I. The Cold Plasma Approximation

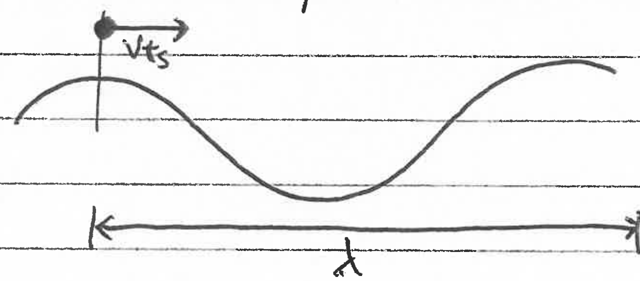
A. 1. MHD is the appropriate description of plasma dynamics for a plasma under strongly collisional, magnetized, and non-relativistic conditions. It provides the low-frequency (compared to ω_{ci}) limit of plasma behavior.

2. Many higher frequency plasma waves are well-described by the cold plasma equations.

B. When is a plasma cold?

1. The cold plasma approximation is appropriate when the thermal spread in velocities of different electrons and ions is ignored.

2. The cold plasma description will work when the distance travelled at the thermal velocity over one wave period is small compared to the wavelength.

a. 

Wave period $T = \frac{2\pi}{\omega}$

Using $k = \frac{2\pi}{\lambda}$

$$\frac{\left(\begin{array}{c}\text{Distance travelled}\\ \text{due to thermal motion}\end{array}\right)}{\text{Wave length}} = \frac{(v_{ts})\left(\frac{2\pi}{\omega}\right)}{\lambda} \ll 1 \Rightarrow v_{ts} \ll \frac{\omega}{k} \equiv v_p$$

b. Thus, thermal velocity must be small compared to the phase velocity of the wave, v_p.

c. Since electrons have a small mass $m_e \ll m_i$, $v_{te} \gg v_{ti}$ for the same temperature $T_i = T_e$.

d. Therefore, we require $\boxed{v_{te} \ll \frac{\omega}{k}}$ Cold Plasma Approximation

2. (Continued)

C. Two Fluid Treatment:

1. The requirement $v_{te} \ll \frac{\omega}{k}$ implies we will be looking at high-frequency phenomena.
   a. At high frequencies, much of the physics occurs because of different ion and electron responses (due to $m_i \gg m_e$). Thus, we need a Two-Fluid Approach
   b. High frequencies also suggest $\omega \gg \nu_{ei}$, so collisional effects may be neglected.
   c. Finally, the thermal pressure term may be neglected.

II. Cold Plasma Equations

A. Neglecting viscosity, pressure, and collisional drag in the Two-Fluid Equations yields the cold Plasma Equations:

Continuity Eq: $\frac{\partial n_s}{\partial t} + \underset{\sim}{U_s} \cdot \nabla n_s = -n_s \nabla \cdot \underset{\sim}{U_s}$

Momentum Eq: $m_s n_s \left[\frac{\partial \underset{\sim}{U_s}}{\partial t} + \underset{\sim}{U_s} \cdot \nabla \underset{\sim}{U_s} \right] = q_s n_s (\underset{\sim}{E} + \underset{\sim}{U_s} \times \underset{\sim}{B})$

Maxwell's Eq's:
$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$$
$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$
$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$
$$\nabla \cdot \underset{\sim}{B} = 0$$

where Charge Density: $\rho_q = \sum_s n_s q_s$
Current Density: $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U_s}$

1. NOTE: There is no closure problem in the Cold Plasma Equations.
   a. 14 Equations for 14 unknowns: $n_i, n_e, \underset{\sim}{U_i}, \underset{\sim}{U_e}, \underset{\sim}{E}, \underset{\sim}{B}$
   b. Effectively, the limit $T_s \rightarrow 0$ (Cold plasma limit) acts as the Equation of State, closing the system

III. Waves in a Cold, Unmagnetized Plasma:

Once again we Linearize, Fourier Transform, and Solve

A. Linearization:

1. Consider a single species, singly ionized plasma.
   We take a homogeneous system with no fields or flows.

   a. $n_{i0} = n_{e0}$
   b. $q_i = -q_e$
   } Neutrality of Equilibrium $\rho_{q0} = \sum_s n_{s0} q_s = q_i n_{i0} + q_e n_{e0} = 0$

