\omega} \ll 1}$$

$\boxed{\text{This separates fast oscillation timescale due to EM wave from slow drift timescale due to amplitude variation}}$

4. $\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})$

Compare these terms:

$$\frac{|\underset{\sim}{v}\times\underset{\sim}{B}|}{|\underset{\sim}{E}|} \sim \frac{|v||B|\cancel{\sin\theta}^{1}}{|E|} \sim \frac{v\left(\frac{\cancel{E_0}}{L\omega}\right)}{\cancel{E_0}} \sim \frac{v}{L\omega} \sim \epsilon \ll 1$$

$$\frac{\partial B}{\partial t} = -\nabla\times\underset{\sim}{E} \Rightarrow \omega B \sim \frac{E_0}{L} \text{ or } B \sim \frac{E_0}{L\omega}$$

I.B. (Continued)

5. Two Timescales: a. t Fast oscillation timescale

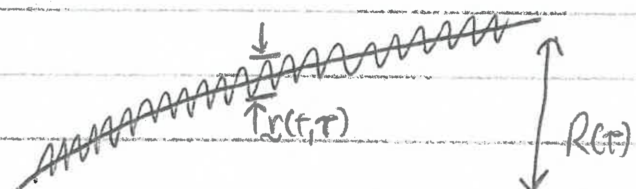
b. $\tau = \epsilon t$ Slow timescale of amplitude variation

c. Thus
$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial t}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}}$$

d. Over each fast oscillation period, $\underset{\sim}{E}$ varies little in amplitude. But small changes each oscillation can sum to produce a long timescale change.

6. Write particle position as slowly varying oscillation center $\underset{\sim}{R}(\tau)$ plus small, rapidly oscillating position $\underset{\sim}{r}(t,\tau)$

$$\boxed{\underset{\sim}{x} = \underset{\sim}{R}(\tau) + \epsilon \underset{\sim}{r}(t,\tau)}$$



7. Velocity:

$$\underset{\sim}{v} = \frac{d\underset{\sim}{x}}{dt} = \frac{d\underset{\sim}{R}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{r}(t,\tau)}{dt}$$

Define $\underset{\sim}{\mathcal{U}} \equiv \frac{d\underset{\sim}{R}(\tau)}{dt} = \epsilon \frac{\partial \underset{\sim}{R}(\tau)}{\partial \tau}$

$\underset{\sim}{u} \equiv \frac{d\underset{\sim}{r}(t,\tau)}{dt}$,

$$\underset{\sim}{v} = \epsilon \underset{\sim}{\mathcal{U}}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)$$

8. Acceleration:

$$\frac{d\underset{\sim}{v}}{dt} = \epsilon \frac{d\underset{\sim}{\mathcal{U}}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{u}(t,\tau)}{dt} = \cancel{\epsilon \frac{\partial \underset{\sim}{\mathcal{U}}(\tau)}{\partial t}}^{\,0} + \epsilon^2 \frac{\partial \underset{\sim}{\mathcal{U}}(\tau)}{\partial \tau} + \epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau}$$

9. Thus, we find:

$$\epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{\mathcal{U}}(\tau)}{\partial \tau} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau} = \epsilon \frac{q}{m}\left[\underset{\sim}{E}(\underset{\sim}{R}+\epsilon \underset{\sim}{r},\tau,t) + \left(\epsilon \underset{\sim}{\mathcal{U}}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)\right) \times \underset{\sim}{B}(\underset{\sim}{R}+\epsilon \underset{\sim}{r},\tau,t)\right]$$

a. NOTE that highest order nonzero term of LHS is $\mathcal{O}(\epsilon)$. Thus, highest term on RHS must be $\mathcal{O}(\epsilon)$ to balance. Hence, we multiply RHS by ϵ to give balance

I.B. (Continued)

