

Lecture #13 Moment Equations

Howes ①

I. Review of Kinetic Description of Plasmas

A. Picture of a Kinetic Plasma.

1. At any point in space, a kinetic plasma contains particles moving with a distribution of velocities



2. The distribution function $f_s(\underline{x}, \underline{v}, t)$ describes a statistical measure of the plasma in a volume of ~~size~~ x_0^3 such that
 $x_0 \gg n^{-1/3}$ larger than particle spacing \Rightarrow many particles
 $x_0 \ll \lambda_D$ smaller than Debye length

3. $f_s(\underline{x}, \underline{v}, t)$ is a number density in 6-D phase space $(\underline{x}, \underline{v})$

B. Evolution of distribution function

1. Along particle orbit, $\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial f_s}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial f_s}{\partial \underline{v}}$

2. The statistical ensemble average yields

Plasma Kinetic Equation

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \left(\frac{df_s}{dt} \right)_{\text{coll}}$$

Statistically smooth description of particles, f_s , and fields \underline{E} & \underline{B} over volume x_0^3

Effect of discrete particles on smaller scales \Rightarrow collisions.

3. Neglect the effect of collisions yields Vlasov Equation

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

I (Continued)

C. Moments of the Distribution Function

1. Summing over velocity space gives useful macroscopic quantities.

a. Density $n_s(\underline{x}, t) = \int d^3\underline{v} f_s(\underline{x}, \underline{v}, t)$

b. Fluid Velocity $\underline{U}_s(\underline{x}, t) = \frac{1}{n_s(\underline{x}, t)} \int d^3\underline{v} \underline{v} f_s(\underline{x}, \underline{v}, t)$

c. Kinetic Energy Density: $\mathcal{E}_s(\underline{x}, t) = \int d^3\underline{v} \frac{1}{2} m |\underline{v}|^2 f_s(\underline{x}, \underline{v}, t)$

d. Pressure Tensor: $\underline{P}_s(\underline{x}, t) = \int d^3\underline{v} m_s (\underline{v} - \underline{U}_s)(\underline{v} - \underline{U}_s) f_s(\underline{x}, \underline{v}, t)$

2. These are the velocity moments of the distribution function.

3. Taking moments of the kinetic distribution function gives "fluid" quantities, or observable quantities.

• Ex: The temperature of a kinetic species s can be defined.

$$\mathcal{E}_s \equiv \frac{3}{2} n_s k T_s \quad \Rightarrow \quad T_s \equiv \frac{2 \mathcal{E}_s}{3 n_s k}$$

NOTE: From thermodynamics, $P_s = n_s k T_s$, so $\mathcal{E}_s = \frac{3}{2} P_s$, as expected.

II. From a Kinetic to a Fluid Description

A. Fluid Description:

1. Often, we don't care about detailed distribution of velocities.
2. Thus, all we want are velocity moments, n_s , \underline{U}_s , \mathcal{E}_s , etc.
3. The velocity moments just define the dependent variables of a fluid description, which depend only on \underline{x} and t .

II. (Continued)

B. Comparison of Kinetic vs. Fluid Descriptions

Kinetic

1. $\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{x}} = 0$

Charge Density $\rho = \sum_s q_s \int d^3v f_s(\underline{x}, \underline{v}, t)$

Current Density $\underline{j} = \sum_s q_s \int d^3v \underline{v} f_s(\underline{x}, \underline{v}, t)$

$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$ $\nabla \cdot \underline{B} = 0$

$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ $\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$

2. Everything depends on $G \rightarrow 0$ distribution function $f_s(\underline{x}, \underline{v}, t)$

Fluid:

1. Moments: $n_s(\underline{x}, t)$

$U_s(\underline{x}, t)$

$\tilde{E}_s(\underline{x}, t)$

etc.

