Lecture #12 Kinetic Description of a Plasma

I. Overview of Plasma Descriptions

Lorentz Force Law

$$m\frac{d\underset{\sim}{v}}{dt} = q(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})$$

$\underset{\sim}{E}$ & $\underset{\sim}{B}$ $\Rightarrow$ $\underset{\sim}{x}$ & $\underset{\sim}{v}$

Particles: Ions and Electrons

$\underset{\sim}{x}, \underset{\sim}{v} \Rightarrow \rho, \underset{\sim}{j}$

$\underset{\sim}{B}$ & $\underset{\sim}{E}$ fields

Maxwell's Equations

$$\nabla\cdot\underset{\sim}{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla\times\underset{\sim}{B} = \mu_0\underset{\sim}{j} + \epsilon_0\mu_0\frac{\partial\underset{\sim}{E}}{\partial t}$$

$$\nabla\times\underset{\sim}{E} = -\frac{\partial\underset{\sim}{B}}{\partial t}$$

$$\nabla\cdot\underset{\sim}{B} = 0$$

ρ & $\underset{\sim}{j}$ $\Rightarrow$ $\underset{\sim}{E}$ & $\underset{\sim}{B}$

A. Inconsistent Models:

1. Single Particle Motion: Lorentz Force Law for specified $\underset{\sim}{E}$ & $\underset{\sim}{B}$

B. Consistent Models:

1. Kinetic Description:
   - a. Describe positions and velocities of all particles $\Rightarrow f(\underset{\sim}{x},\underset{\sim}{v},t)$ Distribution function
   - b. $\boxed{\text{Klimontovich Equation}}$:
     - i. Exact description of every ion and electron in system
     - ii. Together with Maxwell's Equation's, completely deterministic
     - iii. Too detailed for practical application
   - c. Liouville Equation: Another exact description of system $\Rightarrow$ too detailed.
   - d. $\boxed{\text{Plasma Kinetic Equation}}$ (Boltzmann Equation)
     - i. Statistical treatment of a plasma
     - ii. Evolves six-dimensional distribution function $f(\underset{\sim}{x},\underset{\sim}{v},t)$ due to $\underset{\sim}{E}$ & $\underset{\sim}{B}$ fields and collisions.
     - iii. $\boxed{\text{Vlasov Equation}}$: Limit of Plasma Kinetic Equation when collisions $\rightarrow 0$.

2. Fluid Description:
   - a. Velocity moments of the distribution function $f(\underset{\sim}{x},\underset{\sim}{v},t)$ [integrating over velocity space] lead to fluid variables [functions of $(\underset{\sim}{x},t)$ only].

Lecture #2 (Continued)

I. 2. (Continued)

b. Evolution for each moment involves a higher moment
 ⇒ This leads to a closure problem

c. A physically motivated approximation is used to close equations.

d. $\boxed{\text{Two Fluid Equations}}$

i. Ions and electrons are each evolved separately

e. $\boxed{\text{MHD Equations}}$

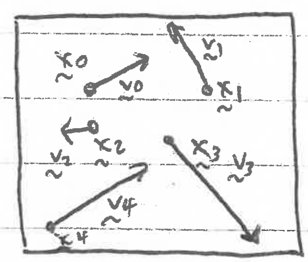
i. Ions and electrons move together, leading to a single fluid theory.

ii. Simple, consistent description of plasma behavior, but only valid when MHD approximation is satisfied.