   But, $\rho_{q1} = q_i n_{i1} + q_e n_{e1} \neq 0$ (in general)

   c. $\underset{\sim}{U}_{i0} = 0$, $\underset{\sim}{U}_{e0} = 0$
   d. $\underset{\sim}{B}_0 = 0$, $\underset{\sim}{E}_0 = 0$

2. Take
$$n_i = n_{i0} + \epsilon n_{i1}, \quad n_e = n_{e0} + \epsilon n_{e1}$$
$$\underset{\sim}{U}_i = \epsilon \underset{\sim}{U}_{i1}, \quad \underset{\sim}{U}_e = \epsilon \underset{\sim}{U}_{e1}$$
$$\underset{\sim}{B} = \epsilon \underset{\sim}{B}_1, \quad \underset{\sim}{E} = \epsilon \underset{\sim}{E}_1$$

3. Continuity Eq:
$$\cancelto{0}{\frac{\partial n_{i0}}{\partial t}} + \epsilon \frac{\partial n_{i1}}{\partial t} + \epsilon \underset{\sim}{U}_{i1} \cdot \overbrace{\cancelto{0}{\nabla n_{i0}}}^{\text{uniform}} + \epsilon^2 \underset{\sim}{U}_{i1} \cdot \nabla n_{i1} = -\epsilon n_{i0} \nabla \cdot \underset{\sim}{U}_{i1} + \epsilon^2 n_{i1} \nabla \cdot \underset{\sim}{U}_{i1}$$

$$\mathcal{O}(\epsilon) \Rightarrow \boxed{\frac{\partial n_{i1}}{\partial t} = -n_{i0} \nabla \cdot \underset{\sim}{U}_{i1}} \quad \text{Similarly} \quad \boxed{\frac{\partial n_{e1}}{\partial t} = -n_{e0} \nabla \cdot \underset{\sim}{U}_{e1}}$$

4. Momentum Eq:
$$\epsilon m_i n_{i0} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} + \epsilon^2 m_i n_{i1} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} + \epsilon^2 m_i n_{i0} \underset{\sim}{U}_{i1} \cdot \nabla \underset{\sim}{U}_{i1} + \epsilon^3 m_i n_{i1} \underset{\sim}{U}_{i1} \cdot \nabla \underset{\sim}{U}_{i1}$$
$$= \epsilon q_i n_{i0} \underset{\sim}{E}_1 + \epsilon^2 q_i n_{i1} \underset{\sim}{E}_1 + \epsilon^2 q_i n_{i0} \underset{\sim}{U}_{i1} \times \underset{\sim}{B}_1 + \epsilon^3 q_i n_{i1} \underset{\sim}{U}_{i1} \times \underset{\sim}{B}_1$$

$$\mathcal{O}(\epsilon): \quad m_i n_{i0} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} = q_i n_{i0} \underset{\sim}{E}_1 \Rightarrow \boxed{\frac{\partial \underset{\sim}{U}_{i1}}{\partial t} = \frac{q_i}{m_i} \underset{\sim}{E}_1} \text{ and } \boxed{\frac{\partial \underset{\sim}{U}_{e1}}{\partial t} = \frac{q_e}{m_e} \underset{\sim}{E}_1}$$

III. A. (Continued)

5. Poisson's Equation: $\epsilon \nabla \cdot \underset{\sim}{E}_1 = \underbrace{\cancel{\frac{(q_i n_{i0} + q_e n_{e0})}{\epsilon_0}}}_{}^{=0 \text{ by Neutrality of equilibrium}} + \epsilon \frac{q_i n_{i1} + q_e n_{e1}}{\epsilon_0}$

$$\boxed{\nabla \cdot \underset{\sim}{E}_1 = \frac{1}{\epsilon_0}(q_i n_{i1} + q_e n_{e1})}$$