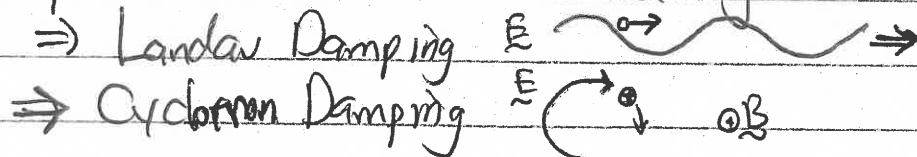
2. Evolution equations (derived ^{to be} here)

3. A number of fluid variables (moments), all of which only depend on \underline{x} and t .

C. When is a fluid description appropriate?

ANSWER: When the details of the velocity distribution do not significantly affect the evolution.

1. Ex: Fluid description is not valid when particles at certain velocities become resonant with fluctuating EM fields.



2. Ex: Fluid description is appropriate for strongly collisional plasmas. Collisions drive towards Local Thermodynamic Equilibrium
 \Rightarrow Velocity distribution becomes Maxwellian.

Lecture #13 (Continued)

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I. (Continued)

D. Maxwellian (and Bi-Maxwellian) Plasmas:

1. Fluid equations are a good description of a plasma when the velocity distributions remain Maxwellian

2. Maxwellian Distribution

$$f_s(\mathbf{x}, \mathbf{v}, t) = \frac{n_s(\mathbf{x}, t)}{\pi^{3/2} v_{Ts}^3} e^{-\frac{m_s |\mathbf{v} - \mathbf{U}(\mathbf{x}, t)|^2}{2kT_s(\mathbf{x}, t)}}$$

a. Fluid is completely described by three fluid variables:

Density $n_s(\mathbf{x}, t)$

Fluid velocity $\mathbf{U}_s(\mathbf{x}, t)$

Temperature $T_s(\mathbf{x}, t)$

b. Here, as usual, the thermal velocity $v_{Ts} = \sqrt{\frac{2kT_s}{m_s}}$

3. Zeroth Moment: Density

a.
$$\int d^3v f_s(\mathbf{x}, \mathbf{v}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{n_s}{\pi^{3/2} v_{Ts}^3} e^{-\frac{|\mathbf{v} - \mathbf{U}|^2}{v_{Ts}^2}}$$

b. NOTE: $|\mathbf{v} - \mathbf{U}|^2 = (v_x - U_x)^2 + (v_y - U_y)^2 + (v_z - U_z)^2$

c. We can split this into three integrals which are identical

$$\frac{1}{\pi^{3/2} v_{Ts}^3} \int_{-\infty}^{\infty} e^{-\frac{(v_x - U_x)^2}{v_{Ts}^2}} dv_x = \frac{1}{\pi^{3/2} v_{Ts}^3} \int_{-\infty}^{\infty} v_{Ts} dy e^{-y^2} = \frac{\sqrt{\pi} v_{Ts}}{\pi^{3/2} v_{Ts}^3} = 1$$

$y = \frac{v_x - U_x}{v_{Ts}} \quad dy = \frac{dv_x}{v_{Ts}}$

d. Thus
$$\int d^3v f_{s,m}(\mathbf{x}, \mathbf{v}, t) = n_s(\mathbf{x}, t)$$

4. First Moment: Fluid Velocity

$$\int d^3v \, v f_{sm}(\underline{x}, \underline{v}, t) = n_s(\underline{x}, t) \underline{U}_s(\underline{x}, t)$$

5. Second Moment: Energy

$$\int d^3v \, \frac{1}{2} m_s |\underline{v}|^2 f_{sm}(\underline{x}, \underline{v}, t) = \underbrace{\frac{3}{2} n_s k T_s}_{\text{Thermal Energy}} + \underbrace{\frac{1}{2} m_s n_s |\underline{U}_s|^2}_{\text{Kinetic Energy of Fluid Flow}}$$

6. The presence of a Mean Magnetic Field \underline{B}_0 often leads to an anisotropic distribution function

Bi-Maxwellian Distribution:
$$f_{s, \text{Bm}}(\underline{x}, \underline{v}, t) = \frac{n_s(\underline{x}, t)}{\pi^{3/2} v_{Ts}^2 v_{T||s}} e^{-\frac{mv_{\perp}^2}{2kT_{s\perp}} - \frac{mv_{||}^2}{2kT_{s||}}}$$

where

$$v_{Ts} = \sqrt{\frac{2kT_{s\perp}}{m_s}} \quad \text{and} \quad v_{T||s} = \sqrt{\frac{2kT_{s||}}{m_s}}$$

a. This can lead to the Chew-Goldberger-Low, or Double Adiabatic, Equation of State

III. Moment Equations

Moments of the Plasma Kinetic Equation determine evolution of velocity moments.