II. Klimontovich Equation:

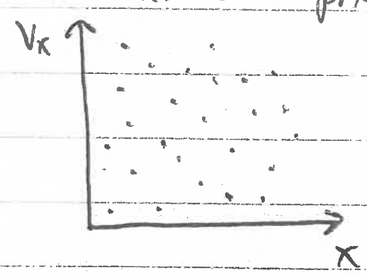
A. Phase Space:

1. A plasma is completely described if we know the position $\underset{\sim}{x}$ and velocity $\underset{\sim}{v}$ of each particle at time t.



a. We can introduce the six-dimensional <u>phase space $(\underset{\sim}{x}, \underset{\sim}{v})$</u> such that each of the N particles occupies a position $(\underset{\sim}{x}_i, \underset{\sim}{v}_i)$ for $i=1,\dots,N$.

b. In this 6-D phase space, each particle occupies a point.



c. In a real plasma, the number of particles is huge, $\sim 10^{10} - 10^{30}$ particles.

2. We can describe this N-particle plasma with the

a. Klimontovich Distribution

$$\boxed{\mathcal{F} = \sum_{i=1}^{N} \delta[\underset{\sim}{x} - \underset{\sim}{x}_i(t)]\, \delta[\underset{\sim}{v} - \underset{\sim}{v}_i(t)]}$$

b. $\underset{\sim}{x}_i(t)$ = position of i^{th} particle

$\underset{\sim}{v}_i(t)$ = velocity of i^{th} particle

II. A.2. (Continued)

c. $\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i$ $\qquad$ $\frac{d\mathbf{v}_i}{dt} = \mathbf{a}_i = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B})$

d. $\mathcal{F}$ describes the density of particles in phase space. If we integrate over all velocities and position, we get number of particles

$$N = \int d^3x \int d^3v \; \mathcal{F}(\mathbf{x}, \mathbf{v}, t)$$

e. 1. $\mathcal{F}$ depends on detailed initial positions and velocities of each particle

2. $\mathcal{F}$ is far ~~too~~ detailed for practical use

3. This level of detail is not needed for most important macroscopic results.

f. How does $\mathcal{F}$ change in time? $\frac{d\mathcal{F}}{dt} = ?$

1. $\mathcal{F}(\mathbf{x}, \mathbf{v}, t)$ depends on 3 position, 3 velocity, and 1 time variables

2. $$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{dv_x}{dt}\frac{\partial}{\partial v_x} + \frac{dv_y}{dt}\frac{\partial}{\partial v_y} + \frac{dv_z}{dt}\frac{\partial}{\partial z}$$

3. If we evaluate $\left.\frac{d\mathcal{F}}{dt}\right|_{\text{orbit}}$ along the particle orbits.

i. $\left.\frac{dx}{dt}\right|_{\text{orbit}} = v_x$, $\left.\frac{dy}{dt}\right|_{\text{orbit}} = v_y$, etc. $\Rightarrow$ $\frac{d\mathbf{x}}{dt} = \mathbf{v}$

ii. Similarly $\left.\frac{dv_x}{dt}\right|_{\text{orbit}} = a_x$, etc. $\Rightarrow$ $\frac{d\mathbf{v}}{dt} = \mathbf{a}$

4. Thus $$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}}$$

g. for a single species s, we get

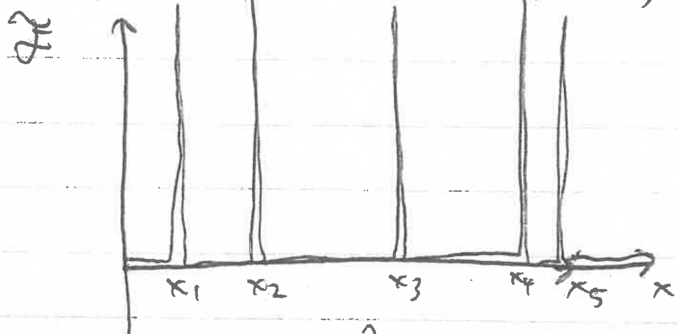
$$\frac{d\mathcal{F}_s}{dt} = \boxed{\frac{\partial \mathcal{F}_s}{\partial t} + \mathbf{v}\cdot\nabla\mathcal{F}_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathcal{F}_s}{\partial \mathbf{v}} = 0} \quad \boxed{\text{Klimontovich Equation}}$$

Lecture #12: (Continued)

Howes ④

II. (Continued)

B. 1. The Klimontovich Distribution is a very spiky function (sum of delta functions) due to particle discreteness.



2. Along the orbit of particles, the distribution does not change $\frac{d\mathcal{F}}{dt} = 0$. Either you are on a particle, and density is "infinite", or not on a particle, density is zero.