10. Taylor Expand Fields about oscillation center $\mathbf{R}$:

a. $$\mathbf{E}(\mathbf{x},\tau,t) = \mathbf{E}(\mathbf{R},\tau,t) + (\mathbf{x}-\mathbf{R})\cdot\nabla\mathbf{E}(\mathbf{R},\tau,t) + \frac{[(\mathbf{x}-\mathbf{R})\cdot\nabla]^2}{2!}\mathbf{E}(\mathbf{R},\tau,t) + \dots$$

b. NOTE: $\mathbf{x}-\mathbf{R} = \mathbf{r}$, so

$$\mathbf{E}(\mathbf{x},\tau,t) = \mathbf{E}(\mathbf{R},\tau,t) + \epsilon\, \mathbf{r}\cdot\nabla\mathbf{E}(\mathbf{R},\tau,t) + \dots$$

and likewise with $\mathbf{B}(\mathbf{x},\tau,t)$

11. Expand all variables and substitute:

a. $$\boldsymbol{\mathcal{U}}(\tau) = \boldsymbol{\mathcal{U}}_1(\tau) + \epsilon\,\boldsymbol{\mathcal{U}}_2(\tau) + \dots$$

$$\mathbf{u}(\tau,t) = \mathbf{u}_1(\tau,t) + \epsilon\,\mathbf{u}_2(\tau,t) + \dots$$

$$\mathbf{r}(\tau,t) = \mathbf{r}_1(\tau,t) + \epsilon\,\mathbf{r}_2(\tau,t) + \dots$$

12. Thus, we get

$$\epsilon\frac{\partial \mathbf{u}_1(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \mathbf{u}_2(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \boldsymbol{\mathcal{U}}_1(\tau)}{\partial \tau} + \epsilon^3\frac{\partial \boldsymbol{\mathcal{U}}_2(\tau)}{\partial \tau} + \epsilon^2\frac{\partial \mathbf{u}_1(t,\tau)}{\partial \tau} + \epsilon^3\frac{\partial \mathbf{u}_2(t,\tau)}{\partial \tau}$$

$$= \frac{q}{m}\Big[\epsilon\,\mathbf{E}(\mathbf{R},\tau,t) + \epsilon^2\,\mathbf{r}_1\cdot\nabla\mathbf{E}(\mathbf{R},\tau,t) + \epsilon^2\,\boldsymbol{\mathcal{U}}_1\times\mathbf{B}(\mathbf{R},\tau,t) + \epsilon^3\,\boldsymbol{\mathcal{U}}_2\times\mathbf{B}(\mathbf{R},\tau,t)$$

$$+ \epsilon^3\,\boldsymbol{\mathcal{U}}_1\times(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{R},\tau,t) + \epsilon^4\,\boldsymbol{\mathcal{U}}_2\times(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{R},\tau,t) + \epsilon^2\,\mathbf{u}_1\times\mathbf{B}(\mathbf{R},\tau,t)$$

$$+ \epsilon^3\,\mathbf{u}_2\times\mathbf{B}(\mathbf{R},\tau,t) + \epsilon^3\,\mathbf{u}_1\times(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{R},\tau,t) + \epsilon^4\,\mathbf{u}_2\times(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{R},\tau,t) + \dots\Big]$$

13. Lowest Order: $O(\epsilon)$

$$\frac{\partial \mathbf{u}_1(t,\tau)}{\partial t} = \frac{q}{m}\mathbf{E}_0(\mathbf{R},\tau)\cos(\omega t - \mathbf{k}\cdot\mathbf{R})$$

$$\frac{\partial \mathbf{r}_1}{\partial t} = \mathbf{u}_1(t,\tau) = \frac{q}{m\omega}\mathbf{E}_0(\mathbf{R},\tau)\sin(\omega t - \mathbf{k}\cdot\mathbf{R}) \longleftarrow \text{Oscillation velocity}$$

$$\mathbf{r}_1(t,\tau) = \frac{-q}{m\omega^2}\mathbf{E}_0(\mathbf{R},\tau)\cos(\omega t - \mathbf{k}\cdot\mathbf{R}) \longleftarrow \text{oscillation position}$$

Lecture #10 (Continued)