A. Zeroth Moment: Noetherian $S_0 = \int d^3v$

$$\int d^3v \, \frac{\partial f_s}{\partial t} + \int d^3v \, \underline{v} \cdot \nabla f_s + \int d^3v \, \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \int d^3v \, \left(\frac{\partial f_s}{\partial t} \right)_{\text{coll}}$$

i. ① = $\frac{\partial}{\partial t} \int d^3v f_s = \frac{\partial}{\partial t} n_s$

Lecture 13 (Continued)

Homework 6

III. A. (Continued)

$$2. \textcircled{2} = \int_{\underline{V}} \underline{v} \cdot \nabla f_s = \nabla \cdot \int_{\underline{V}} \underline{v} f_s = \nabla \cdot (n_s \underline{U}_s)$$

a. NOTE: \underline{x} and \underline{v} are independent variables, so $\nabla \cdot (\underline{v} f_s) = \underline{v} \cdot \nabla f_s + f_s \nabla \cdot \underline{v}$

$$3. \textcircled{3} = \frac{q_s}{m_s} \int_{\underline{V}} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \frac{q_s}{m_s} \int_{\underline{V}} \nabla_{\underline{v}} \cdot [(\underline{E} + \underline{v} \times \underline{B}) f_s]$$

a. NOTE: $\underline{E}(\underline{x}, t)$ & $\underline{B}(\underline{x}, t)$ are independent of \underline{v} : $\frac{\partial}{\partial \underline{v}} \cdot (\underline{E} f_s) = \underline{E} \cdot \frac{\partial f_s}{\partial \underline{v}} + f_s \nabla_{\underline{v}} \cdot \underline{E}$

$$b. \frac{\partial}{\partial \underline{v}} \cdot (\underline{v} \times \underline{B}) f_s = \underline{v} \times \underline{B} \cdot \frac{\partial f_s}{\partial \underline{v}} + f_s \frac{\partial}{\partial \underline{v}} \cdot (\underline{v} \times \underline{B})$$

Since each component is like $\frac{\partial}{\partial v_x} (\underline{v} \times \underline{B})_x = \frac{\partial}{\partial v_x} (v_y B_z - v_z B_y) = 0$

c. Using NRL p. 5 (28) Gauss's Thm: $\int_{\underline{V}} d^3x \nabla \cdot \underline{A} = \int_S dS \cdot \underline{A}$

$$\Rightarrow \textcircled{3} = \frac{q_s}{m_s} \int d\underline{S}_{\underline{v}} \cdot [(\underline{E} + \underline{v} \times \underline{B}) f_s] = 0$$

d. At $v \rightarrow \pm \infty$, $f_s \rightarrow 0$, so surface integral vanishes at infinity.

$$4. \textcircled{4} = \left(\frac{\partial}{\partial t} \int_{\underline{V}} f_s \right)_{\text{coll}} = \left(\frac{\partial n_s}{\partial t} \right)_{\text{coll}} = 0$$

Collisions don't create or destroy particles (number is conserved).

5. Thus, we are left with

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) = 0$$

Continuity Equation

a. Expresses conservation of particles

b. Integrate over a small volume \underline{V} : $\int_{\underline{V}} d^3x \frac{\partial n_s}{\partial t} = - \int_{\underline{V}} d^3x \nabla \cdot (n_s \underline{U}_s)$

i) Once again, use Gauss's Thm on RHS.

ii) $N = \text{number of particles in } \underline{V} \Rightarrow \frac{\partial N}{\partial t} = - \int_S dS \cdot (n_s \underline{U}_s)$ ← Particles moving through surface.