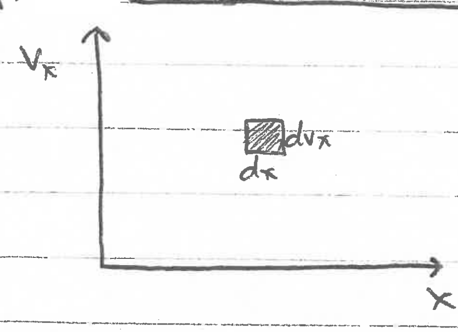
3. Evolving the Klimontovich Distribution is equivalent to an N-body problem.

   a. For practical situations with $N \sim 10^{20}$, this is not possible

⇒ 4. We prefer a statistical description that averages over many particles, smoothing out the spiky-ness due to discrete particles, yielding a smooth solution.

III. Averaging to Yield Plasma Kinetic Equation

A. Coarse Grained Average:



1. Create a smoothed distribution by averaging over a small volume $d^3\underline{x}\, d^3\underline{v}$
2. This volume contains many particles, but is small enough that average doesn't change much over the volume.

3. For example, average with spherical exponential weighting

"Weight Function" $W(\underline{x}', \underline{v}') = e^{-\left(\frac{|\underline{x}'|}{x_0}\right)^2 - \left(\frac{|\underline{v}'|}{v_0}\right)^2}$ (characteristic sizes x_0, v_0.)

$x_0 \ll \lambda_D$ but $x_0 \gg n^{-1/3}$

III. A (Continued)

4.

a. $$\underbrace{f(\underset{\sim}{x},\underset{\sim}{v},t)}_{\text{Distribution Function}} = \int d^3\underset{\sim}{x}' \int d^3\underset{\sim}{v}'\, W(\underset{\sim}{x}',\underset{\sim}{v}')\, \mathcal{F}(\underset{\sim}{x}-\underset{\sim}{x}', \underset{\sim}{v}-\underset{\sim}{v}', t)$$

b. The distribution function $f(\underset{\sim}{x},\underset{\sim}{v},t)$ is a statistical density in 6-D phase space over small volume $d^3\underset{\sim}{x}\, d^3\underset{\sim}{v}$

$\Rightarrow$ Smooths out "discreteness" of Klimontovich.

c. Enough particles in $d^3\underset{\sim}{x}\, d^3\underset{\sim}{v}$ to yield a good statistical average $f(\underset{\sim}{x},\underset{\sim}{v},t)$.

5. a. An alternative approach uses the $\boxed{\text{Liouville Equation}}$ to describe a system of particles

b. An ensemble average of such systems leads to a similar statistical description of kinetic plasmas.

B. Separating Smooth and Fluctuating parts:

1. a. The Klimontovich Equation describes the evolution of all plasma particles exactly.

b. We wish to find an evolution equation for the distribution function $f(\underset{\sim}{x},\underset{\sim}{v},t)$

2. Moments of Klimontovich Distribution: Charge and Current Densities.

a. Charge Density

$$\rho(\underset{\sim}{x},t) = \sum_s \int d^3\underset{\sim}{v}\, q_s \mathcal{F}_s(\underset{\sim}{x},\underset{\sim}{v},t)$$ "zeroth" velocity moment

b. Current Density

$$\underset{\sim}{j}(\underset{\sim}{x},t) = \sum_s \int d^3\underset{\sim}{v}\, q_s \underset{\sim}{v}\, \mathcal{F}_s(\underset{\sim}{x},\underset{\sim}{v},t)$$ "First" velocity moment

