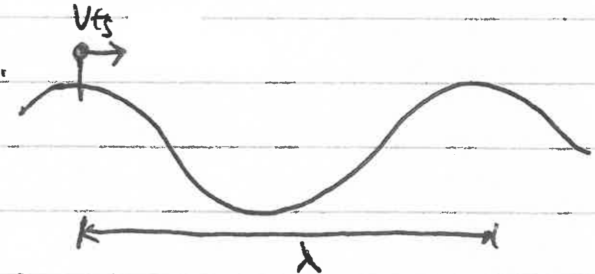
Lecture #23: Properties of Waves in Cold, Unmagnitized Plasmas

I. Review of the Derivation of the Cold, Unmagnetized Plasma Dispersion Relation

A. Cold Plasma Approximation

1. $\boxed{V_{te} \ll \dfrac{\omega}{k}}$ ← wave phase velocity

2.



Particle moves only a fraction of wavelength λ in one period $T = \frac{2\pi}{\omega}$.

3. Ions & electrons may respond differently

B. Cold Plasma Equations

1. Continuity: $\dfrac{\partial n_s}{\partial t} + \underset{\sim}{U_s} \cdot \nabla n_s = -n_s \nabla \cdot \underset{\sim}{U_s}$

2. Momentum: $m_s n_s \left[\dfrac{\partial \underset{\sim}{U_s}}{\partial t} + \underset{\sim}{U_s} \cdot \nabla \underset{\sim}{U_s} \right] = q_s n_s (\underset{\sim}{E} + \underset{\sim}{U_s} \times \underset{\sim}{B})$

3. Maxwell's Eqs:

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0} \qquad \rho_q = \sum_s n_s q_s$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \qquad \underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U_s}$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

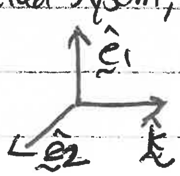
4. Cold Plasma Closure $\underset{\approx}{P_s} = 0 \Rightarrow$ Leads to closed set of equations for $n_i, n_e, \underset{\sim}{U_i}, \underset{\sim}{U_e}, \underset{\sim}{E}, \underset{\sim}{B}$

C. Cold, Unmagnitized Plasma Dispersion Relation

1. Linearize, Fourier Transform, and Solve

2.
$$\begin{pmatrix} \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) - c^2k^2 & 0 & 0 \\ 0 & \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) - c^2k^2 & 0 \\ 0 & 0 & \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2) \end{pmatrix} \begin{pmatrix} E_{T_1} \\ E_{T_2} \\ E_L \end{pmatrix} = 0$$

3. In unmagnitized system, $\underset{\sim}{k}$ is only specified direction:



$$\underset{\sim}{\hat{e}_1} \times \underset{\sim}{\hat{e}_2} = \underset{\sim}{\hat{k}}$$

I.C. (Continued)

4. NOTE: A Common Point of Confusion

a. In a magnetized plasma (such as MHD), we refer to components relative to the mean magnetic field $\underset{\sim}{B}_0$ as parallel & perpendicular

b. In an unmagnetized plasma, we refer to components relative to the wave vector $\underset{\sim}{k}$ as longitudinal & transverse.

5. Solutions to the Cold, Unmagnetized Plasma Dispersion Relation are (1) Longitudinal Plasma Oscillation
(2) Transverse Modified Light Waves.

II. Properties of Longitudinal Mode in a Cold, Unmagnetized Plasma

A. Characteristics of Plasma Oscillations

1. Longitudinal modes occur when $\underset{\sim}{E}_T = 0$ and $E_L \neq 0$.

a. In this case, the frequency must satisfy the ~~wod~~ dispersion relation

$$\boxed{\omega^2 = \omega_{pi}^2 + \omega_{pe}^2 \equiv \omega_p^2}$$

b. These are $\boxed{\text{Plasma Oscillations}}$ at the plasma frequency ω_p

2. $\underset{\sim}{E}_1 = E_L \hat{\underset{\sim}{k}}$

a. Thus $\underset{\sim}{k} \times \underset{\sim}{E}_1 = \underset{\sim}{k} \times E_L \hat{\underset{\sim}{k}} = 0 \Rightarrow$ Longitudinal $\boxed{\underset{\sim}{k} \times \underset{\sim}{E}_1 = 0}$

b. The linearized, Fourier Transformed Faraday's Law gives

$$\underset{\sim}{B}_1 = \frac{\underset{\sim}{k}}{\omega} \times \underset{\sim}{E}_1 = 0$$