III. B. First Moment: $\int \underline{v} \underline{v}$ on Kinetic Equation

$$\int \underline{v} \underline{v} \frac{d\underline{f}_s}{dt} + \int \underline{v} \underline{v} \cdot \nabla \underline{f}_s + \int \underline{v} \underline{v} \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{d\underline{f}_s}{d\underline{v}} = \int \underline{v} \underline{v} \left(\frac{d\underline{f}_s}{dt} \right)_{coll} \quad (1)$$

$$1. \textcircled{1} = \frac{\partial}{\partial t} \int \underline{v} \underline{v} \underline{f}_s = \frac{\partial}{\partial t} (n_s \underline{U}_s)$$

2. This term involves Tensor mathematics.

$$\textcircled{2} = \int \underline{v} \underline{v} \underline{v} \cdot \nabla \underline{f}_s = \nabla \cdot \left(\int \underline{v} \underline{v} \underline{v} \underline{f}_s \right) \quad \text{since } \underline{x} \text{ \& } \underline{v} \text{ are independent.}$$

a. Use $\underline{v} = \underline{U}_s + \underline{v} - \underline{U}_s$ to get $\underline{v} \underline{v} = \underbrace{\underline{U}_s \underline{U}_s}_{\textcircled{a}} + \underbrace{\underline{U}_s (\underline{v} - \underline{U}_s)}_{\textcircled{b}} + \underbrace{(\underline{v} - \underline{U}_s) \underline{U}_s}_{\textcircled{c}} + \underbrace{(\underline{v} - \underline{U}_s) (\underline{v} - \underline{U}_s)}_{\textcircled{d}}$

b. $\textcircled{a} = \int \underline{U}_s \underline{U}_s \underline{f}_s = n_s \underline{U}_s \underline{U}_s$

c. $\textcircled{b} = \int \underline{U}_s (\underline{v} - \underline{U}_s) \underline{f}_s = \underline{U}_s \int \underline{v} \underline{f}_s - \underline{U}_s \underline{U}_s n_s = \underline{U}_s \underline{U}_s n_s - \underline{U}_s \underline{U}_s n_s = 0$

d. \textcircled{c} same as $\textcircled{b} = 0$

e. \textcircled{d} Using $\underline{P}_s = \int \underline{v} m_s (\underline{v} - \underline{U}_s) (\underline{v} - \underline{U}_s) \underline{f}_s$, $\textcircled{d} = \frac{\underline{P}_s}{m}$

$$\underline{r} \underline{T}_{ms} \textcircled{2} = \nabla \cdot (n_s \underline{U}_s \underline{U}_s) + \frac{1}{m_s} \nabla \cdot \underline{P}_s$$

$$3. \textcircled{3} = \frac{q_s}{m_s} \int \underline{v} \underline{v} \frac{\partial}{\partial \underline{v}} \cdot \left[(\underline{E} + \underline{v} \times \underline{B}) \underline{f}_s \right] \quad \text{Using same trick as Zeroth Moment.}$$

a. Nine integrals for each pairing of v_x, v_y, v_z & $\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}$

i) Six have $v_i \frac{\partial}{\partial v_j}$ where $i \neq j$, thus are similar to

$$\int dv_x \int dv_y \int dv_z v_i \frac{\partial}{\partial v_j} \left[(\underline{E} + \underline{v} \times \underline{B}) \underline{f}_s \right]$$

(i) v_z integral gives $(\underline{E} + \underline{v} \times \underline{B}) \underline{f}_s \Big|_{-\infty}^{\infty} = 0$ as $v \rightarrow \infty$ since $\underline{f}_s \rightarrow 0$.