Lecture #12 (Continued) Howes ⑥

III. B. (Continued)

3. Maxwell's Equations: ρ and $\mathbf{j}$ are sources for $\mathbf{E}$ & $\mathbf{B}$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

4. Split into smoothed and fluctuating parts.

a. Take $\mathcal{F}_s = \underbrace{f_s}_{\text{Smoothed part}} + \underbrace{(\mathcal{F}_s - f_s)}_{\text{Fluctuating due to particle discreteness}}$

b. Thus $\rho(\mathbf{x},t) = \bar{\rho}(\mathbf{x},t) + \tilde{\rho}(\mathbf{x},t)$

where $\bar{\rho}(\mathbf{x},t) = \int d^3\mathbf{v}\, q_s f_s(\mathbf{x},\mathbf{v},t)$

$\tilde{\rho}(\mathbf{x},t) = \int d^3\mathbf{v}\, q_s \left[\mathcal{F}_s(\mathbf{x},\mathbf{v},t) - f_s(\mathbf{x},\mathbf{v},t)\right]$

c. Similarly $\mathbf{j}(\mathbf{x},t) = \bar{\mathbf{j}}(\mathbf{x},t) + \tilde{\mathbf{j}}(\mathbf{x},t)$

5. Maxwell's Equations are linear, so we can perform the same separation for the fields $\mathbf{E}$ & $\mathbf{B}$

a. For example $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Rightarrow \begin{cases} \nabla \cdot \bar{\mathbf{E}} = \frac{\bar{\rho}}{\epsilon_0} \\ \nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\rho}}{\epsilon_0} \end{cases}$ and the same for $\mathbf{B}$.

C. Averaging the Klimontovich Equation:

1. Now, we perform the same averaging procedure on $\frac{d\mathcal{F}}{dt} = 0$ to yield:

$$\boxed{\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\bar{\mathbf{E}} + \mathbf{v} \times \bar{\mathbf{B}}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \underbrace{\frac{q_s}{m_s}\left\langle \left(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}}\right) \cdot \frac{\partial \tilde{\mathcal{F}}_s}{\partial \mathbf{v}} \right\rangle}_{= \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}}}$$

$\boxed{\text{Plasma Kinetic Equation}}$ (or Boltzmann Equation)

Lecture #12 (Continued) Howes (7)

III C. (Continued)

2. a. The LHS contains only terms that vary smoothly in $(\underset{\sim}{x}, \underset{\sim}{v})$ space, i.e. f_s, $\underset{\sim}{E}$, $\underset{\sim}{B}$.
   $\Rightarrow$ Collective effects of plasma

   b. The RHS contains very spiky quantities due to particle discreteness. $\Rightarrow$ Collisional effects of plasma

3. Recall the ratio of $\frac{\text{Collective effects}}{\text{Collisional effects}} \sim \frac{\omega_{pe}}{\nu_c} \sim N_D$

   Thus, collisional effects are typically weak.

4. We can neglect the effect of collisions, a good approximation for most plasmas, to give

$$\boxed{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0}$$

Vlasov Equation

$$\nabla \cdot \underset{\sim}{E} = \frac{1}{\epsilon_0} \overbrace{\sum_s \int d^3\underset{\sim}{v}\, q_s\, f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}^{=\rho}$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \overbrace{\sum_s \int d^3\underset{\sim}{v}\, q_s v_s f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}^{=\underset{\sim}{j}} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

Maxwell's Equations

These $\boxed{\text{Vlasov-Maxwell Equations}}$ describe the <u>collisionless</u> kinetic evolution of a plasma.

• "Integro-differential" system of equations.