$\Rightarrow$ $\boxed{\text{Plasma Oscillations have no associated magnetic field Fluctuation}}$

II. A.2 (Continued)

c. $\underset{\sim}{k} \times \underset{\sim}{E}_1 = 0 \Rightarrow \nabla \times \underset{\sim}{E}_1 = 0$

So, we can write the electric field as the gradient of an <u>electrostatic potential</u> ϕ, $\underset{\sim}{E}_1 = -\nabla \phi$

$\Rightarrow$ $\boxed{\text{Plasma Oscillations are Electrostatic}}$

d. Poisson's Eq: $i \underset{\sim}{k} \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$. Since $k \cdot E \neq 0$, $\boxed{\text{Plasma Oscillations have charge density fluctuations } \rho_q}$

3. Group velocity: $\underset{\sim}{v}_g = \frac{\partial \omega}{\partial \underset{\sim}{k}} = 0$

$\Rightarrow$ $\boxed{\text{Plasma Oscillations do not propagate}}$

4. Poynting Flux: $\underset{\sim}{S} \equiv \frac{1}{\mu_0} \underset{\sim}{E} \times \underset{\sim}{B} = 0$

$\Rightarrow$ $\boxed{\text{There is no flow of Electromagnetic Energy}}$

5. We'll see in the next lecture that a finite temperature (a warm, rather a cold, plasma) causes plasma oscillations to propagate.

III. <u>Properties of Transverse Modes in a Cold, Unmagnetized Plasma</u>

A. <u>Characteristics of Modified Light Waves</u>

1. Transverse Mode occurs when either a) $E_{T_1} \neq 0$, $E_{T_2} = 0$, $E_L = 0$

OR b) $E_{T_1} = 0$, $E_{T_2} \neq 0$, $E_L = 0$

$\boxed{\omega^2 = \omega_p^2 + c^2 k^2}$

$\Rightarrow$ There are two polarizations, analogous to the two polarizations of light.

2. Consider $\underset{\sim}{E}_1 = E_{T_1} \hat{\underset{\sim}{e}}_1$

$\hat{\underset{\sim}{e}}_1 \times \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{k}}$

a. $\underset{\sim}{k} \times \underset{\sim}{E} = k \hat{\underset{\sim}{k}} \times E_{T_1} \hat{\underset{\sim}{e}}_1 = k E_{T_1} \hat{\underset{\sim}{e}}_2$

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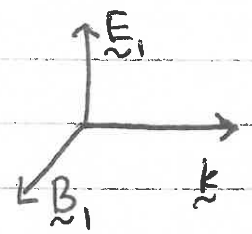
III. A. 2. (Continued)

b. Faraday's Law gives $\underset{\sim}{B}_1 = \frac{\underset{\sim}{k}}{\omega} \times \underset{\sim}{E}_1 = \frac{k E_{T_1}}{\omega} \hat{\underset{\sim}{e}}_2$

Therefore, $\underset{\sim}{E}_1$, $\underset{\sim}{B}_1$, and $\underset{\sim}{k}$ form an orthogonal triad. $\Rightarrow$



Modified Light Waves are Electromagnetic

Second Polarization:

c. For $\underset{\sim}{E}_1 = E_{T_2} \hat{\underset{\sim}{e}}_2 \Rightarrow \underset{\sim}{B}_1 = \frac{-k E_{T_2}}{\omega} \hat{\underset{\sim}{e}}_1$, so picture looks the same if we rotate about $\underset{\sim}{k}$ by $\frac{\pi}{2}$.

3. Poisson's Equation: $i \underset{\sim}{k} \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$

a. $\underset{\sim}{k} \cdot \underset{\sim}{E} = \underset{\sim}{k} \cdot (E_{T_1} \hat{e}_1) = 0 \quad \Rightarrow \rho_q = 0$

