Lecture #11 Collisions and Resistivity

I. Single Particle Motion and Collisions

A. 1. So far, we have considered the motion of a single charged particle in ~~a~~ prescribed (non-self-consistent) $\underline{E}$ & $\underline{B}$ fields

2. Another effect that can affect the motion of a particle is the collision with another particle.

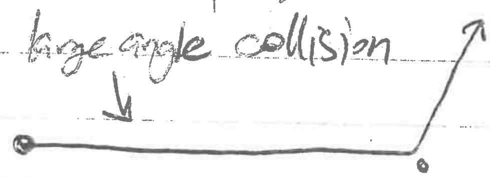
   a. This is not a collective effect, such as the collective motion of ions & electrons producing current and charge densities and leading to $\underline{E}$ & $\underline{B}$ fields

   b. Although a single charged particle may collide with many other particles (as we shall see) these interactions are independent, and do not act cooperatively, so collisions belongs with single particle motion discussion

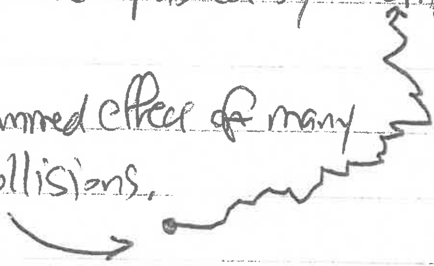
II. Single Large Angle vs. Many Small Angle Collisions.

A. Def: Collision time $\tau_c \equiv$ Time required for particle trajectory to be deflected by $\pi/2$.

1. This may be accomplished by a single large angle collision



2. Or by the summed effect of many small angle collisions.



3. We will see, in fact, the small angle collisions dominate.

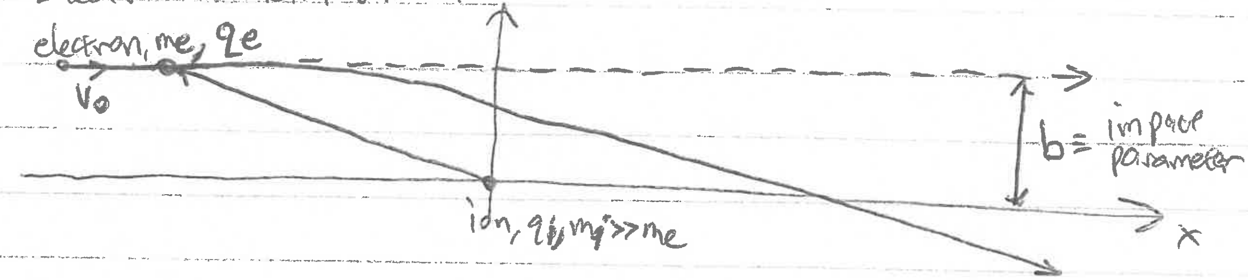
   ⇒ Coulomb force is long range, so particle can interact with many particles at once

   ⇒ But Debye shielding limits long-range interactions, leaving possible interactions with N_D particles within Debye sphere.

II. (Continued)

B. Large-Angle Collision Frequency $\nu_L = \frac{1}{\tau_L}$

1. Electron collision with ion

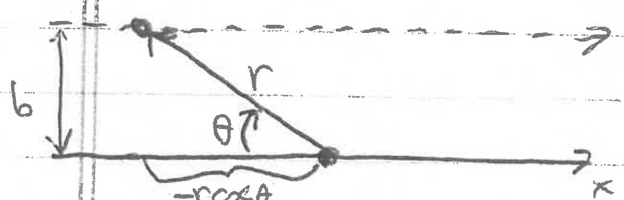


2. Consider the perpendicular velocity $v_\perp$ caused by a small-angle collision with massive ion $m_i \gg m_e$ (effectively, take $m_i \to \infty$).

