

Lecture #21 Force-Balanced MHD Equilibria

Hewes ①

I. Review of MHD Equilibria

At Nonzero Eq:

$$\text{Force/Vol} = \rho \frac{\partial \underline{U}}{\partial t} + \rho \underline{U} \cdot \nabla \underline{U} = \underbrace{-\nabla p + \underline{j} \times \underline{B} - \rho \nabla \Phi_G}_{\text{Equilibrium has zero net force}} = 0$$

2. Neglecting gravity (generally valid for laboratory plasmas)

$$\nabla p = \underline{j} \times \underline{B} \quad \text{required for MHD equilibrium}$$

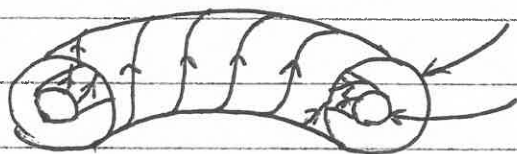
a. ZF $\underline{j} \times \underline{B} \approx 0$ ($|\nabla p| \ll |\underline{j} \times \underline{B}|$) Force-Free

b. ZF $\underline{j} \times \underline{B} \neq 0$ Force-Balanced

3. Hopf's Theorem:

a. A torus is simplest topological surface satisfying $\nabla \cdot \underline{B} = 0$ & $\underline{B} \cdot \nabla \alpha = 0$

Thus, to contain a hot plasma, magnetic field lines lie on closed surfaces of constant pressure:



Nested Toroidal surfaces of constant pressure.

4. Force-Free Equilibrium Solutions

a. Flux Ropes

b. Reverse Field Pinch (RFP)

II. Force-Balanced Equilibria:

A.1. Using Ampere's law, we may write force balance as

2. For $p = p(x)$, $\underline{B} = B(x)\hat{x}$, there exist an infinite number of solutions.

⇒ Let's focus on cylindrical and toroidal geometries

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = \frac{(\underline{B} \cdot \nabla) \underline{B}}{\mu_0}$$

III. Force-Balanced Equilibria in Cylindrical Geometries

A. General Cases

1. We'll focus on the family of solutions with a magnetic field with no radial component and only radial dependence,

$$\vec{B} = B_\phi(r) \hat{\phi} + B_z(r) \hat{z} \quad (B_r = 0)$$

2. The radial component of the force balance (using NRL p. 6-7) gives

$$\frac{d}{dr} \left(p(r) + \frac{[B_\phi(r)]^2}{2\mu_0} + \frac{[B_z(r)]^2}{2\mu_0} \right) = - \frac{[B_\phi(r)]^2}{\mu_0 r}$$

NOTE: Magnetic tension term depends only on B_ϕ .

3. Ampere's Law $\vec{j} = \frac{1}{\mu_0} (\nabla \times \vec{B})$ gives

$$\vec{j} = -\frac{1}{\mu_0} \frac{dB_z}{dr} \hat{\phi} + \frac{1}{\mu_0 r} \frac{d}{dr} (r B_\phi) \hat{z}$$

4. Force balance depends on 3- functions

$$p(r), B_\phi(r), B_z(r)$$

- a. If two are given, the third is determined by force balance (solution to differential equation)
- b. We still have an infinite number of possible solutions \Rightarrow Certain limiting cases simplify the force balance equations.

B. The Z-Pinch:

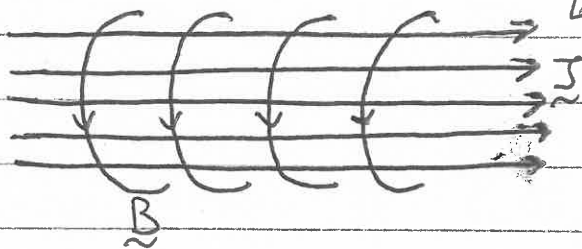
1. Here, we take $B_z = 0$. Therefore $j_\phi = 0$.