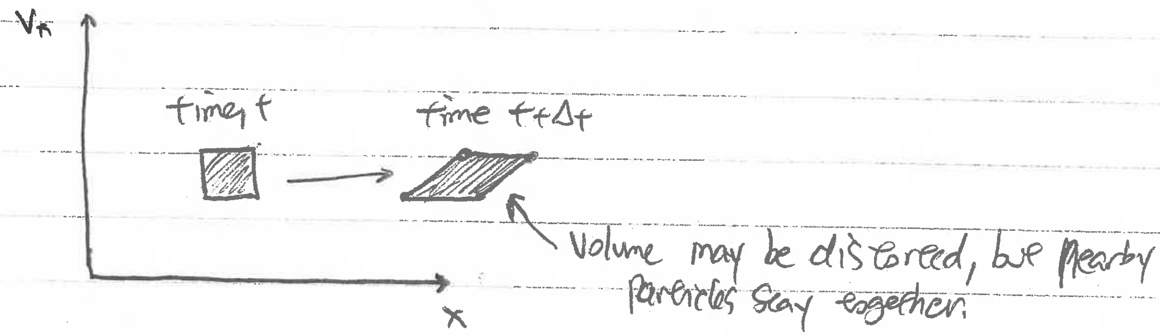
Lecture #12 (Continued)

IV. The Distribution Function $f_s(\underline{x}, \underline{v}, t)$

A. Intuitive Picture:

1. $d^3\underline{x}\, d^3\underline{v}\, f_s(\underline{x}, \underline{v}, t)$ is the number of particles of species s in a infinitesimal volume in 6-D phase space, $\Delta v_x \Delta v_y \Delta v_z \Delta x \Delta y \Delta z$.

Thus $f_s(\underline{x}, \underline{v}, t) = \dfrac{\text{Number}}{\text{Volume}}$ is a number density in 6-D phase space.

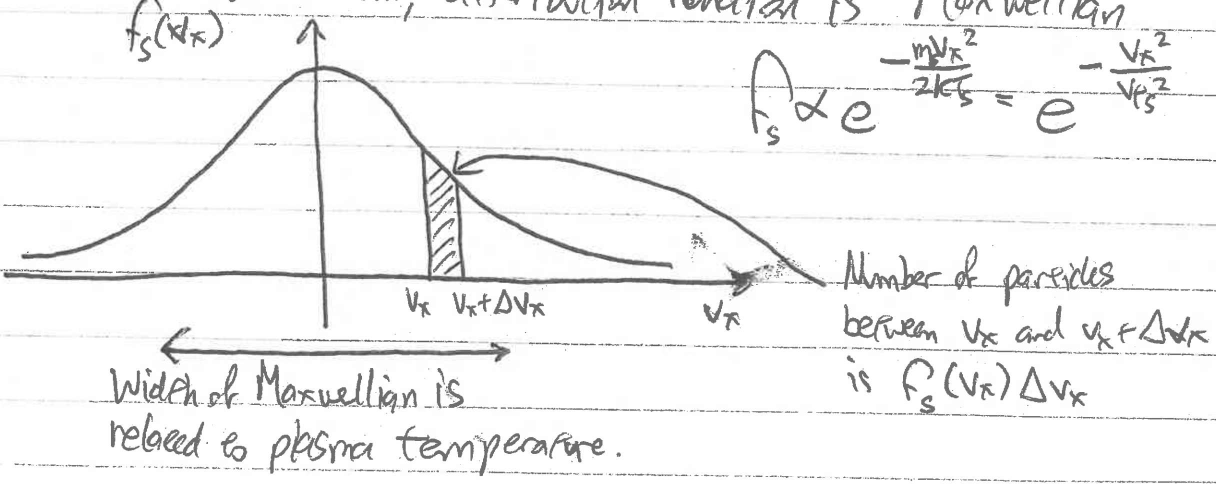
2. 

a. Particles at nearby points in phase space move together along their trajectories

b. $\dfrac{df_s}{dt} = 0$ (collisionless) suggests the distribution function is the density of an incompressible, 6-D fluid.

