

I. Longitudinal Modes in Warm vs. Cold Unmagnetized Plasmas

A. Longitudinal Modes

1. Last time, we found that, in a cold plasma, longitudinal modes, or Plasma Oscillations, do not propagate.

2. If a plasma has a finite temperature — typically the case in laboratory, space, and astrophysical plasmas — longitudinal modes do propagate as Langmuir waves or Ion acoustic waves.

3. To demonstrate this, we will calculate the dispersion relation when the electron temperature is finite $T_e \neq 0$.

a. We still take $T_i = 0$ (or $T_i \ll T_e$), a cold ion approximation to keep the mathematics simple.

b. In laboratory plasmas, it is common to have $T_i < T_e$.

ASIDE: Why?

1) To ionize a laboratory plasmas, often a hot cathode is used.

2) The hot cathode emits very fast electrons which stream along the magnetic field line leading to ionization of the plasma (collisional ionization)

3) From Lecture #1, we learned about the timescale to achieve thermal equilibrium between species via collisions.

a. Plasma electrons can easily gain energy from (hot electron) - (electron) collisions on a timescale $\tau_{ee} \sim \frac{1}{\nu_{ee}}$

b. But, for ions to reach thermal equilibrium with hot electrons requires a time $\tau_{ie} \sim \frac{m_i}{m_e} \tau_{ee} \sim 1836 \tau_{ee} \gg \tau_{ee}!$

II. Dispersion Relation for a Warm, Unmagnetized Plasma with $T_i \ll T_e$

A. Equations.

1. We now may not neglect the electron pressure term in the electron momentum equation, but we'll continue to neglect ion pressure.

Continuity: $\frac{\partial n_i}{\partial t} + \underline{U}_i \cdot \nabla n_i = -n_i \nabla \cdot \underline{U}_i$ $\frac{\partial n_e}{\partial t} + \underline{U}_e \cdot \nabla n_e = -n_e \nabla \cdot \underline{U}_e$

Momentum: $m_i n_i \left[\frac{\partial \underline{U}_i}{\partial t} + \underline{U}_i \cdot \nabla \underline{U}_i \right] = q_i n_i (\underline{E} + \underline{U}_i \times \underline{B})$

$$m_e n_e \left[\frac{\partial \underline{U}_e}{\partial t} + \underline{U}_e \cdot \nabla \underline{U}_e \right] = -\nabla p_e + q_e n_e (\underline{E} + \underline{U}_e \times \underline{B})$$

Maxwell's Eqs: $\nabla \cdot \underline{E} = \frac{\rho_e}{\epsilon_0}$ $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$
 $\nabla \cdot \underline{B} = 0$ $\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$

2. Closure: We need an additional equation to relate p_e to the other variables to solve this system.

3. Isothermal Equation of State for electrons

a. $\frac{d}{dt} \left(\frac{p_e}{n_e} \right) = 0$

- b. This time we take $\gamma = 1$, corresponding to an isothermal eq. of state.

c. $\Rightarrow p_e = n_e \overset{\uparrow}{\text{constant}} C$ we take the constant $C = k T_e = \text{const}$
 Thus $T_e \uparrow$ is constant.

$$\Rightarrow \boxed{p_e = n_e k T_e}$$

- d. Because electrons can move very rapidly in any direction in an unmagnetized plasma, for slow motions (low freq) an isothermal eq. of state is often used.

III (Continued)

B. Linearization and Fourier Transform of Equations

1. This has been done in detail in lecture #22, so we'll only treat the new electron momentum equation.

2. Assume: $n_e = n_{e0} + \epsilon n_{e1}$

$$U_e = \epsilon U_{e1}$$

$$p_e = p_{e0} + \epsilon p_{e1}$$

$$\underline{E} = \epsilon \underline{E}_1$$

$$\underline{B} = \epsilon \underline{B}_1$$

3. Linearize:

$$\epsilon m_e n_{e0} \frac{\partial U_{e1}}{\partial t} + \epsilon^2 m_e n_{e1} \frac{\partial U_{e1}}{\partial t} + \epsilon^2 \tilde{u}_{e1} \nabla U_e + \epsilon^3 m_e n_{e1} U_{e1} \nabla U_{e1}$$

$$= -\cancel{\nabla p_{e0}} + \epsilon \nabla p_{e1} + \epsilon q_e n_{e0} \underline{E}_1 + \epsilon^2 q_e n_{e1} \underline{E}_1 + \epsilon^2 q_e n_{e0} U_{e1} \nabla \underline{B}_1 + \epsilon^2 q_e n_{e1} U_{e1} \nabla \underline{B}_1$$

$$\mathcal{O}(\epsilon) \Rightarrow \boxed{m_e n_{e0} \frac{\partial U_{e1}}{\partial t} = -\nabla p_{e1} + q_e n_{e0} \underline{E}_1}$$