2. $\frac{d}{dr} \left(p + \frac{B_\phi^2}{2\mu_0} \right) = - \frac{B_\phi^2}{\mu_0 r}$
 - a. The radial pressure gradient is entirely confined by B_ϕ .

- b. B_ϕ is entirely generated by an axial current j_z .

III. B_0 (Continued) The z-Pinch

3.



Strong axial current through plasma produces a confining azimuthal B_ϕ field.

4. Consider a plasma of radius a with a constant axial current density inside the plasma, $j_z = j_0$.

a. Total Current: $I_0 = \int_0^a \int_0^{2\pi} r dr d\phi j_0 = \pi a^2 j_0 \Rightarrow j_0 = \frac{I_0}{\pi a^2}$

b. Calculate the resulting B_ϕ from Ampere's Law:

i) $j_z = \frac{1}{\mu_0 r} \frac{d}{dr} (r B_\phi)$

ii) $\int_0^r \frac{d}{dr'} (r' B_\phi) dr' = \int_0^r \mu_0 r' j_z dr'$ where $j_z = \begin{cases} \frac{I_0}{\pi a^2} & r \leq a \\ 0 & r > a \end{cases}$

$$= \begin{cases} \int_0^r \frac{\mu_0 I_0}{\pi a^2} r' dr' & r \leq a \\ \int_0^a \frac{\mu_0 I_0}{\pi a^2} r' dr' & r > a \end{cases} = \begin{cases} \frac{\mu_0 I_0 r^2}{2\pi a^2} & r \leq a \\ \frac{\mu_0 I_0 a^2}{2\pi a^2} & r > a \end{cases}$$

iii) Thus $B_\phi(r) = \begin{cases} \frac{\mu_0 I_0}{2\pi a^2} r & r \leq a \\ \frac{\mu_0 I_0}{2\pi a} & r > a \end{cases}$

c. Calculate the pressure $p(r)$

i) $\frac{dp}{dr} = -\frac{d}{dr} \left(\frac{B_\phi^2}{2\mu_0} \right) - \frac{B_\phi^2}{\mu_0 r} = \begin{cases} -\frac{2\mu_0 I_0^2 r}{4\pi^2 a^4} & r \leq a \\ 0 & r > a \end{cases}$

ii) At the edge of the plasma, $r=a$, we take $p(a) = 0$.

Lecture #21 (Continued)

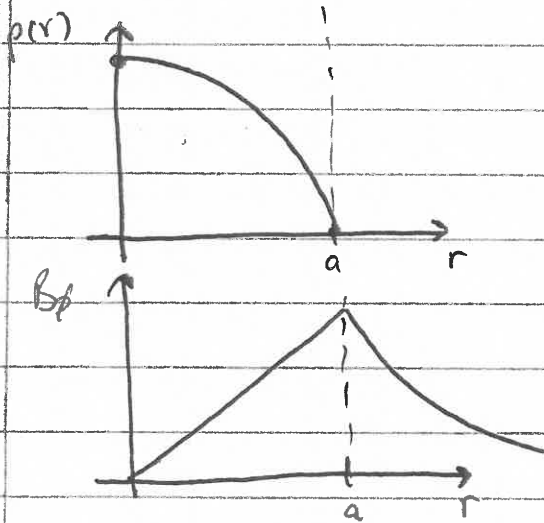
III. B. f. c. (Continued)

Howes ⊕

iii) Thus $\int_r^a \frac{dp}{dr'} dr' = p(a) - p(r) = \int_r^a \frac{-2\mu_0 I_0^2 r}{4\pi^2 a^4} dr = -\frac{\mu_0 I_0^2}{4\pi^2 a^2} (1 - \frac{r^2}{a^2})$ By boundary conditions