3. Perpendicular Impulse $\quad m_e v_\perp = \int_{-\infty}^{\infty} dt \, F_\perp$

a. For a small-angle collisions, final parallel velocity $v_\parallel \simeq v_0$, so we can take unperturbed orbit to calculate impulse, $x = v_0 t$.

b. Define θ as angle of radial vector:



$b = r\sin\theta$

c. We know $m_e \frac{d^2 x}{dt^2} = \frac{q_e q_i}{4\pi\epsilon_0 r^2}\hat{r} \Rightarrow F_\perp = \frac{q_e q_i}{4\pi\epsilon_0 r^2}\sin\theta = \frac{q_e q_i}{4\pi\epsilon_0 b^2}\sin^3\theta$

d. From unperturbed orbit $x = v_0 t = r\cos\theta = -b\frac{\cos\theta}{\sin\theta}$

$$dt = \frac{-b}{v_0}\left(\frac{-\sin\theta \, d\theta}{\sin\theta} - \frac{\cos^2\theta \, d\theta}{\sin^2\theta}\right) = \frac{b}{v_0}\frac{d\theta}{\sin^2\theta}$$

e. Thus $m_e v_\perp = \int_0^\pi \frac{q_e q_i}{4\pi\epsilon_0 b^2}\sin^3\theta \frac{b \, d\theta}{v_0 \sin^2\theta} = \frac{2 q_e q_i}{4\pi\epsilon_0 b v_0} \Rightarrow v_\perp = \frac{q_e q_i}{2\pi\epsilon_0 m_e v_0 b}$

f. Define b_0 as value of b when $v_\perp = v_0$ $\Rightarrow \boxed{b_0 \equiv \frac{q_e q_i}{2\pi\epsilon_0 m_e v_0^2}} \Rightarrow \boxed{\frac{v_\perp}{v_0} = \frac{b_0}{b}}$

$\Rightarrow$ Large Angle Collision

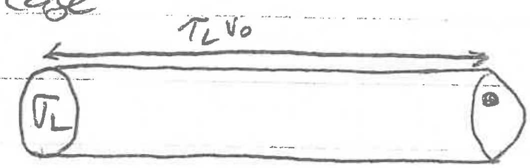
II. B. (Continued)

4. Any impact parameter $b \leq b_0$ will yield a large-angle collision.

   a. <u>Define:</u> Cross-Section for Large-Angle Collision: $\sigma_L \equiv \pi b_0^2$

5. One large-angle collision will occur in a plasma of density n_0 for the following case

   a. $\tau_L V_0 n_0 \sigma_L = 1$



   b. $\tau_L v_0 n_0 \pi \frac{q_e^2 q_i^2}{4\pi^2 \epsilon_0^2 m^2 v_0^4} = \tau_L \frac{n_0 q_e^2 q_i^2}{4\pi\epsilon_0^2 m^2 v_0^3} = 1$

6. Collision Frequency: Take $q_i = -q_e = e$

<u>Define:</u> $\boxed{\nu_L \equiv \frac{1}{\tau_L} = \frac{n_0 e^4}{4\pi\epsilon_0^2 m_e^2 v_0^3}}$

C. Small-Angle Collision Frequency:

1. For a number of small angle collisions, each collision will be independent, leading to a random walk in velocity.

   a. RANDOM WALK: For N independent steps of size Δx_j, the total distance moved Δx is

$$(\Delta x)^2 = N(\Delta x_j)^2$$

   b. We want to find the rate of change of $\Delta v_\perp$, so

$$\frac{d}{dt}(\Delta v_\perp^{tot})^2 = \frac{dN}{dt}(\Delta v_\perp)^2$$

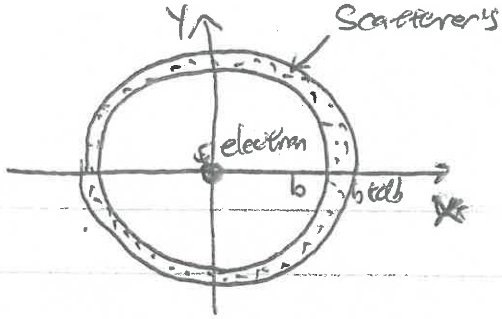
where $(\Delta v_\perp)^2 = \frac{b_0^2 v_0^2}{b^2}$ using $\frac{v_\perp}{v_0} = \frac{b_0}{b}$