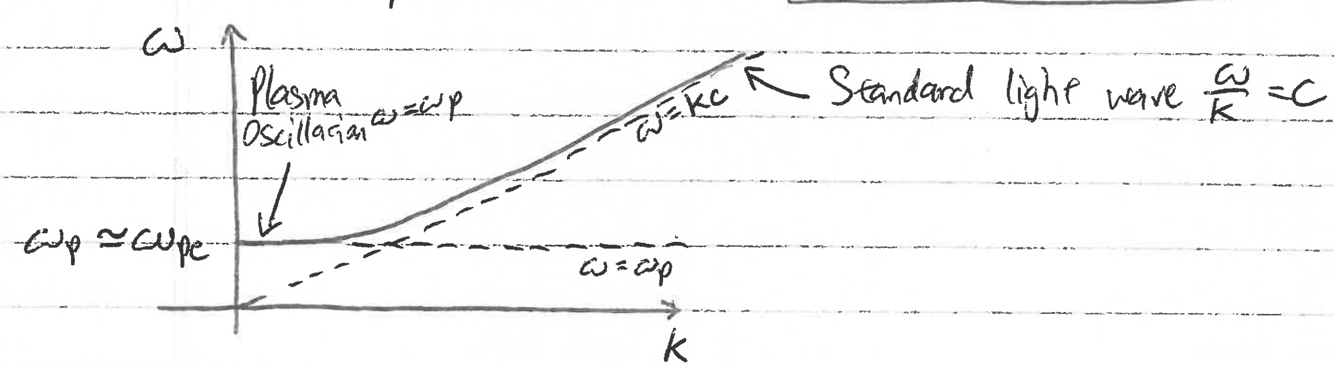
$\Rightarrow$ Modified Light Waves have no charge density fluctuations

4. Poynting Flux: $\underset{\sim}{S} = \frac{1}{\mu_0} \underset{\sim}{E} \times \underset{\sim}{B} = \frac{1}{\mu_0} (E_{T_1} \hat{\underset{\sim}{e}}_1) \times \left(\frac{k E_{T_1}}{\omega} \hat{\underset{\sim}{e}}_2 \right)$

$$\underset{\sim}{S} = \frac{k E_{T_1}^2}{\mu_0 \omega} \hat{\underset{\sim}{k}}$$

$\Rightarrow$ The Poynting Flux of Electromagnetic Energy is in the direction of the wavevector $\underset{\sim}{k}$

5. The dispersion relation is $\boxed{\omega^2 = \omega_p^2 + c^2 k^2}$



III. A. (Continued)

6. What happens for $\omega < \omega_{pe}$?

a. If we attempt to drive a modified light wave in a plasma (see next Section IV.) at $\omega < \omega_p$, what will happen?

i. Solve for k as a function of ω:

$$k = \pm\sqrt{\frac{\omega^2 - \omega_p^2}{c^2}} = \pm \frac{i}{c}\sqrt{\omega_p^2 - \omega^2}$$

ii. Remember, our solution goes as $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$

- In this case, we specify ω and find $k(\omega)$.
- For $\underset{\sim}{k} = k\hat{\underset{\sim}{k}}$, we get $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} = e^{i(kz - \omega t)}$

where we take $\hat{\underset{\sim}{k}} = \hat{z}$ direction. (Thus $\hat{\underset{\sim}{e}}_1 = \hat{x}$, $\hat{\underset{\sim}{e}}_2 = \hat{y}$ for example)

- Then $\underset{\sim}{E}(\underset{\sim}{x}) = \underset{\sim}{E}_0(\underset{\sim}{k}) e^{i(kz - \omega t)}$ and $\underset{\sim}{E}_0(\underset{\sim}{k}) = E_{T_1}\hat{\underset{\sim}{x}}$

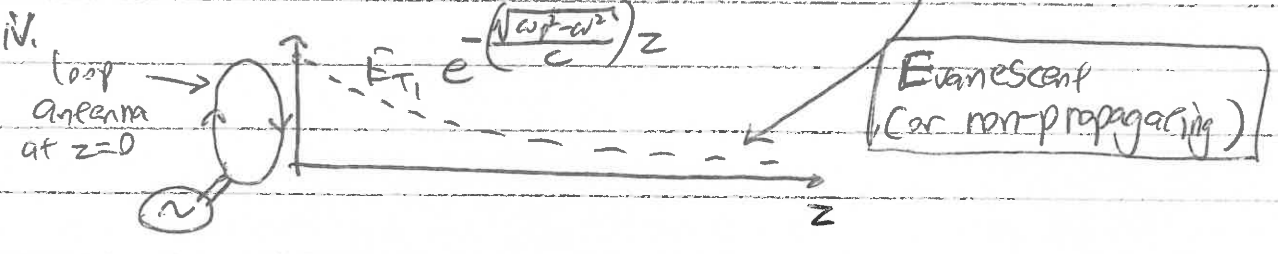
iii. Substituting in for k, we find

$$\underset{\sim}{E}(\underset{\sim}{x}) = E_{T_1} e^{i\left(\left[\frac{i}{c}\sqrt{\omega_p^2 - \omega^2}\right]z - \omega t\right)}\hat{\underset{\sim}{x}}$$

$\leftarrow$ Boundary conditions specify $\oplus$ root.

$$= E_{T_1} e^{-i\omega t} \underbrace{e^{-\frac{z}{c}\sqrt{\omega_p^2 - \omega^2}}}_{\text{This solution decays exponentially in } z!} \hat{\underset{\sim}{x}}$$

iv.



b. In this case, we take real ω and find complex $k(\omega)$

$\Rightarrow$ This is often more appropriate to laboratory experiments

c. Alternatively, theorists usually take k real and find complex $\omega(\underset{\sim}{k})$.

d. Conclusion: $\boxed{\text{Waves don't propagate at } \omega < \omega_p \text{ in a cold, unmagnetized plasma}}$

III A. (Continued)

7. Aside: Compare this to the solution for the parallel-propagating ($\underset{\sim}{k} = k_\parallel \hat{\underset{\sim}{z}}$) Alfven wave in Resistive MHD:

a. $$\omega = \pm k_\parallel v_A \sqrt{1 - \frac{\eta^2 k_\parallel^2}{4\mu_0^2 v_A^2}} - i\frac{\eta k_\parallel^2}{2\mu_0}$$

b. Let's simplify this in the weakly resistive limit $\frac{\eta^2 k_\parallel^2}{4\mu_0^2 v_A^2} \ll 1$,

$$\omega \simeq \pm k_\parallel v_A - i\frac{\eta k_\parallel^2}{2\mu_0}$$

c. Thus, we find

$$\underset{\sim}{U}(\underset{\sim}{x}) = U_y(\underset{\sim}{k}) e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} \hat{\underset{\sim}{y}} = U_y e^{i(k_\parallel z \mp k_\parallel v_A t) - \frac{\eta k_\parallel^2}{2\mu_0} t} \hat{\underset{\sim}{y}}$$

$$\underset{\sim}{U}(\underset{\sim}{x},t) = \underbrace{U_y e^{ik_\parallel(z \mp v_A t)}}_{\text{Alfven waves travelling up \& down field at } z = \pm v_A t} \underbrace{e^{-\frac{\eta k_\parallel^2}{2\mu_0} t}}_{\text{Exponential decay in time.}}$$

$\boxed{\text{Wave propagates in } z \text{ and decays in time}}$

B. Why is propagation of waves cut off at plasma frequency?

1. Physically, it depends on the strength of the conduction current to the displacement current.

2. Ampere-Maxwell Law: $$\nabla \times \underset{\sim}{B} = \underbrace{\mu_0 \underset{\sim}{j}}_{\text{Conduction Current}} + \underbrace{\mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}}_{\text{Displacement current}}$$

3. In free space (no plasma), $\underset{\sim}{j} = 0$ and light waves are supported by the displacement current.

4. When a plasma is present, the conduction current can act to short out the displacement current (mostly electron motion).

III. B. (Continued)

5. As we solved for the Cold, Unmagnetized Plasma Dispersion relation, the Ampere-Maxwell law became:

$$-c^2 \underset{\sim}{k} \times \underset{\sim}{B}_1 = \underbrace{\omega \underset{\sim}{E}_1}_{\text{Displacement}} - \underbrace{\frac{\omega_p^2}{\omega} \underset{\sim}{E}_1}_{\text{Conduction}}$$

6. Ratio of

$$\frac{\text{Conduction Current}}{\text{Displacement Current}} = \frac{-\frac{\omega_p^2}{\omega}\cancel{\underset{\sim}{E}_1}}{\omega \cancel{\underset{\sim}{E}_1}} = -\frac{\omega_p^2}{\omega^2}$$

a. At $\omega \gg \omega_p$, the conduction current has a small effect on the displacement current $\Rightarrow$ $\omega^2 \simeq c^2 k^2$ Light Wave

b. The minus sign indicates that the conduction current opposes the displacement current.

c. At $\omega = \omega_p$, the conduction current cancels out the displacement current of the light wave, so it ceases to propagate.

