Lecture #2 : Characteristic Scales in a Plasma Howes ①

Disclaimer: Please accept a blanket disclaimer for the inevitable errors that appear in these notes.

I. Characteristic Length and Time Scales in a Plasma

A. Debye Length and Shielding:

1. Consider a plasma in a volume L^3 with N ions (mass m_i, charge $q_i = e$) and N electrons (~~charge~~ mass m_e, charge $q_e = -e$)

a. This plasma is $\boxed{\text{quasi-neutral}}$

$$n_i = n_e = \frac{N}{L^3} \equiv n_0$$

b. Add a test particle, an ion with mass m_i, charge $q_i = e$. The potential due to this test particle alone is

$$\phi(r) = \frac{e}{4\pi\epsilon_0 r} \quad \text{where } r \text{ is spherical coordinate.}$$

Electrons will be attracted to this positive charge!

c. $\boxed{\text{Heavy ion approximation}}$ For protons, $\frac{m_i}{m_e} = 1836 \gg 1$, so taking $\frac{m_i}{m_e} \gg 1$, we simplify the problem by assuming the "infinitely" heavy ions don't move; they only provide a neutralizing background.

d. If we allow ample time to reach thermal equilibrium, statistical mechanics tells us the electron density will be governed by Maxwell-Boltzmann distribution

$$n_e = n_0 e^{+\frac{e\phi}{kT_e}} \quad \text{where } T_e \text{ is electron temperature.}$$

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I. A. 1 (Continued)

d. (Continued) Here $\phi(r)$ is the potential of the test particle plus the disturbed electron distribution.

e. Solve using Poisson's Eq. $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0}$

where charge density $\rho = q_i n_i + q_e n_e = e n_0 - e n_e$

i) The electrostatic potential $\underset{\sim}{E} = -\nabla \phi$, so

$$\nabla^2 \phi = \frac{-e}{\epsilon_0}\left(n_0 - n_0 e^{+\frac{e\phi}{kT_e}}\right)$$

Valid for $\frac{e\phi}{kT_e} \ll 1$

ii) We'll Taylor expand the exponential, $e^{+\frac{e\phi}{kT_e}} \simeq 1 + \frac{e\phi}{kT_e} + \ldots$ H.O.T. (higher order terms)

iii) Substituting in and dropping higher order terms:

$$\nabla^2 \phi = \frac{-e n_0}{\epsilon_0}\left[\cancel{1} - \left(\cancel{1} + \frac{e\phi}{kT_e}\right)\right] = \frac{e^2 n_0}{\epsilon_0 k T_e}\phi$$

iv) Using NRL p.8 for Laplacian ∇^2 in spherical coordinates, and taking spherical symmetry,

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = \frac{e^2 n_0}{\epsilon_0 k T_e}\phi}$$

f. To solve for $\phi(r)$, we'll use a trick to simplify

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \frac{1}{r}\frac{d^2}{dr^2}(rf)$$ [Prove this in HW]

g. Using $y = r\phi$, and defining $\frac{1}{\lambda_D^2} = \frac{e^2 n_0}{\epsilon_0 k T_e}$, we find

$$\frac{d^2 y}{dr^2} = \frac{y}{\lambda_D^2}$$

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I. A. 1. (Continued)

h. This equation has two solutions, so a linear combination is

$$y = Ae^{-\frac{r}{\lambda_D}} + Be^{r/\lambda_D}$$

i) On physical grounds, we cannot have $y \to \infty$ as $r \to \infty$, so $B = 0$.

ii) Thus, $y = Ae^{-\frac{r}{\lambda_D}}$ or $\boxed{\phi = \frac{A}{r} e^{-\frac{r}{\lambda_D}}}$

i. We must now determine the constant A.
We do this by matching to the solution as $r \to 0$ as given by the potential of the test particle,

$$\phi = \frac{e}{4\pi\epsilon_0 r}$$

For $r \ll \lambda_D$, $\phi = \frac{A}{r}$ $\Rightarrow$ $A = \frac{e}{4\pi\epsilon_0}$

j. Our solution is therefore $\boxed{\phi(r) = \frac{e}{4\pi\epsilon_0 r} e^{-\frac{r}{\lambda_D}}}$