Lecture #11 (Continued)

Hawes ④

II C. (Continued)

2.



$$\frac{dN}{dt} = 2\pi b\,db\,n_0 V_0$$

3. Thus

$$\frac{d}{dt}\left(\Delta V_\perp^{tot}\right)^2 = 2\pi b\,db\,n_0 V_0 \left(\frac{b_0^2 V_0^2}{b^2}\right) = 2\pi n_0 V_0^3 b_0^2 \;\frac{db}{b}$$

4. We want to integrate to get the total summed effect of many small angle collisions from b_{min} to b_{max}.

$$\frac{d}{dt}\left(\Delta V_\perp^{tot}\right)^2 = 2\pi n_0 V_0^3 b_0^2 \int_{b_{min}}^{b_{max}} \frac{db}{b}$$

a. Debye shielding suggests we should cutoff our distant interactions at $b_{max} = \lambda_D$

b. We'll take $b_{min} = b_0$ as the limit of large-angle scattering.

c. Thus

$$\frac{d}{dt}\left(\Delta V_\perp^{tot}\right)^2 = 2\pi n_0 V_0^3 b_0^2 \ln\left(\frac{\lambda_D}{b_0}\right)$$

d. Taking $q_i = +e$, $q_e = -e$, we find

$$\frac{\lambda_D}{b_0} = \frac{\lambda_D\, 2\pi \epsilon_0 m_e V_0^2}{e^2} \underset{\uparrow}{=} \lambda_D\, 4\pi \overbrace{\left(\frac{\epsilon_0 k T_e}{n_0 e^2}\right)}^{\lambda_D^2} n_0 = 3\left(\frac{4\pi n_0}{3}\lambda_D^3\right) = 3N_D$$

$$V_0^2 = V_{te}^2 = \frac{2kT_e}{m_e}$$

5. Take $\Delta V_\perp^{tot} = V_0$ to yield $\frac{\pi}{2}$ deflection, and $\frac{d}{dt} \sim \nu_c$ collision frequency.

a. $$\nu_c V_0^2 = 2\pi n_0 V_0^3 \left(\frac{e^4}{4\pi^2 \epsilon_0^2 m_e^2 V_0^4}\right) \ln 3N_D$$

b. Noting $\ln 3N_D = \ln 3 + \ln N_D \simeq \ln N_D$ since $N_D \gg 1$.

c. $$\boxed{\nu_c = \frac{n_0 e^4}{2\pi \epsilon_0^2 m_e^2 V_0^3} \ln N_D}$$ Collision rate due to summed small-angle collisions.

II. (Continued)

D. Summary:

1. $\nu_c = 2 \ln N_D \, \nu_L$

a. For $N_D = 10^6$, $\ln N_D \simeq 14$, so $\nu_c \gg \nu_L$.

$\boxed{\text{Small-angle collisions dominate over large-angle collisions}}$

2. Effectively, any particle is suffering N_D collisions simultaneously with all particles in the Debye sphere.

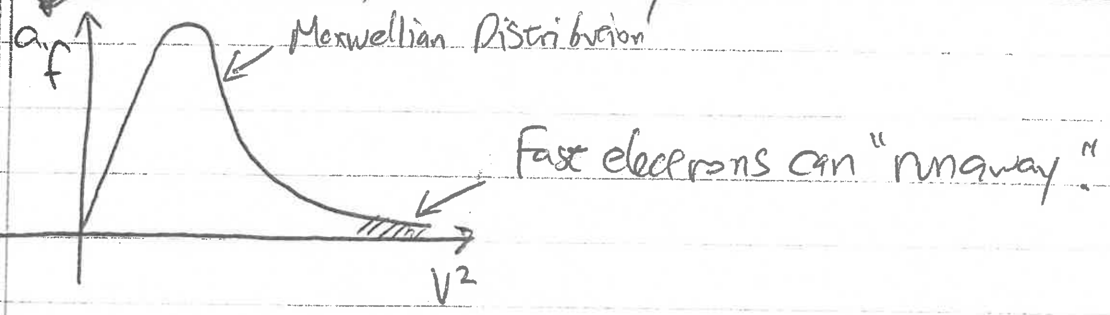
3. For $v_0^2 = v_{te}^2 = \frac{2T_e}{m_e}$, we find