I. B. (Continued)

14. ~~Next~~ Order: $O(\epsilon^2)$

a. $$\underbrace{\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}}_{1} + \underbrace{\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}}_{2} + \underbrace{\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial \tau}}_{3} = \frac{q}{m}\Big[\underbrace{\underset{\sim}{r}_1\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t)}_{4} + \underbrace{\underset{\sim}{U}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{5} + \underbrace{\underset{\sim}{u}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{6}\Big]$$

b. We now average over oscillation period to anihilate terms $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt$.

i) Assume $\underset{\sim}{u}_2(t,\tau)$ is periodic over $T=\frac{2\pi}{\omega}$

This should be checked a posteriori.

c. Term 1: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}dt = 0$ by assumed periodicity

d. Term 2: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}dt = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}\cancelto{1}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt} = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}$

e. Term 3: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial}{\partial\tau}\Big[\frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]dt$

$$= \frac{q}{m\omega}\frac{\partial \underset{\sim}{E}_0(\underset{\sim}{R},\tau)}{\partial\tau}\,\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt = 0$$

f. Term 4: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\frac{q}{m}\Big[\frac{-q}{m\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt$

$$= \frac{-q^2}{m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\,\frac{\omega}{2\pi}\underbrace{\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt}_{=\frac{\pi}{\omega}} = \frac{-q^2}{2m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)$$

g. Term 5: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\frac{q}{m}\underset{\sim}{U}_1(\tau)\times\Big[\frac{-1}{\omega}\Big\{\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R}) - \underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big\}\Big]dt$

$$= \frac{-q}{m\omega}\Big[\underset{\sim}{U}_1(\tau)\times\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt} - \underset{\sim}{U}(\tau)\times\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt}\Big]$$

$= 0$

I. B.14 (Continued)

h. Term (6): $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q}{m}\left[\frac{q}{m\omega}\mathbf{E}_0(\mathbf{R},\tau)\sin(\omega t-\mathbf{k}\cdot\mathbf{R})\right]\times\left\{\frac{-1}{\omega}\left[\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\sin(\omega t-\mathbf{k}\cdot\mathbf{R})\right.\right.$

$$\left.\left. -\mathbf{k}\times\mathbf{E}_0(\mathbf{R},\tau)\cos(\omega t-\mathbf{k}\cdot\mathbf{R})\right]\right\}dt$$

$$=\frac{-q^2}{m^2\omega^2}\left\{\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\,\frac{\omega}{2\pi}\underbrace{\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t-\mathbf{k}\cdot\mathbf{R})\,dt}_{\frac{\pi}{\omega}}\right.$$

$$\left. -\mathbf{E}_0(\mathbf{R},\tau)\times\mathbf{k}\times\mathbf{E}_0(\mathbf{R},\tau)\,\frac{\omega}{2\pi}\underbrace{\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\mathbf{k}\cdot\mathbf{R})\cos(\omega t-\mathbf{k}\cdot\mathbf{R})\,dt}_{0}\right\}$$

$$=-\frac{q^2}{2m^2\omega^2}\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)$$

i. Putting solution together, we find

$$\frac{\partial \mathbf{u}_1(\tau)}{\partial\tau}=-\frac{q^2}{2m^2\omega^2}\left[\mathbf{E}_0(\mathbf{R},\tau)\cdot\nabla\mathbf{E}_0(\mathbf{R},\tau)+\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\right]$$

j. NOTE: NRL p. 4 (12) gives

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$$

Taking $\mathbf{A}=\mathbf{B}=\mathbf{E}$, we get $\nabla\left(\frac{|\mathbf{E}|^2}{2}\right)=\mathbf{E}\times(\nabla\times\mathbf{E})+(\mathbf{E}\cdot\nabla)\mathbf{E}$

k. Thus, we find:

$$\boxed{\frac{\partial \mathbf{u}_1(\tau)}{\partial\tau}=\frac{-q^2}{2m^2\omega^2}\nabla\left(\frac{|\mathbf{E}_0(\mathbf{R},\tau)|^2}{2}\right)}$$

Ponderomotive Force

I. (Continued)

C. Properties of the Ponderomotive Force

1. $$\boxed{\underset{\sim}{F}_{pond} = m\frac{d\underset{\sim}{u}}{dt} = \frac{-q^2}{4m\omega^2}\nabla|\underset{\sim}{E}_0|^2}$$ Pushes away from regions of intense field.

a. We can write this as a potential force

$$\underset{\sim}{F}_{pond} = -\nabla\Phi_{pond}$$

where $$\boxed{\Phi_{pond} = \frac{q^2}{4m\omega^2}|\underset{\sim}{E}_0|^2}$$

b. Note that the average fast oscillation energy is

$$\frac{1}{2}m\overline{|\underset{\sim}{v}|^2} = \frac{m}{2}\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\frac{q^2}{m^2\omega^2}|\underset{\sim}{E}_0|^2\underbrace{\sin^2(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R})\,dt}_{=\frac{\pi}{\omega}} = \frac{m}{2}\frac{\omega}{2\pi}\frac{q^2}{m^2\omega^2}|\underset{\sim}{E}_0|^2\frac{\pi}{\omega}$$

$$= \frac{q^2}{4m\omega^2}|\underset{\sim}{E}_0|^2$$

c. Thus

$$\Phi_{pond} = \frac{1}{2}m\overline{|\underset{\sim}{v}|^2}$$

Ponderomotive potential is the average fast oscillation kinetic energy.

d. For center of oscillation $$\boxed{\mathcal{E}_{osc} = \frac{1}{2}mu^2 + \Phi_{pond} = \frac{1}{2}mu^2 + \frac{1}{2}m\overline{|\underset{\sim}{v}|^2}}$$

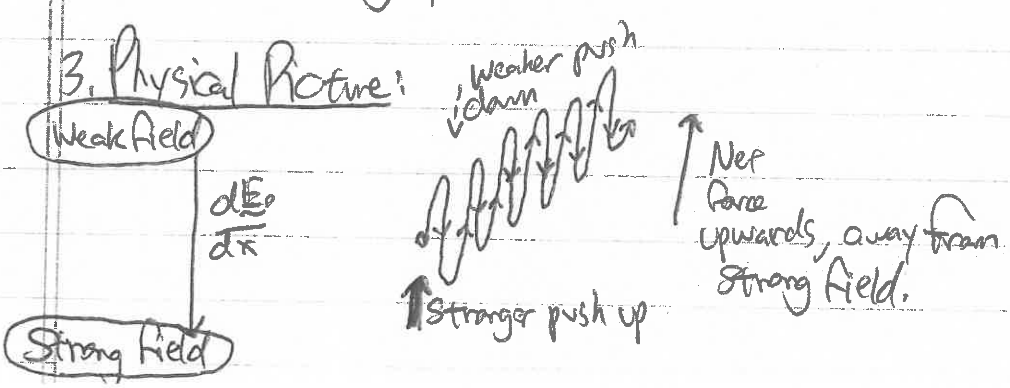
2. a. Force is independent of charge

⇒ Repels both ions and electrons from high field regions.

b. Because $m_e \ll m_i$, electrons are pushed aside much more easily.

⇒ Resulting polarization electric field acts to pull ions out.

3. Physical Picture:



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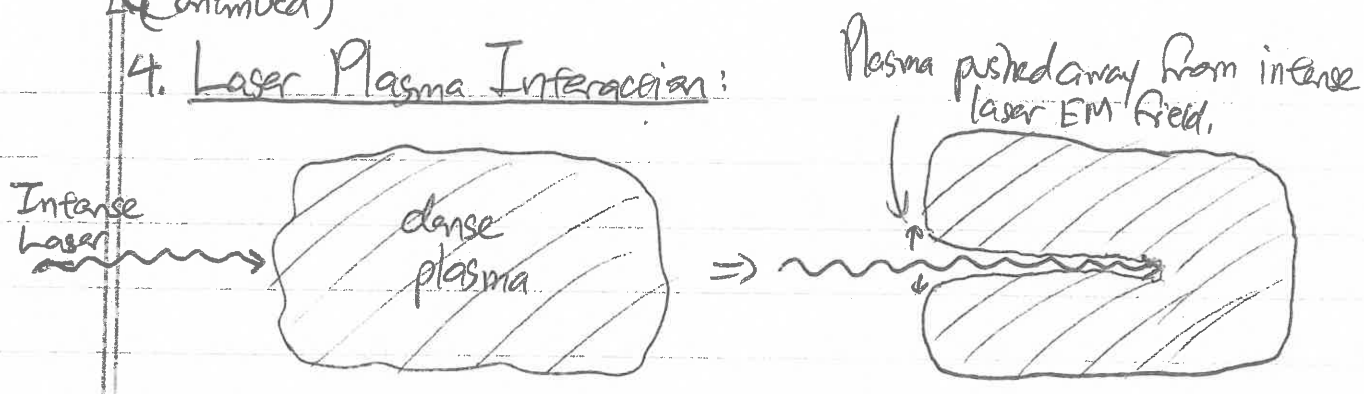
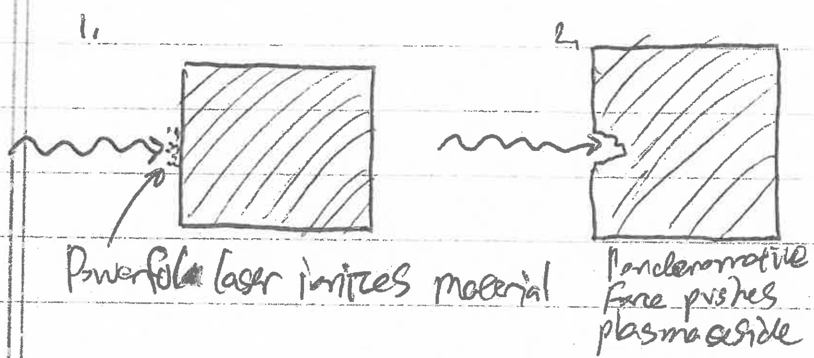
4. Laser Plasma Interaction:



a. This can lead to self-focusing of laser light in a plasma.

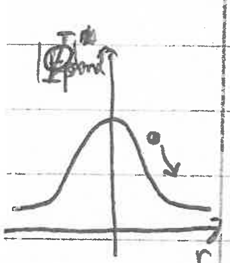
1. Powerful laser pushes aside electrons (and ions) due to ponderomotive force.
2. The resulting depression in plasma density acts as a convex lens, focusing the laser light into the evacuated channel.

b. Lasers can bore holes in materials by this mechanism.



3. Laser can propagate if light frequency $\omega > \omega_{pe}$ in plasma.

5. Example: A particle of charge q & mass m is initially at rest at the center of a Gaussian laser beam with $|\underset{\sim}{E}_0(\underset{\sim}{x})| = E_0\, e^{-\frac{r^2}{r_0^2}}$. Find linear of oscillation velocity as a function of position.



$$\mathcal{E}_0 = \frac{1}{2} m u^2 + \Phi_{pond} \quad \text{where } \Phi_{pond} = \frac{q^2}{4m\omega^2}|\underset{\sim}{E}_0|^2 = \frac{q^2}{4m\omega^2} E_0^2\, e^{-\frac{2r^2}{r_0^2}}$$

At $r=0$, $\mathcal{E}_0 = \frac{1}{2} m \cancel{u^2}^{\,0} + \frac{q^2}{4m\omega^2} E_0^2$.

Thus
$$u = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\Phi_{pond}}{m}} = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\mathcal{E}_0}{m} e^{-\frac{2r^2}{r_0^2}}} = \sqrt{\frac{2\mathcal{E}_0}{m}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}$$

$$\boxed{u(r) = \frac{qE_0}{m\omega\sqrt{2}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}}$$