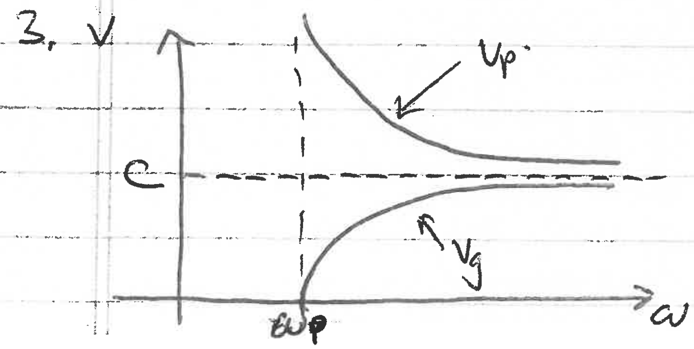
d. For a light frequency $\omega < \omega_p$, the conduction current is able to completely oppose the electric field of the light wave. The light wave becomes evanescent!

C. Group velocity vs. Phase velocity $\quad \omega^2 = \omega_p^2 + c^2 k^2$

1. $V_p = \frac{\omega}{k}$ a. $\frac{\omega^2 - \omega_p^2}{\omega^2} = \frac{c^2 k^2}{\omega^2} \Rightarrow \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$ ($> c$ for $\omega > \omega_p$)

2. $V_g = \frac{\partial \omega}{\partial k}$ a. $2\omega\, \partial\omega = 2c^2 k\, dk \Rightarrow \frac{\partial \omega}{\partial k} = c\frac{ck}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$ ($< c$ for $\omega < \omega_p$)

3.



4. NOTE: $V_p V_g = c^2$

Analogous to light propagation in waveguide.

VI. Driving of Cold, Unmagnetized Plasma Wave Modes

A. External Source Current: Add $\underset{\sim}{j}_{ext}$ to Ampere-Maxwell Law

1. $\nabla \times \underset{\sim}{B} = \mu_0(\underset{\sim}{j} + \underset{\sim}{j}_{ext}) + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$

2. Adding j_{ext} and finding dispersion relation yields

$$\frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\underset{\sim}{E}_1 + \underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_1) = -i\omega\mu_0 \underbrace{\underset{\sim}{j}_{ext}}_{\text{This is Fourier Transform of external current, } \underset{\sim}{j}_{ext}(\underset{\sim}{k})}$$

3. Separating $\underset{\sim}{j}_{ext} = \overset{\text{Transverse}}{\underset{\sim}{j}_{T ext}} + \overset{\text{Longitudinal}}{j_{L ext}\,\hat{\underset{\sim}{k}}}$, the source is decoupled just as the waves to give:

$$(\omega^2 - \omega_p^2 - c^2k^2)\,\underset{\sim}{E}_T = \frac{-i\omega}{\epsilon_0}\,\underset{\sim}{j}_{T ext} \Longleftarrow \text{Excites Modified Light Waves}$$

$$(\omega^2 - \omega_p^2)\,E_L = \frac{-i\omega}{\epsilon_0}\,j_{L ext} \Longleftarrow \text{Excites Plasma Oscillations}$$

4. Interpretation:

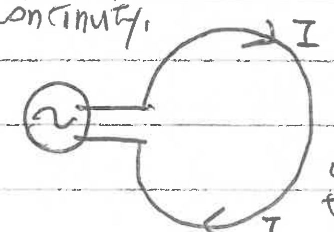
a. $\underset{\sim}{j}_{T ext}$ and $j_{L ext}$ are Fourier transforms of the external current, so interpretation is complicated.

b. But, from the charge-weighted sum of continuity equations,

$$\left.\begin{aligned} q_i\frac{\partial n_i}{\partial t} + \nabla\cdot(q_i n_i \underset{\sim}{U}_i) &= 0 \\ q_e\frac{\partial n_e}{\partial t} + \nabla\cdot(q_e n_e \underset{\sim}{U}_e) &= 0 \end{aligned}\right\} \Rightarrow \underbrace{\frac{\partial \rho_q}{\partial t} + \nabla\cdot\underset{\sim}{j} = 0}_{\text{Charge Continuity.}}$$

c. $\nabla\cdot\underset{\sim}{j} \Rightarrow \underset{\sim}{k}\cdot\underset{\sim}{j}$ i) $\underset{\sim}{k}\cdot\underset{\sim}{j}_{T ext} = 0$

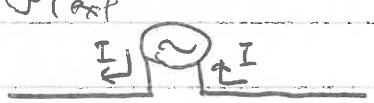
Thus no charge density ρ_q from $\underset{\sim}{j}_{T ext}$



Loop Antenna Can only excite transverse waves (no ρ_q)

ii) $\underset{\sim}{k}\cdot(j_{L ext}\hat{\underset{\sim}{k}}) \neq 0$

Thus charge density ρ_q from $j_{L ext}$



Dipole antenna can excite both Transverse & longitudinal waves