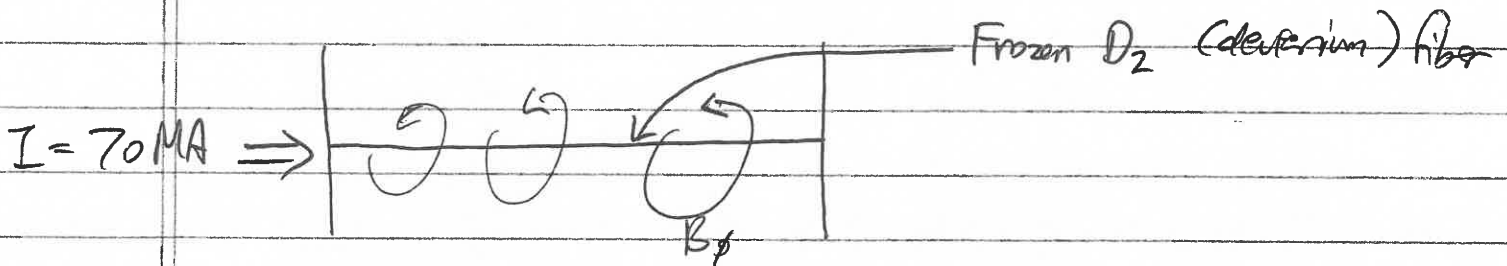
iv) Finally $p(r) = \mu_0 \left(\frac{I_0}{4\pi a}\right)^2 \left(1 - \frac{r^2}{a^2}\right)$ $r \leq a$

5. Profiles of p & B_θ

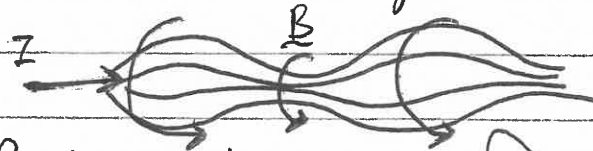


Magnetic pressure and magnetic tension confine thermal pressure.

6. The Z-Pinch at Sandia National Laboratories



- a. Enormous magnetic pressure (and tension) due to axial current confines plasma of deuterium in small volume.
- b. Unstable to "Sausage" instability



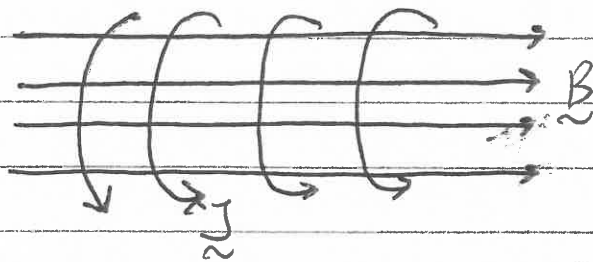
(Next semester we'll look at MHD Stability)

a. produces copious x-rays from hot plasma. Useful for studies of high-energy-density plasmas!

II. (Continued)

C. The "Theta" Pinch

- In this case, we take $B_\phi = 0$ and can solve for $p(r)$ in terms of $B_z(r)$.

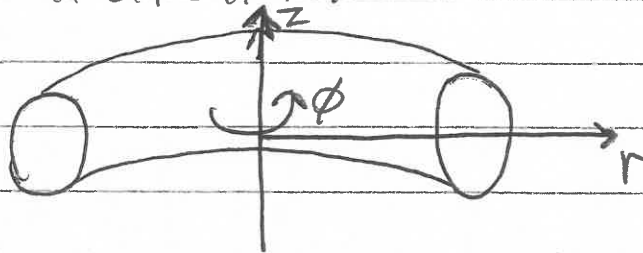


This is a homework problem!

Current in ϕ direction produces B_z that confines pressure.

III. Force-Balanced Equilibria in Toroidal Geometries

- Although the cylindrical cases give us a good intuition of Force-Balanced MHD Equilibrium, it is toroidal geometries that are necessary to have confinement in 3-D (Rohr's theorem tells us we must at least have a torus).
- We now consider toroidal geometries with symmetry in the toroidal direction ϕ .