6. Faraday Law: $\boxed{\nabla \times \underset{\sim}{E}_1 = -\frac{\partial \underset{\sim}{B}_1}{\partial t}}$

7. Ampere-Maxwell Law:

$$\epsilon \nabla \times \underset{\sim}{B}_1 = \epsilon \mu_0 (q_i n_{i0} \underset{\sim}{U}_{i1} + q_e n_{e0} \underset{\sim}{U}_{e1}) + \epsilon^2 \mu_0 (q_i n_{i1} \underset{\sim}{U}_{i1} + q_e n_{e1} \underset{\sim}{U}_{e1}) + \epsilon \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}_1}{\partial t}$$

$O(\epsilon)$: $\boxed{\nabla \times \underset{\sim}{B}_1 = \mu_0 (q_i n_{i0} \underset{\sim}{U}_{i1} + q_e n_{e0} \underset{\sim}{U}_{e1}) + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}_1}{\partial t}}$

8. Divergence of B: $\boxed{\nabla \cdot \underset{\sim}{B}_1 = 0}$

Remember:

$\boxed{\frac{\partial}{\partial t} \Rightarrow -i\omega \quad \nabla \Rightarrow i\underset{\sim}{k}}$

B. Fourier Transform: Taking plane wave solutions $\propto e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}$

1. Continuity Eq: $-i\omega n_{i1} = -i \underset{\sim}{k} \cdot \underset{\sim}{U}_{i1} n_{i0} \Rightarrow$ $\boxed{\omega n_{i1} = n_{i0} \underset{\sim}{k} \cdot \underset{\sim}{U}_{i1}}$

$\boxed{\omega n_{e1} = n_{e0} \underset{\sim}{k} \cdot \underset{\sim}{U}_{e1}}$

2. Momentum Eq: $\boxed{-i\omega \underset{\sim}{U}_{i1} = \frac{q_i}{m_i} \underset{\sim}{E}_1}$ $\boxed{-i\omega \underset{\sim}{U}_{e1} = \frac{q_e}{m_e} \underset{\sim}{E}_1}$

3. Poisson's Eq: $\boxed{i \underset{\sim}{k} \cdot \underset{\sim}{E}_1 = \frac{1}{\epsilon_0}(q_i n_{i1} + q_e n_{e1})}$

4. Faraday's Law: $i \underset{\sim}{k} \times \underset{\sim}{E}_1 = +i\omega \underset{\sim}{B}_1 \Rightarrow$ $\boxed{\omega \underset{\sim}{B}_1 = \underset{\sim}{k} \times \underset{\sim}{E}_1}$

5. Ampere-Maxwell Law: $i \underset{\sim}{k} \times \underset{\sim}{B}_1 = \mu_0 (q_i n_{i0} \underset{\sim}{U}_{i1} + q_e n_{e0} \underset{\sim}{U}_{e1}) - i\omega \mu_0 \epsilon_0 \underset{\sim}{E}_1$

$$\boxed{\underset{\sim}{k} \times \underset{\sim}{B}_1 = -i \mu_0 (q_i n_{i0} \underset{\sim}{U}_{i1} + q_e n_{e0} \underset{\sim}{U}_{e1}) - \omega \mu_0 \epsilon_0 \underset{\sim}{E}_1}$$

Lecture #22 (Continued)

III. B. (Continued)

6. Div $\cdot \underset{\sim}{B}$: $\quad i\underset{\sim}{k}\cdot\underset{\sim}{B}_1 = 0 \quad \Rightarrow \quad \boxed{\underset{\sim}{k}\cdot\underset{\sim}{B}_1 = 0}$

7. Collecting our Linearized, Fourier Transformed Equations, we have:

Continuity: $\omega n_{i1} = n_{i0}\,\underset{\sim}{k}\cdot\underset{\sim}{U}_{i1}$

$\omega n_{e1} = n_{e0}\,\underset{\sim}{k}\cdot\underset{\sim}{U}_{e1}$

Momentum: $\omega \underset{\sim}{U}_{i1} = i\frac{q_i}{m_i}\underset{\sim}{E}_1$

$\omega \underset{\sim}{U}_{e1} = i\frac{q_e}{m_e}\underset{\sim}{E}_1$

Poisson: $i\underset{\sim}{k}\cdot\underset{\sim}{E}_1 = \frac{1}{\epsilon_0}(q_i n_{i1} + q_e n_{e1})$

Faraday's: $\omega \underset{\sim}{B}_1 = \underset{\sim}{k}\times\underset{\sim}{E}_1$

Ampere-Maxwell: $\omega \underset{\sim}{E}_1 = -i\frac{1}{\epsilon_0}(q_i n_{i0}\underset{\sim}{U}_{i1} + q_e n_{e0}\underset{\sim}{U}_{e1}) - \frac{1}{\mu_0\epsilon_0}\underset{\sim}{k}\times\underset{\sim}{B}_1$

C. Solve for Equations in Matrix Form:

1. As it turns out, we don't need all of the equations to solve for our system.
We need only Momentum Eq's, Faraday's & Ampere-Maxwell's Laws.

2. Eliminate $\underset{\sim}{U}_{s1}$ from Ampere-Maxwell using Momentum Equations: $\underset{\sim}{U}_{s1} = i\frac{q_s}{m_s\omega}\underset{\sim}{E}_1$

a. $\omega \underset{\sim}{E}_1 = \frac{-i}{\epsilon_0}\left[q_i n_{i0}\left(\frac{iq_i}{m_i\omega}\underset{\sim}{E}_1\right) + q_e n_{e0}\left(\frac{iq_e}{m_e\omega}\underset{\sim}{E}_1\right)\right] - \frac{1}{\mu_0\epsilon_0}\underset{\sim}{k}\times\underset{\sim}{B}_1$

b. Noting that $\omega_{ps}^2 = \frac{n_{s0}q_s^2}{\epsilon_0 m_s}$ and $\frac{1}{\mu_0\epsilon_0} = c^2$, this is

$$\boxed{\omega \underset{\sim}{E}_1 = \left(\frac{\omega_{pi}^2 + \omega_{pe}^2}{\omega}\right)\underset{\sim}{E}_1 - c^2\,\underset{\sim}{k}\times\underset{\sim}{B}_1}$$

Lecture #23 (Continued)

III. C. (Continued)

3. Eliminate $\mathbf{B}_1$ using Faraday's Law:

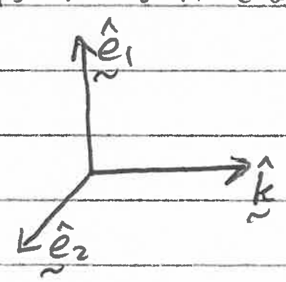
a. $$\omega \mathbf{E}_1 = \left(\frac{\omega_{pi}^2 + \omega_{pe}^2}{\omega}\right)\mathbf{E}_1 - c^2 \mathbf{k} \times \left[\frac{\mathbf{k}\times\mathbf{E}_1}{\omega}\right]$$

b. Simplifying

$$\boxed{\left[\omega^2 - (\omega_{pi}^2 + \omega_{pe}^2)\right]\mathbf{E}_1 + c^2\, \mathbf{k}\times(\mathbf{k}\times\mathbf{E}_1) = 0}$$

4. Let's define an orthonormal coordinate system
$\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{k}}$ such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{k}}$

where $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$ is the unit vector in the direction of the wave vector.



5. Specify components of $\mathbf{E}_1$: $\mathbf{E}_1 = E_{T_1}\hat{\mathbf{e}}_1 + E_{T_2}\hat{\mathbf{e}}_2 + E_L \hat{\mathbf{k}}$

a. Here, the Longitudinal component E_L is in the $\hat{\mathbf{k}}$ direction.

b. The Transverse component $\mathbf{E}_T = E_{T_1}\hat{\mathbf{e}}_1 + E_{T_2}\hat{\mathbf{e}}_2$