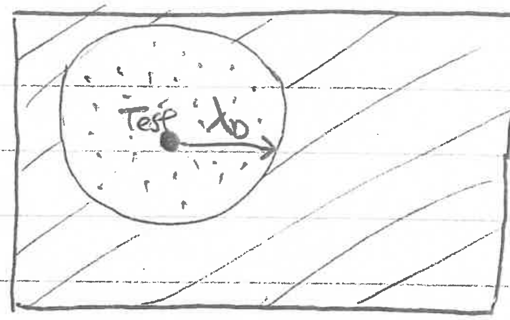
where the Debye Length $\boxed{\lambda_D = \left(\frac{\epsilon_0 K T_e}{e^2 n_0}\right)^{\frac{1}{2}}}$

2. Summary:

a. Here we have used Asymptotic matching to connect the solution $\phi(r)$ at large $r \gtrsim \lambda_D$ to the potential dominated by the test particle at $r \ll \lambda_D$.

I. A. (Continued)

3. Interpretation



Debye Shielding

a. Electrons act to shield out the Coulomb field of the test ion

b. The net charge inside sphere of radius λ_D is zero.

4. Plasma Parameter, N_D:

a. For the above picture to be correct, we need many particles in a Debye sphere.

$$\boxed{N_D \equiv \frac{4\pi}{3}\lambda_D^3 n}$$

b. $N_D \gg 1$ "Weak coupling", usual requirement for collective behavior

$N_D \ll 1$ "Strong coupling", stellar interiors, very dense plasmas, etc.

c. NOTE that
$$N_D = \left(\frac{4\pi}{3}\frac{\epsilon_0^{3/2} K^{3/2}}{e^3}\right)\frac{T_e^{3/2}}{n_0^{1/2}} \propto \frac{T^{3/2}}{n^{1/2}}$$

$\Rightarrow$ High temperature or low density gives $N_D \gg 1$.

I.A. (Continued)

5. Check validity of $\frac{e\phi}{KT} \ll 1$ for outer solution at $r \gtrsim \lambda_D$

a. $$\frac{e\phi}{KT} = \frac{1}{4\pi}\underbrace{\left(\frac{e^2 n_0}{\epsilon_0 KT}\right)}_{=\frac{1}{\lambda_D^2}} \frac{1}{r n_0} e^{-\frac{r}{\lambda_D}} = \frac{\lambda_D}{\left(\frac{4\pi}{3}\lambda_D^3 n_0\right) 3r} e^{-\frac{r}{\lambda_D}}$$

b. $$\frac{e\phi}{KT} \sim \frac{1}{N_D}\frac{\lambda_D}{3r} e^{-\frac{r}{\lambda_D}} \ll 1 \quad \checkmark$$

B. Average Particle Spacing:

1. For a plasma of density n_0, average particle spacing is $n_0^{-\frac{1}{3}}$

2. How do kinetic and potential energies compare?

a. $$P.E. \sim \frac{e^2}{4\pi\epsilon_0 n_0^{-\frac{1}{3}}}$$

b. $$K.E. \sim \frac{3}{2}KT$$

c. Thus $$\frac{K.E.}{P.E.} \sim \frac{\frac{3}{2}KT}{\frac{e^2}{4\pi\epsilon_0 n_0^{-\frac{1}{3}}}} = 4\pi\overbrace{\left(\frac{\epsilon_0 KT}{e^2 n_0}\right)}^{=\lambda_D^2}\frac{3}{2} n_0\, n_0^{-\frac{1}{3}} = \left(\frac{4\pi}{3}\lambda_D^3 n_0\right)\frac{9}{2}\frac{n_0^{-\frac{1}{3}}}{\lambda_D}$$

$$= N_D \frac{9}{2}\frac{n_0^{-\frac{1}{3}}}{\lambda_D} \gg 1$$

3. a. The kinetic energy of a particle dominates over the potential energy due to the interaction with one other particle.

b. Most of the potential energy arises from the many particles between $n_0^{-\frac{1}{3}}$ and λ_D.

c. The smoothed field of many particles dominates over discrete particle effects $\Rightarrow$ Statistical description appropriate.

C. System Size: L

1. Another important length scale is the scale over which we view the system, or the system size L.

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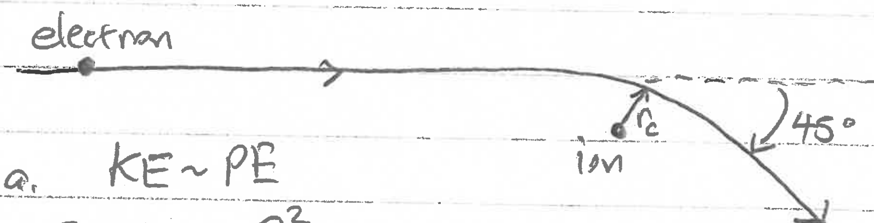
I. C. (Continued)

2\. a. To observe the "collective behavior" characteristic of plasmas, we require $\boxed{\lambda_D \ll L}$

b. Otherwise, for $L \ll \lambda_D$, particles interact little and follow relatively straight-line trajectories.