B. Magnetic Flux Coordinates:

- For $\frac{\partial}{\partial \phi} = 0$, $\nabla \cdot \underline{B} = 0$ implies $\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial B_z}{\partial z} = 0$.
- If we use the vector potential $\underline{B} = \nabla \times \underline{A}$, then $\nabla \cdot \underline{B} = 0$ is automatically satisfied.

Lecture #21 (Continued)
 III. B. (Continued)

Hawes 6

3. The toroidal symmetry implies a simplification ($\frac{\partial}{\partial \phi} = 0$)

From NRL p.61

$$B_r = (\nabla \times \underline{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} = -\frac{\partial A_\phi}{\partial z}$$

$$B_z = (\nabla \times \underline{A})_z = \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi} = \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi)$$

Both B_r & B_z depend only on A_ϕ !

4. Defining a flux function $\psi(r, z) = r A_\phi$, we get

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}$$

$$B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

5. NOTE: For $\psi(r, z)$, $\underline{B} \cdot \nabla \psi = B_r \frac{\partial \psi}{\partial r} + B_z \frac{\partial \psi}{\partial z} = B_r(B_z r) + B_z(-B_r r) = 0$

a. Thus,

$$\underline{B} \cdot \nabla \psi = 0$$

Magnetic field lines must lie on surfaces of $\psi = \text{constant}$.

b. But, we also know

$$\underline{B} \cdot \nabla p = 0,$$

from the force balance, so

$$p = p(\psi)$$

6. Consider the ϕ -component of $\underline{j} \times \underline{B} = \nabla p$:

$$a. j_z B_r - j_r B_z = \frac{1}{r} \frac{\partial p}{\partial \phi} \quad \text{but} \quad \frac{1}{r} \frac{\partial p(\psi)}{\partial \phi} = 0 \quad \text{since} \quad \psi = \psi(r, z).$$

NRL p.6

$$b. \text{Ampere's Law gives: } j_r = \frac{1}{\mu_0} (\nabla \times \underline{B})_r = \frac{1}{\mu_0} \left[\frac{1}{r} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right]$$

$$j_z = \frac{1}{\mu_0} (\nabla \times \underline{B})_z = \frac{1}{\mu_0} \left[\frac{1}{r} \frac{\partial}{\partial r}(r B_\phi) - \frac{1}{r} \frac{\partial B_r}{\partial \phi} \right]$$

As with B_z in terms of A_ϕ , j depends only on $B_\phi(r, z)$.

c. Analogously, we define a second flux function $F = r B_\phi$ so
 succinctly $j_z B_r - j_r B_z = 0$.

Lecture #21 (Continued)
 III. B. G. (Continued)

Howes (7)

d. It follows that $B_r \frac{\partial F}{\partial r} + B_z \frac{\partial F}{\partial z} = 0 \Rightarrow \underline{B} \cdot \nabla F = 0$

and so $\boxed{F = F(\psi)}$

7. Thus, we can express the toroidally symmetric magnetic field in terms of two scalar Flux Functions $\psi(r, z), F(r, z)$.

a. $\boxed{\underline{B} = \left(-\frac{1}{r} \frac{\partial \psi}{\partial z}\right) \hat{r} + \left(\frac{1}{r} \frac{\partial \psi}{\partial r}\right) \hat{z} + \left(\frac{F(\psi)}{r}\right) \hat{\phi}}$

Magnetic Field expressed using Magnetic Flux Coordinates

b. This can also be expressed $\underline{B} = \frac{\nabla \psi}{r} \times \hat{\phi} + \frac{F(\psi)}{r} \hat{\phi}$

C. The Grad-Shafranov Equation:

1. Now, let's consider the r-component of the force balance.

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial r}$$

j_ϕ

2. First use Ampere's law to calculate j_ϕ

$$\mu_0 j_\phi = (\nabla \times \underline{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \equiv -\frac{1}{r} \Delta^* \psi$$