NOTE: T_e in energy units (implicitly includes Boltzmann constant K.

$$\boxed{\nu_{c_{e\text{-}i}} = \frac{e^4}{2^{5/2}\pi\epsilon_0^2 m_e^{1/2}} \frac{n_{0i}}{(T_e)^{3/2}} \ln N_D}$$

4. $\boxed{\text{Unlike in a solid, collisionality decreases as } T_e \text{ increases.}}$

a.



5. Compare collision frequency to electron plasma frequency:

$$\frac{\nu_c}{\omega_{pe}} = \frac{n_0 e^4}{2^{5/2}\pi\epsilon_0^2 m_e^{1/2}(T_e)^{3/2}}\left(\frac{\epsilon_0 m_e}{n_0 e^2}\right)^{1/2} = \frac{1}{4\sqrt{2}\pi n_0}\left(\frac{n_0 e^2}{\epsilon_0 T_e}\right)^{3/2} = \frac{1}{3\sqrt{2}\left(\frac{4\pi}{3} n_0 \lambda_D^3\right)}$$

$$= \frac{1}{3\sqrt{2} N_D} \Rightarrow \boxed{\frac{\nu_c}{\omega_{pe}} \simeq \frac{1}{N_D}}$$

Single Particle Collisions much less important than collective effects.

II (Continued)

E. Collisional Equilibration Times:

1. Collision Frequency for species s on species r

$$\nu_{sr} = \frac{e^4 n_{0r}}{2^{5/2}\pi\epsilon_0^2 m_s^{1/2} (T_s)^{3/2}} \ln N_D$$

2. Electron-Ion collisions: ν_{ei} calculated as before.

3. Electron-electron collisions:

a. Need to transform to center-of-mass frame. May introduce a few factors of 2, but generally $\boxed{\nu_{ee} \simeq \nu_{ei}}$

4. Ion-Ion collisions:

a. Same as electron-electron collisions, except we must replace m_e by m_i in denominator (taking $T_i = T_e$)

$$\boxed{\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ee}}$$

5. Ion-electron collisions:

a. Center-of-mass frame calculation introduces another factor of $\left(\frac{m_e}{m_i}\right)^{1/2}$, so $\boxed{\nu_{ie} \simeq \left(\frac{m_e}{m_i}\right) \nu_{ee}}$

NOTE: For proton-electron plasma $\boxed{m_i/m_e = 1836.}$

6. For a plasma with arbitrary velocity distributions for both protons & electrons and unequal temperatures $T_i \neq T_e$,

a. Electrons thermalize on timescale $\tau_{ee} \sim \frac{1}{\nu_{ee}} \sim \frac{1}{\nu_{ei}}$

b. Ions thermalize on timescale $\tau_{ii} \sim \left(\frac{m_i}{m_e}\right)^{1/2} \tau_{ee} \simeq 43\,\tau_{ee}$

c. Ions & electrons come to same temperature $\tau_{ie} \sim \frac{m_i}{m_e} \tau_{ee} = 1836\,\tau_{ee}$

Lecture #11 (Continued) Howes ⑦

III. Resistivity and Collisions:

A. Consider an unmagnetized, quasineutral plasma of ions and electrons

1. In response to an applied Electric field $\underset{\sim}{E}$, a current will flow in the plasma.

a. Current density $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{v}_s = n_{0i} e \underset{\sim}{v}_i - n_{0e} e \underset{\sim}{v}_e$ $\boxed{\underset{\sim}{j} = e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e)}$

b. For equivalent temperatures (or energies)

$$\frac{1}{2} m_e v_e^2 = \frac{1}{2} m_i v_i^2 \Rightarrow v_e = \left(\frac{m_i}{m_e}\right)^{1/2} v_i$$

For protons and electrons $v_e \simeq 43\, v_i$

c. Thus, **Current in a plasma is carried mostly by electrons.**

2. a. Because of conservation of momentum, electron-electron collisions do not lead to resistivity.

b. ~~If~~ Electron-ion collisions are responsible for resistivity.