D. Mean Free Path and Collisions:

1\. An electron is deflected by $\sim 45°$ when it passes close enough to an ion such that $KE \sim PE$.



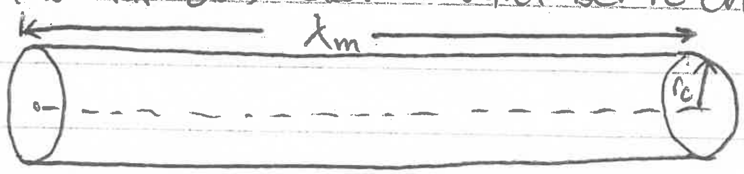
a. $KE \sim PE$

$$\frac{3}{2} k T_e \sim \frac{e^2}{4\pi\epsilon_0 r_c}$$

b. Solve for r_c: $r_c \sim \frac{2}{3} \underbrace{\left(\frac{e^2 n_0}{\epsilon_0 k T_e}\right)}^{\frac{1}{\lambda_D^2}} \frac{1}{4\pi n_0} = \frac{2}{3 \cdot 3 \cdot \frac{4\pi}{3} \lambda_D^3 n_0} \lambda_D \simeq \frac{2}{9} \frac{\lambda_D}{N_D}$

Distance of closest approach $r_c \sim \cancel{\frac{2}{9}} \frac{\lambda_D}{N_D} \sim \frac{\lambda_D}{N_D}$

2\. How far does electron travel before encountering ion within r_c?



a. It must pass through a volume $\pi r_c^2 \lambda_m$ such that $\pi r_c^2 \lambda_m n_0 = 1$

b. $\lambda_m = \frac{1}{\pi r_c^2 n_0} \sim \frac{N_D^2}{\pi n_0 \lambda_D^2} \sim \frac{N_D \cancel{\frac{4\pi}{3}} \lambda_D^{\cancel{3}} \cancel{n_0}}{\cancel{\pi n_0} \cancel{\lambda_D^2}} \sim \underline{\frac{4}{3} \lambda_D N_D \gg \lambda_D}$

c. Particles travel a long distance before colliding.

3\. Systems are: $\lambda_m \gg L$ "collisionless"

$\lambda_m \ll L$ collisional

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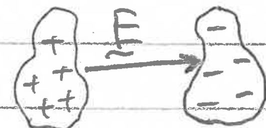
I. (Continued)

E. Plasma Frequency:

1. Consider a quasi-neutral plasma ($n_i = n_e \equiv n_0$) with infinitely heavy ions ($m_i/m_e \gg 1$).
2. a. Take the electrons in a small region and displace them



   b. A strong electrostatic field will arise to restore neutrality.
   c. This restoring force will lead to plasma oscillations at a characteristic frequency, the $\boxed{\text{plasma frequency } \omega_{pe} = \left(\frac{n_0 e^2}{\epsilon_0 m_e}\right)^{\frac{1}{2}}}$
3. Any applied electric field with a frequency $\omega < \omega_{pe}$ will be "Shorted out" by the rapid electron response.

F. Observation Time: τ

1. a. If we observe the plasma for a time τ, we are most sensitive to dynamics on that timescale, or at frequency $f \sim \frac{1}{\tau}$ (or $\omega \sim \frac{2\pi}{\tau}$).
   b. Dynamics occuring on a slower timescale will not be apparent.

G. Collision Frequency: ν

1. If the mean free path λ_m and typical velocity are known, we can determine the frequency of collisions.
2. Typical velocity: Thermal Velocity
   a. Take $\frac{1}{2} m_s v^2 = KT_s$ for a species s.
   b. Define Thermal Velocity $\boxed{v_{ts} \equiv \left(\frac{2KT_s}{m_s}\right)^{\frac{1}{2}}}$

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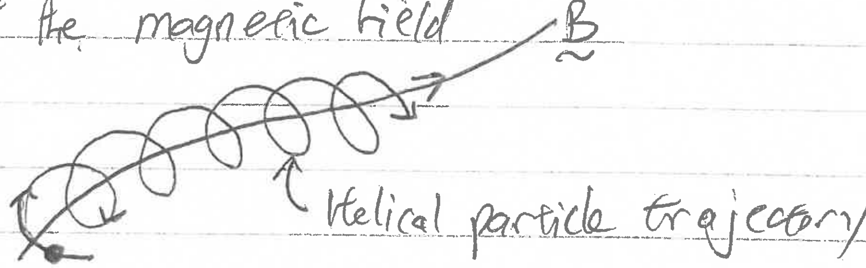
I. G. (Continued)

3. Thus, collision frequency for species s

$$\boxed{\nu_s = \frac{v_{ts}}{\lambda_{mfp}}}$$

H. Magnetized Plasmas:

1. Two additional important scales occur when a plasma is magnetized.
   a. As we know, charged particles exhibit cyclotron motion about the magnetic field

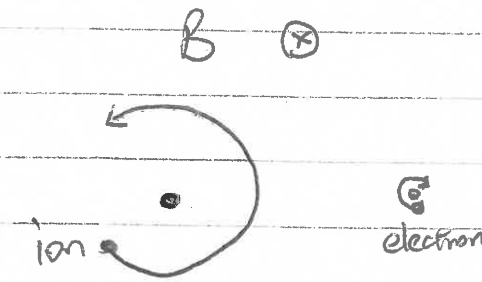


2. Cyclotron Frequency: ω_{cs}
   a. The characteristic frequency of gyration is given by

$$\boxed{\omega_{cs} = \frac{q_s B}{m_s}}$$

3. Thermal Larmor Radius: r_{Ls}
   a. For particles with a characteristic thermal velocity v_{ts}, the radius of this gyration is

$$\boxed{r_{Ls} \equiv \frac{v_{ts}}{\omega_{cs}}}$$



II. Dimensionless Parameters of a Plasma

A. Why dimensionless?

1. Plasma physicists often use dimensionless quantities to characterize the nature of a given plasma.
2. These dimensionless numbers connect directly to the dynamics of the plasma, regardless of the order of magnitude of the characteristic dimensional quantities
3. Allows for comparison of plasmas in very different environments.
4. The plasma parameter N_D is one such dimensionless number.

B. Plasma Beta

1. $$\beta \equiv \frac{\text{Thermal Pressure}}{\text{Magnetic Pressure}} = \frac{P_{th}}{P_m}$$
   a. $P_{th} = n_0 K(T_i + T_e)$ (for quasineutral plasma $n_i = n_e$)
   b. $P_m = \frac{B_0^2}{2\mu_0}$
   c. $$\boxed{\beta \equiv \frac{2\mu_0 n_0 K(T_i + T_e)}{B_0^2}}$$
2. Low beta plasmas, $\beta \ll 1$ are magnetically dominated.
   Ex. magnetic fusion plasmas, laboratory plasmas, solar corona
3. High beta plasmas, $\beta \gg 1$, have a magnetic field that can be highly deformed by the plasma motions
   Ex: Black hole accretion disks

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II (Continued)

C. Magnetization:

1. Whether a plasma can be considered magnetized is determined by $\frac{r_{Li}}{L}$
   a. $\frac{r_{Li}}{L} \ll 1$ magnetized
   b. $\frac{r_{Li}}{L} \gg 1$ unmagnetized

D. Collisionality:

1. Compare collision frequency ν to observation "frequency" $\frac{1}{\tau}$

$$\frac{\nu}{\left(\frac{1}{\tau}\right)} = \nu\tau$$

   a. $\nu\tau \ll 1$ "collisionless"
   b. $\nu\tau \gg 1$ collisional

2. NOTE: Since $\nu_s = \frac{v_{ts}}{\lambda_m}$ (and suppressing species subscript's")
   a. $\nu\tau = \frac{v_t}{\lambda_m}\tau = \frac{L}{\lambda_m}$ if $v_t \equiv \frac{L}{\tau}$
   b. This agrees with definitions of collisionality earlier in terms of L & λ_m

III. Summary of length scales, time scales, and dimensionless quantities.

Length:		Time/Frequency:		Dimensionless:	
Particle Spacing	$n_0^{-\frac{1}{3}}$	Plasma Frequency	ω_p	Plasma parameter	N_D
Debye Length	λ_D	Cyclotron Frequency	ω_c	Plasma Beta	β
Larmor Radius	r_L	Collision Frequency	ν	Magnetization	r_{Li}/L
Mean Free Path	λ_m	Observation "frequency"	$\frac{1}{\tau}$	Collisionality	$\nu\tau$
System Size	L				