6. Simplify: $\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_1) = k^2\, \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{E}_1)$

$$\underset{\text{NRL p.4 (2)}}{=} k^2\left[\hat{\mathbf{k}}(\mathbf{E}_1\cdot\hat{\mathbf{k}}) - \mathbf{E}_1(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})\right] = k^2\left[E_L\hat{\mathbf{k}} - \mathbf{E}_1\right]$$

$$= -k^2\, \mathbf{E}_T$$

III. C. (Continued)

7. Thus, we find

a. $\left[\omega^2 - (\omega_{pi}^2 + \omega_{pe}^2)\right] \underset{\sim}{E}_1 - c^2k^2 \underset{\sim}{E}_T = 0$

b. In Matrix Form:

$$\begin{pmatrix} \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) - c^2k^2 & 0 & 0 \\ 0 & \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) - c^2k^2 & 0 \\ 0 & 0 & \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) \end{pmatrix} \begin{pmatrix} E_{T_1} \\ E_{T_2} \\ E_L \end{pmatrix} = 0$$

8. Setting the determinant of this matrix $D(\omega, \underset{\sim}{k}) = 0$ yields the dispersion relation:

$$\boxed{D(\omega, \underset{\sim}{k}) = \left[\omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) - c^2k^2\right]^2 \left[\omega^2 - (\omega_{pi}^2 + \omega_{pe}^2)\right] = 0}$$

9. a. NOTE: Because the matrix is diagonal, each of the components E_{T_1}, E_{T_2} & E_L is decoupled from the others

b. We can investigate each of these waves separately.

D. Longitudinal Wave: Plasma Oscillations

1. For $E_{T_1} = E_{T_2} = 0$ and $E_L \neq 0$, we have the solution

$$\boxed{\omega^2 = \omega_{pi}^2 + \omega_{pe}^2}$$

2. This is a plasma oscillation at the plasma frequency $\boxed{\omega_p^2 \equiv \omega_{pi}^2 + \omega_{pe}^2}$

a. NOTE: $\dfrac{\omega_{pe}^2}{\omega_{pi}^2} = \dfrac{\frac{n_{e0} \cancel{e^2}}{\cancel{\epsilon_0} m_e}}{\frac{\cancel{n_{i0} q_i^2}}{\cancel{\epsilon_0} m_i}} = \dfrac{m_i}{m_e} \gg 1$ $\left(\dfrac{m_i}{m_e} = 1836 \text{ for protons}\right)$

b. Thus, we often take $\omega_p^2 \simeq \omega_{pe}^2$

c. Plasma oscillations are dominated by the electrons!

III. D. (Continued)

3. a. NOTE: $\underset{\sim}{v_g} = \frac{\partial \omega}{\partial \underset{\sim}{k}} = 0$ for plasma oscillations (No dependence on $\underset{\sim}{k}$)

b. These oscillations do not propagate.

c. This is why we call it an oscillation, rather than a wave.

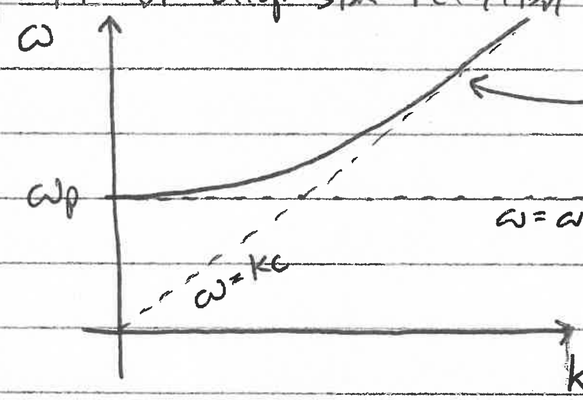
E. Transverse Wave: $\boxed{\text{Modified Light Wave}}$

1. a. The two transverse components are the same.

b. Let's consider $E_{T_1} \neq 0$ $E_{T_2} = E_L = 0$.

2. Solution: $\boxed{\omega^2 = \underbrace{(\omega_{pi}^2 + \omega_{pe}^2)}_{\equiv \omega_p^2 \simeq \omega_{pe}^2} + c^2 k^2}$

3. Plot of dispersion relation $\omega = \pm\sqrt{\omega_p^2 + c^2 k^2}$



a. For $ck \gg \omega_p^2$, we get $\omega \simeq \pm ck$.

This is just the dispersion relation for light waves.