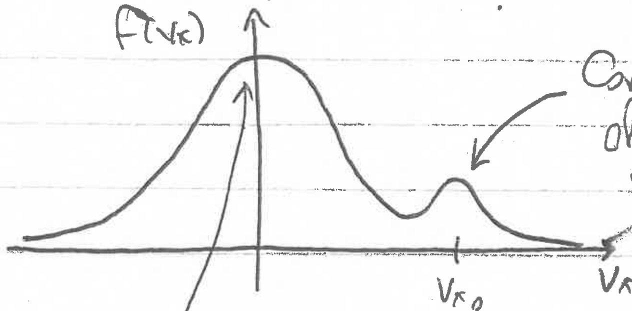
3. In Thermal Equilibrium, distribution function is Maxwellian

$$f_s \propto e^{-\frac{m v_x^2}{2kT_s}} = e^{-\frac{v_x^2}{v_{ts}^2}}$$



IV. A. (Continued)

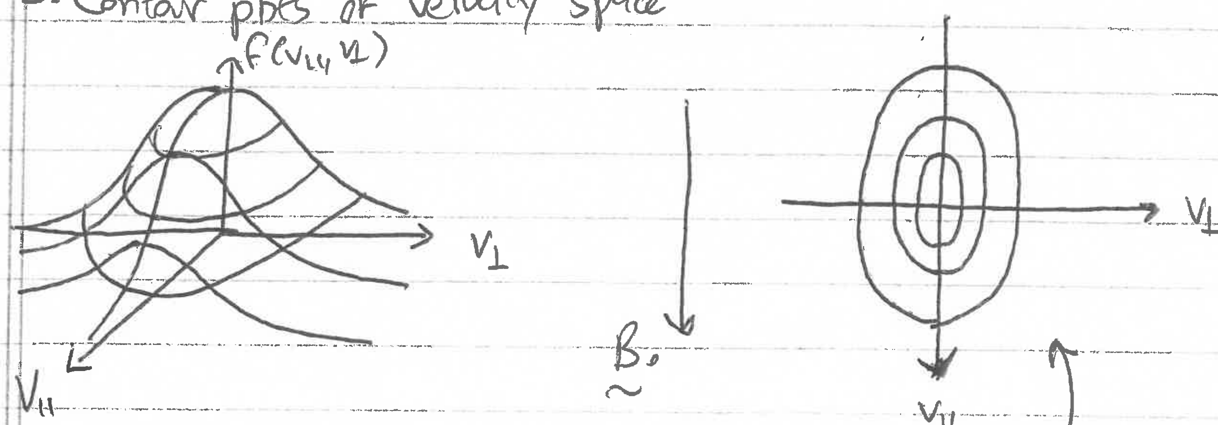
4. Non-Maxwellian Distribution function



Corresponds to an added "beam" of particles moving at speed v_{x0} with respect to Maxwellian plasma at rest

Maxwellian plasma at rest

5. Contour plots of velocity space



Here, temperature in the direction parallel to $\underset{\sim}{B}_0$ is greater than perpendicular temp.

B. Moments of the distribution function:

1. Density: $n_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\, f_s(\underset{\sim}{x},\underset{\sim}{v},t)$ $=$ $\dfrac{\text{Number of particles}}{\text{unit volume}}$

2. Fluid Velocity: $\underset{\sim}{U}_s(\underset{\sim}{x},t) = \dfrac{\int d^3\underset{\sim}{v}\, \underset{\sim}{v}\, f_s(\underset{\sim}{x},\underset{\sim}{v},t)}{n_s(\underset{\sim}{x},t)}$

3. Kinetic Energy Density: $\mathcal{E}(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\, \frac{1}{2} m |\underset{\sim}{v}|^2 f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

4. Pressure Tensor: $\underset{\approx}{P}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\, (\underset{\sim}{v}-\underset{\sim}{U}_s)(\underset{\sim}{v}-\underset{\sim}{U}_s)\, f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

Next time, we'll use velocity moments to derive fluid equations.