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

j_z

3. Similarly

$$\mu_0 j_z = (\nabla \times \underline{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) - \frac{1}{r} \frac{\partial B_r}{\partial \phi} = \frac{1}{r} \frac{\partial F}{\partial r}$$

$F = r B_\phi$

4. Thus $-\frac{1}{r} \Delta^* \psi B_z - \frac{1}{r} \frac{\partial F}{\partial r} B_\phi = \mu_0 \frac{\partial p}{\partial r}$

a. Again, substituting for B_z & B_ϕ gives

$$-\frac{1}{r} \Delta^* \psi \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) - \frac{1}{r} \frac{\partial F}{\partial r} \left(\frac{F}{r} \right) = \mu_0 \frac{\partial p}{\partial r}$$

Lecture #21 (Continued)
 III. C. (Continued)

Hawes (8)

5. NOTE: $F = F(\psi)$ and $p = p(\psi)$, so

$$\frac{\partial(F^2)}{\partial r} = \frac{d(F^2)}{d\psi} \frac{\partial\psi}{\partial r} \quad \text{and} \quad \frac{\partial p}{\partial r} = \frac{dp}{d\psi} \frac{\partial\psi}{\partial r}$$

Thus

$$-\frac{1}{r^2} \Delta^* \psi \left(\frac{\partial\psi}{\partial r} \right) - \frac{1}{2r^2} \frac{d(F^2)}{d\psi} \left(\frac{\partial\psi}{\partial r} \right) = \mu_0 \frac{dp}{d\psi} \left(\frac{\partial\psi}{\partial r} \right)$$

6. Finally, we obtain

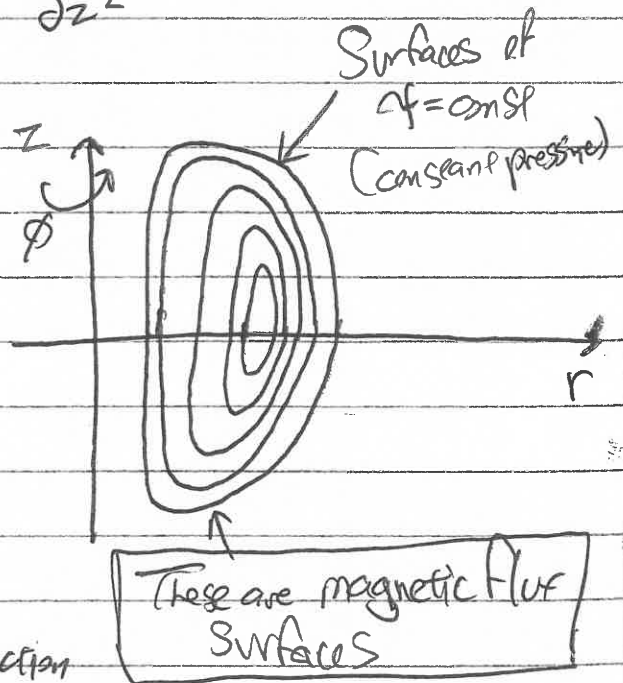
$$\Delta^* \psi = -\mu_0 r^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi}$$

Grad-Shafranov Equation

where $\Delta^* \psi \equiv r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2}$

D. Application

1. Grad-Shafranov equation is used to calculate Magnetostatic Equilibria in axisymmetric toroidal systems.



2. In practice:

- a. Specify $p(\psi)$ and $F(\psi)$
 Pressure \nearrow Toroidal Field Function \nwarrow

b. Solve Grad-Shafranov Eq (Numerically) with specified boundary conditions for $\psi(r, z)$

c. Pressure profile is then determined $p = p(\psi(r, z))$

3. Exact analytical solutions, known as Solov'ev equilibria, are often used in analysis of toroidal magnetic fusion devices.