3. Electron Momentum Equation (in Unmagnetized plasma) $n_0 = n_e = n_i$

a. $$m_e n_0 \frac{d\underset{\sim}{v}_e}{dt} = -e n_0 \underset{\sim}{E} + \underbrace{m_e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e) \nu_{ei}}_{\text{Collisional term}}$$

b. In steady state, $\frac{d\underset{\sim}{v}_e}{dt} = 0$, so

$$\underset{\sim}{E} = \frac{m_e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e) \nu_{ei}}{+ e n_0} = \frac{\overbrace{e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e)}^{=\underset{\sim}{j}} m_e \nu_{ei}}{e^2 n_0} = \left(\frac{m_e \nu_{ei}}{e^2 n_0}\right) \underset{\sim}{j}$$

c. $\boxed{\text{Ohm's Law } \underset{\sim}{E} = \eta \underset{\sim}{j}}$ where $\eta = \frac{m_e \nu_{ei}}{e^2 n_0}$

is the Resistivity (specific)

Lecture#11 (Continued) Hawes ⑧

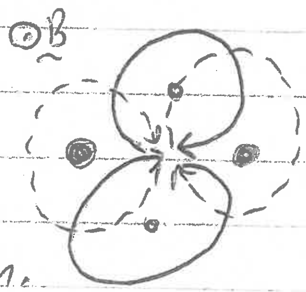
III A. (Continued)

4. $$\eta = \frac{m_e}{e^2 \cancel{N_0}}\left[\frac{\cancel{N_0}\, e^4 \cdot \ln N_0}{2^{3/2}\pi\epsilon_0^2 m_e^{1/2}(kT_e)^{3/2}}\right] = \boxed{\frac{e^2 m_e^{1/2} \ln N_0}{2^{5/2}\pi\epsilon_0^2 (kT_e)^{3/2}} = \eta}$$

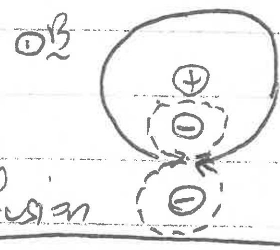
a. Resistivity is independant of density!
b. Resistivity decreases with increasing temperature!

IV. Collisions and Magnetic Confinement

A. Like-Particle Collisions: Center-of-mass remains stationary
⇒ Like-particle collisions give little diffusion across magnetic field lines.

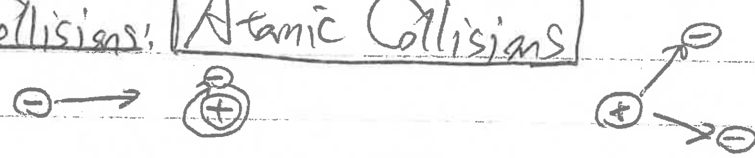


B. Unlike-Particle Collisions:
Center-of-mass is shifted
⇒ Unlike particle collisions give rise to diffusion across magnetic field lines ⇒ $\boxed{\text{LOSS OF CONFINEMENT}}$



V. Other Types of Collisions: $\boxed{\text{Atomic Collisions}}$

1. Ionization:

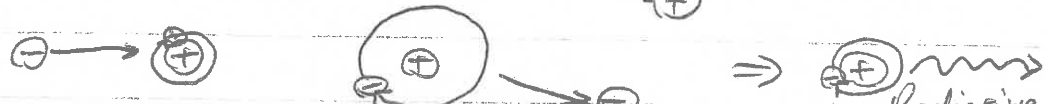


2. Recombination:

a.  Radiative

b.  Three-Body

3. Excitation:



Radiative De-excitation

4. Charge Exchange:

5. Photoionization: