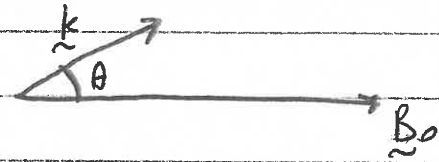
Lecture #19 More About MHD Waves

I. Review

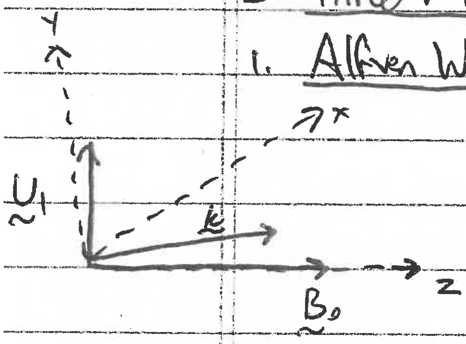
A. Last time, we linearized the MHD equations, assumed plane wave (Fourier) Solutions, and solved to obtain the MHD Dispersion Relation:

$$(\omega^2 - k^2\cos^2\theta\, V_A^2)\left[\omega^4 - \omega^2 k^2 (C_s^2 + V_A^2) + k^4 \cos^2\theta\, C_s^2 V_A^2\right] = 0$$

where 

$\underset{\sim}{B}_0 \cdot \underset{\sim}{k} = B_0 k \cos\theta$

B. Three Wave Modes:

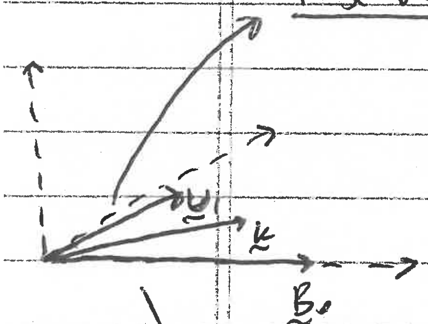


1. Alfven Waves: a. $\omega^2 = k_\parallel^2 V_A^2$

   b. Motion out of the plane defined by $\underset{\sim}{B}_0$, $\underset{\sim}{k}$

   c. Incompressible

   d. Restoring Force: Magnetic Tension alone



2. Fast Waves: a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + V_A^2) + \frac{1}{2}\sqrt{(C_s^2+V_A^2)^2 - 4C_s^2 V_A^2 \cos^2\theta}$

   b. Motion in the plane of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$

   c. Compressible (usually)

   d. Restoring Force: i) Thermal and Magnetic Pressure Add!

      ii) Magnetic Tension

3. Slow Waves: a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + V_A^2) - \frac{1}{2}\sqrt{(C_s^2+V_A^2)^2 - 4C_s^2 V_A^2 \cos^2\theta}$

   b. Motion in the plane of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$

   c. Compressible

   d. Restoring Force i) Thermal and Magnetic Pressure Subtract!

      ii) Magnetic Tension

Lecture #19 (Continued)

II. Polar Plot of MHD Wave Phase Speeds

A. Dimensionless Linear MHD Dispersion Relation

1. $(\omega^2 - k^2\cos^2\theta\, V_A^2)\left[\omega^4 - \omega^2 k^2 (C_s^2 + V_A^2) + k^4 \cos^2\theta\, C_s^2 V_A^2\right] = 0$

2. Def: Dimensionless frequency $\boxed{\bar{\omega} \equiv \frac{\omega}{k V_A}}$

3. Divide by $k^6 V_A^6$

$$\left(\frac{\omega^2}{k^2 V_A^2} - \cos^2\theta\right)\left[\left(\frac{\omega}{kV_A}\right)^4 - \left(\frac{\omega}{kV_A}\right)^2\left(1 + \frac{C_s^2}{V_A^2}\right) + \frac{C_s^2}{V_A^2}\cos^2\theta\right] = 0$$

4. Def: MHD Plasma Beta: $\boxed{\beta \equiv \frac{C_s^2}{V_A^2}}$

a. NOTE: $C_s^2 = \frac{\gamma p_0}{\rho_0}$, $V_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$

b. $\beta = \frac{\frac{\gamma p_0}{\rho_0}}{\frac{B_0^2}{\mu_0 \rho_0}} = \frac{\gamma}{2}\left[\frac{2\mu_0 p_0}{B_0^2}\right]$ Kinetic Definition of total plasma beta $\equiv \frac{\text{thermal pressure}}{\text{magnetic pressure}}$.

c. Fluid $\leftrightarrow$ Kinetic requires specifying adiabatic index γ!

5. Thus, we obtain

see (Klein, et al., ApJ 755:159, 2012) $\rightarrow$ $$\boxed{(\bar{\omega}^2 - \cos^2\theta)\left[\bar{\omega}^4 - \bar{\omega}^2(1+\beta) + \beta\cos^2\theta\right] = 0}$$

6. MHD Dispersion relation depends only on two parameters: (β, θ)

$$\boxed{\frac{\omega}{kV_A} = \bar{\omega}_{MHD}(\beta, \theta)}$$

Parameters: (i) MHD Plasma Beta $\beta \equiv \frac{C_s^2}{V_A^2}$

(ii) Wavevector Angle $\theta = \cos^{-1}\left[\frac{\underset{\sim}{k} \cdot \underset{\sim}{B_0}}{k B_0}\right]$

II. A. (Continued)

7. Validity of MHD Approximation:

a. Remember $r_{Li} \ll L$, so if $L \sim \frac{1}{k}$, this means $\boxed{k r_{Li} \ll 1}$

b. Also

i. $v_0 = \frac{L}{T} \Rightarrow r_{Li} \ll L = T v_0$

ii. For $v_0 \sim v_{ti}$ and using $r_{Li} = \frac{v_{ti}}{\omega_{ci}}$, we get $\frac{\cancel{v_{ti}}}{\omega_{ci}} \ll T \cancel{v_{ti}}$

iii. Take $\omega \sim \frac{1}{T}$, giving us $\boxed{\omega \ll \omega_{ci}}$

c. Thus $\frac{\omega}{k v_A} = \bar{\omega}_{MHD}(\beta, \theta)$ is valid when $\frac{\omega}{\omega_{ci}} \ll 1$ and $k r_{Li} \ll 1$.

(Large scales, low frequencies)

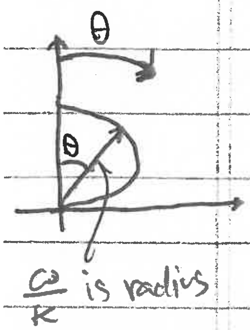
B. Limits of $\frac{\omega}{k}$ at $\theta = 0$.

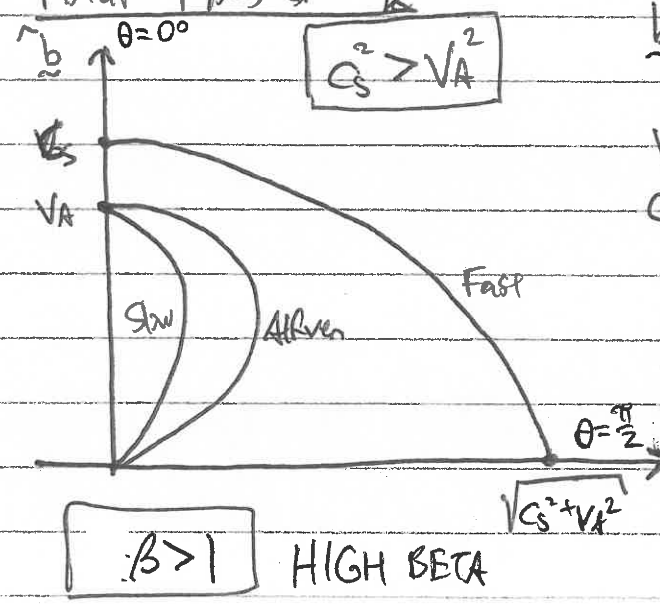
1. Phase velocity $v_p = \frac{\omega}{k}$ for waves at $\theta = 0$

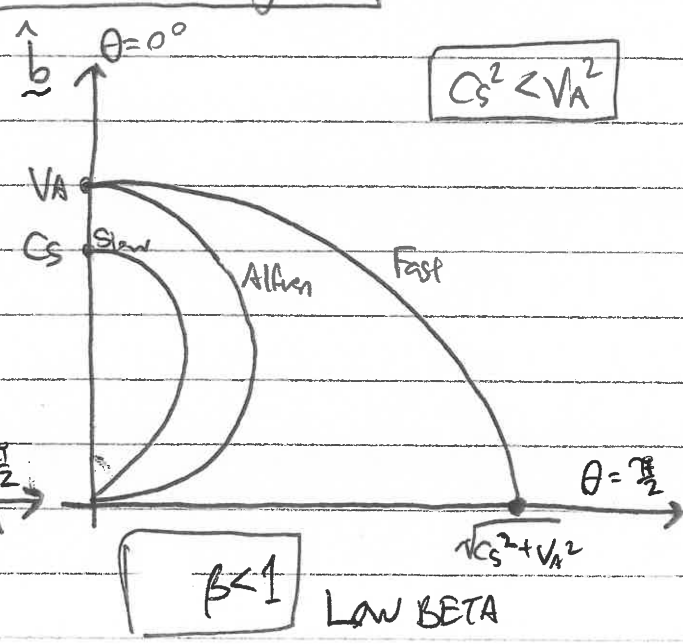
	When $C_s^2 > V_A^2$:	When $C_s^2 < V_A^2$:
Fast	$\frac{\omega}{k} = C_s^2$	$\frac{\omega}{k} = V_A^2$
Alfven	$\frac{\omega}{k} = V_A^2$	$\frac{\omega}{k} = V_A^2$
Slow	$\frac{\omega}{k} = V_A^2$	$\frac{\omega}{k} = C_s^2$

MHD Friedrichs Diagram

C. Polar Plots of $\frac{\omega}{k}$:







III. Conservation of Energy in Ideal MHD:

A.1. The Ideal MHD Equations can be manipulated to give a law for the Conservation of Energy:

$$\frac{\partial}{\partial t}\Big(\underbrace{\tfrac{1}{2}\rho U^2}_{\text{Kinetic Energy}} + \underbrace{\frac{p}{\gamma-1}}_{\text{Internal (Thermal) Energy}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic Energy}}\Big) + \nabla\cdot\Big(\underbrace{\tfrac{1}{2}\rho U^2 \underset{\sim}{U}}_{\text{Flux of Kinetic Energy}} + \underbrace{\frac{\gamma p}{\gamma-1}\underset{\sim}{U}}_{\text{Enthalpy Flux}} + \underbrace{\frac{1}{\mu_0}\underset{\sim}{E}\times\underset{\sim}{B}}_{\text{Poynting Flux}}\Big) = 0$$

2a. Integrating over all space, the volume integral of 2nd term can be converted to a surface integral by divergence theorem

NRL p.5 (28)

b. For surface at infinity, you get

$$\frac{\partial E}{\partial t} = 0 \quad \text{with} \quad \boxed{E = \tfrac{1}{2}\rho U^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}}$$

Conserved Energy in Ideal MHD.

IV. The Entropy Mode:

A.1. The MHD Equations give 8 equations for 8 unknowns: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, p$.

2. But, we found only 6 solutions to the dispersion relation.

3. In fact, a more careful analysis give two roots with $\omega = 0$. What do these roots correspond to?

B. Divergencelessness of B:

1. Remember, we must always satisfy $\nabla\cdot\underset{\sim}{B} = 0$, so there is really an additional constraint, so we only have 7 unknowns, and thus seven solutions.

C. The Entropy Mode:

1. We define DEF: Specific Entropy $\boxed{S = C\frac{p}{\rho^\gamma}}$ where C is some constant.

IV.C. (Continued)

2. Thus, the Adiabatic Equation of State is $\frac{ds}{dt} = 0$,
   $\Rightarrow$ Thus, entropy is conserved by these adiabatic fluctuations.

3. a. If we ~~are~~ consider fluctuations, $p = p_0 + p_1$
   $s = s_0 + s_1$, etc.
   b. The other $\omega = 0$ mode is a zero frequency entropy mode.
   $s_1 \neq 0$, but $p_1 = 0$ (and so are $\underset{\sim}{U}_1 = 0$ & $\underset{\sim}{B}_1 = 0$).

4. Consider the ideal gas law: $pV = NkT$, or $p = nkT = \frac{\rho k T}{m}$
   a. We can have $p_1 = 0$ if $\rho_1 T_1 = $ const.
   b. Thus density & temperature can vary to give constant pressure, $p_1 = 0$.

5. $\boxed{\text{The existence of the (oft-neglected) Entropy mode should not be forgotten}}$

6. There are 7 solutions to ideal MHD dispersion relation
   a. Six waves ($\pm$ Fast, $\pm$ Alfven, $\pm$ Slow)
   b. One zero-frequency entropy mode

V. Eigenfunctions of the MHD Eigenmodes

A. How do we determine eigenfunctions (ρ_1, $\underset{\sim}{U}_1$, $\underset{\sim}{B}_1$, p_1) for a given wave mode?
1. We must go back to the simplified matrix equation for MHD.
2. Choose a value for one component.
3. Solve for all other quantities.

V. (Continued)

B. Example: Eigenfunctions for $k_\parallel = k_\perp = k_0$ ($\theta = 45°$)

1. In this case, the vector equation for $\underset{\sim}{U}$ (normalized dimensionless) is

$$\begin{pmatrix} \bar{\omega}^2 - \frac{1}{2}(2+\beta) & 0 & -\frac{1}{2}\beta \\ 0 & \bar{\omega}^2 - \frac{1}{2} & 0 \\ -\frac{1}{2}\beta & 0 & \bar{\omega}^2 - \frac{1}{2}\beta \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

a. NOTE: $k^2 = k_\perp^2 + k_\parallel^2 = 2k_0^2$, so $\bar{\omega}^2 = \frac{\omega^2}{k^2 v_A^2} = \frac{\omega^2}{2k_0^2 v_A^2}$

2. The matrix shows that U_y (Alfven) is decoupled from U_x, U_z (fast/slow)

3. Let's find the fast & slow eigenfunctions for $U_x = U_0$

a. Take $U_y = 0$ $\quad \begin{pmatrix} \bar{\omega}^2 - \frac{1}{2}(2+\beta) & -\frac{1}{2}\beta \\ -\frac{1}{2}\beta & \bar{\omega}^2 - \frac{1}{2}\beta \end{pmatrix} \begin{pmatrix} U_x \\ U_z \end{pmatrix} = 0$

b. Either equation can be used to solve for U_z in terms of U_x:

$$-\frac{1}{2}\beta U_x + (\bar{\omega}^2 - \tfrac{1}{2}\beta) U_z = 0 \qquad U_x = U_0$$

$$U_z = \frac{\frac{1}{2}\beta}{\bar{\omega}^2 - \frac{1}{2}\beta} U_x = \boxed{\frac{\beta}{2\bar{\omega}^2 - \beta} U_0 = U_z}$$

c. Solution for ω for fast/slow waves (normalized to $\bar{\omega} = \frac{\omega}{k v_A}$)

i) $\bar{\omega}^2 = \frac{1}{2}(1+\beta) \pm \frac{1}{2}\sqrt{(1+\beta)^2 - 4\beta\cos^2\theta}$

ii) For $\theta = 45°$, $\cos^2\theta = \frac{1}{2}$, so $\bar{\omega}^2 = \frac{1+\beta}{2} \pm \frac{1}{2}\sqrt{(1+\beta)^2 - 2\beta}$

iii) $(1+\beta)^2 - 2\beta = 1 + \cancel{2\beta} + \beta^2 - \cancel{2\beta} = 1 + \beta^2$

iv) Thus $\boxed{\bar{\omega}^2 = \frac{\omega^2}{2k_0^2 v_A^2} = \frac{1}{2}(1+\beta) \pm \frac{1}{2}\sqrt{1+\beta^2}}$ $\quad + \rightarrow$ Fast, $- \rightarrow$ slow

v) Substituting into U_z: $\boxed{U_z = \frac{\beta}{1 \pm \sqrt{1+\beta^2}} U_0}$

V B. (Continued)

4. Density perturbation: $\omega \rho_1 = \rho_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) = \rho_0 (k_0 U_x + k_0 U_z)$

a. $\frac{\rho_1}{\rho_0} = \frac{k_0}{\omega}(U_x + U_z) = \frac{k_0}{\omega}\left(U_0 + \frac{\beta}{1 \pm \sqrt{1+\beta^2}} U_0\right)$ $\leftarrow \omega = \sqrt{2}\,\bar{\omega} k_0 v_A$

b. $$\boxed{\frac{\rho_1}{\rho_0} = \frac{1}{\sqrt{2}\,\bar{\omega}} \frac{1+\beta \pm \sqrt{1+\beta^2}}{1 \pm \sqrt{1+\beta^2}} \left(\frac{U_0}{v_A}\right)}$$

5. Similarly $\omega p_1 = \gamma p_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1)$, so analogously

$$\boxed{\frac{p_1}{p_0} = \frac{\gamma}{\sqrt{2}\,\bar{\omega}} \frac{1+\beta \pm \sqrt{1+\beta^2}}{1 \pm \sqrt{1+\beta^2}} \left(\frac{U_0}{v_A}\right)}$$

6. Magnetic Field: $\omega \underset{\sim}{B}_1 = \underset{\sim}{B}_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) - (\underset{\sim}{B}_0 \cdot \underset{\sim}{k}) \underset{\sim}{U}_1$

a. $\omega B_x = -k_0 B_0 U_x$, $B_y = 0$

b. $\omega B_z = (k_0 U_x + \cancel{k_0 U_z}) B_0 - \cancel{k_0 B_0 U_z} = B_0 k_0 U_x$

c. Thus $\frac{B_x}{B_0} = -\frac{k_0}{\omega} U_0 = -\frac{1}{2}\left(\frac{2 k_0 v_A}{\omega}\right) \frac{U_0}{v_A} = \frac{-1}{\sqrt{2}\,\bar{\omega}} \left(\frac{U_0}{v_A}\right)$

$\frac{B_z}{B_0} = \frac{k_0}{\omega} U_0 = \frac{1}{\sqrt{2}\,\bar{\omega}} \left(\frac{U_0}{v_A}\right)$

d. NOTE: $\nabla \cdot \underset{\sim}{B} = 0 \rightarrow \underset{\sim}{k} \cdot \underset{\sim}{B}_1 = 0 \Rightarrow k_0 B_x + k_0 B_z = k_0 \left(\frac{-k_0 U_0}{\omega}\right) B_0 + k_0 \left(\frac{k_0 U_0}{\omega}\right) B_0 = 0$!

7. Thus, Fast/Slow Wave Eigenfunction is given by

$$\boxed{\begin{array}{ll}
\frac{\rho_1}{\rho_0} = \frac{1}{\sqrt{2}\,\bar{\omega}} \frac{1+\beta \pm \sqrt{1+\beta^2}}{1 \pm \sqrt{1+\beta^2}} \frac{U_0}{v_A} & \frac{p_1}{p_0} = \frac{\gamma}{\sqrt{2}\,\bar{\omega}} \frac{1+\beta \pm \sqrt{1+\beta^2}}{1 \pm \sqrt{1+\beta^2}} \frac{U_0}{v_A} \\
\frac{U_x}{v_A} = \frac{U_0}{v_A} & \frac{B_x}{B_0} = \frac{-1}{\sqrt{2}\,\bar{\omega}} \left(\frac{U_0}{v_A}\right) \\
\frac{U_y}{v_A} = 0 & \frac{B_y}{B_0} = 0 \\
\frac{U_z}{v_A} = \frac{\beta}{1 \pm \sqrt{1+\beta^2}} \frac{U_0}{v_A} & \frac{B_z}{B_0} = \frac{1}{\sqrt{2}\,\bar{\omega}} \left(\frac{U_0}{v_A}\right)
\end{array}}$$

IV B. (Continued)

$V_p = \frac{\omega}{k}$

8. Compare fast & slow wave phase speeds for $\beta = 1$ ($c_s = v_A$)

a. $\bar{\omega}^2 = \frac{\omega^2}{2k_0^2 V_A^2} = \frac{1}{2}(1+\beta) \pm \frac{1}{2}\sqrt{1+\beta^2} \overset{\beta=1}{=} 1 \pm \frac{\sqrt{2}}{2}$

b. Since $k^2 = 2k_0^2$, we have

$$\frac{\omega^2}{k^2} = V_A^2\left(1 \pm \frac{\sqrt{2}}{2}\right)$$

c. Fast wave $\boxed{\frac{\omega}{k} = \left(1 + \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} V_A}$

Slow wave $\boxed{\frac{\omega}{k} = \left(1 - \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} V_A}$