(ii) Three have $v_i \frac{\partial}{\partial v_i} \Rightarrow \left(\int dv_x \int dv_y \int dv_z v_i \frac{\partial}{\partial v_i} \left[(\underline{E} + \underline{v} \times \underline{B}) \underline{f}_s \right] \right)$

Lecture #13 (Continued)

Howes (8)

III. B3, (Continued)

$$\int dv_z v_z \frac{\partial}{\partial v_z} \left[(\underline{E} + \underline{v} \times \underline{B}) f_s \right] = v_z (\underline{E} + \underline{v} \times \underline{B}) f_s \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\underline{E} + \underline{v} \times \underline{B}) f_s dv_z$$

$$u = v_z \quad dv = \frac{\partial}{\partial v_z} (\underline{E} + \underline{v} \times \underline{B}) f_s dv_z$$

$$du = dv_z \quad v = (\underline{E} + \underline{v} \times \underline{B}) f_s$$

b. We get $\textcircled{3} = -\frac{n_s q_s}{m_s} (\underline{E} + \underline{U}_s \times \underline{B})$

$$4. \textcircled{4} = \frac{1}{m_s} \int \underline{v} m_s \underline{v} \left(\frac{\partial f_s}{\partial t} \right)_{\text{coll}} = \frac{1}{m_s} \left(\frac{\partial}{\partial t} \int \underline{v} m_s \underline{v} f_s \right)_{\text{coll}} = \frac{1}{m_s} \left(\frac{\partial n_s m_s \underline{U}_s}{\partial t} \right)_{\text{coll}}$$

a. This represents a DRAG FORCE due to collisions between species

b. Same species collisions conserve momentum, so produce no drag.

5. Thus,

$$\frac{\partial}{\partial t} (n_s \underline{U}_s) + \nabla \cdot (n_s \underline{U}_s \underline{U}_s) = -\frac{1}{m_s} \nabla \cdot \underline{P}_s + \frac{n_s q_s}{m_s} (\underline{E} + \underline{U}_s \times \underline{B}) + \text{Collisional } (\underline{F}_{ps})$$

a. NOTE: LHS can be written, using tensor identity $\nabla \cdot (\underline{A} \underline{B}) = (\nabla \cdot \underline{A}) \underline{B} + (\underline{A} \cdot \nabla) \underline{B}$

$$n_s \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \frac{\partial n_s}{\partial t} + \underline{U}_s \nabla \cdot (n_s \underline{U}_s) + n_s \underline{U}_s \cdot \nabla \underline{U}_s$$

$$= \underline{U}_s \left[\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) \right]$$

Continuity eq.

6. Multiplying by m_s , we get

Momentum Equation

$$m_s n_s \frac{\partial \underline{U}_s}{\partial t} + m_s n_s \underline{U}_s \cdot \nabla \underline{U}_s = -\nabla \cdot \underline{P}_s + n_s q_s (\underline{E} + \underline{U}_s \times \underline{B}) + \underline{F}_{ps}$$

7. Second Moment Taking $\int \underline{v} \frac{1}{2} m_s v^2$ of Kinetic Equation gives,

$$\frac{\partial \mathcal{E}_s}{\partial t} + \nabla \cdot \underline{Q}_s - \underline{E} \cdot \underline{j}_s = \left(\frac{\partial \mathcal{E}_s}{\partial t} \right)_{\text{coll}}$$

Energy Equation

III. C. (Continued)

1. Here \underline{Q}_S is a heat flux

Heat Flux $\underline{Q}_S \equiv \int d^3v \frac{1}{2} m_s |\underline{v}|^2 \underline{v} f_s(\underline{x}, \underline{v}, t)$ (Third Velocity Moment)

2. $E \cdot \underline{j}_S$ represents Joule heating of species S

3. $\left(\frac{\partial E_S}{\partial t}\right)_{coll}$ is collisional heating due to collisions with other species.

a. Same-species conserve energy within species.

D. Closure Problem:

1. The evolution equation for the n^{th} moment involves the $(n+1)^{th}$ moment.

a. $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) = 0$
 (Zeroth) (First)

b. $n_s \left[\frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = -\nabla \cdot \underline{P}_s + n_s q_s (\underline{E} + \underline{v} \times \underline{B}) + \underline{F}_s$
 (First) (Second)

2. For N moment equations, a closed system can only be specified if the $(N+1)^{th}$ moment is related to the first N moments.

Ex: We may specify an equation of state relating pressure in terms of density.

Adiabatic Equation of State

$P_s n_s^{-\gamma} = \text{constant}$

$\Rightarrow \frac{d}{dt} \left(\frac{P}{n^\gamma} \right) = 0$