4. NOTE: Equation of State: $p_{e0} + \epsilon p_{e1} = n_{e0} k T_e + \epsilon n_{e1} k T_e$

a. In our equilibrium, we assume $p_{e0} = n_{e0} k T_e$, so we are left with

$$p_{e1} = n_{e1} k T_e \quad (\text{Remember } T_e = \text{const}) \text{ Isothermal}$$

5. Fourier Transform: Plane wave solutions $\propto e^{i(\underline{k} \cdot \underline{x} - \omega t)}$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega$$

$$\nabla \Rightarrow i\underline{k}$$

a. $-i m_e n_{e0} \omega U_{e1} = -i \underline{k} p_{e1} + q_e n_{e0} \underline{E}_1$

b.
$$\boxed{\omega U_{e1} = \underline{k} \frac{p_{e1}}{m_e n_{e0}} + i \frac{q_e}{m_e} \underline{E}_1}$$

Lecture #24 (Continued)

Haines (4)

II. (Continued)

C. System of Equations:

$$\textcircled{1} \omega n_{i1} = n_{i0} \underline{k} \cdot \underline{U}_{i1}$$

$$\textcircled{2} \omega n_{e1} = n_{e0} \underline{k} \cdot \underline{U}_{e1}$$

$$\textcircled{3} \omega \underline{U}_{i1} = \frac{iq_i}{m_i} \underline{E}_1$$

$$\textcircled{4} \omega \underline{U}_{e1} = \underline{k} \frac{p_{e1}}{m_e n_{e0}} + i \frac{qe}{m_e} \underline{E}_1$$

$$\textcircled{5} i \underline{k} \cdot \underline{E}_1 = \frac{1}{\epsilon_0} (q_i n_{i1} + q_e n_{e1})$$

$$\textcircled{6} \omega \underline{B}_1 = \underline{k} \times \underline{E}_1$$

$$\textcircled{7} \omega \underline{E}_1 = \frac{1}{\epsilon_0} (q_i n_{i0} \underline{U}_{i1} + q_e n_{e0} \underline{U}_{e1}) - c^2 \underline{k} \times \underline{B}_1 \quad \textcircled{8} p_{e1} = n_{e1} k T_e$$

D. Solve in terms of \underline{E}_1 only to yield matrix equation

1. Using $\textcircled{1}$, $\textcircled{3}$, $\textcircled{5}$, and $\textcircled{8}$, we can solve $\textcircled{4}$ in terms of \underline{U}_{e1} and \underline{E}_1 only. The result is

$$\omega \underline{U}_{e1} = \frac{iq_e}{m_e} \left[\frac{c^2}{\omega p_e^2} \left(1 - \frac{\omega p_i^2}{\omega^2} \right) \underline{k} (\underline{k} \cdot \underline{E}_1) + \underline{E}_1 \right] \quad \textcircled{9}$$

where we have used the definition

$$c_e^2 \equiv \frac{k T_e}{m_e}$$

and definitions for the plasma frequency $\omega_{ps}^2 \equiv \frac{n_{s0} q_s^2}{\epsilon_0 m_s}$

2. Using $\textcircled{3}$, $\textcircled{6}$, and $\textcircled{9}$, we can solve $\textcircled{7}$ solely in terms of \underline{E}_1 , obtaining

$$\left[\omega^2 - \omega_p^2 - c^2 k^2 \right] \underline{E}_1 - \left[c_e^2 \left(1 - \frac{\omega p_i^2}{\omega^2} \right) + c^2 \right] \underline{k} (\underline{k} \cdot \underline{E}_1) = 0$$

3. NOTE: $\underline{E}_1 = E_{T1} \hat{e}_1 + E_{T2} \hat{e}_2 + E_L \hat{k}$

where $\omega_p^2 = \omega_{p_i}^2 + \omega_{p_e}^2$

and $\underline{k} (\underline{k} \cdot \underline{E}_1) = k^2 \hat{k} (\hat{k} \cdot \underline{E}_1) = k^2 E_L \hat{k}$

II.D. (Continued)

4. The three equations are then:

$$\hat{e}_1: (\omega^2 - \omega_p^2 - c^2 k^2) E_{T1} = 0$$

$$\hat{e}_2: (\omega^2 - \omega_p^2 - c^2 k^2) E_{T2} = 0$$

$$\hat{k}: (\omega^2 - \omega_p^2 - c^2 k^2 - k^2 c_e^2 [1 - \frac{\omega_{pi}^2}{\omega^2}] + c^2 k^2) E_L = 0$$

5. Thus, our Matrix equation is

$$\begin{pmatrix} \omega^2 - \omega_p^2 - c^2 k^2 & 0 & 0 \\ 0 & \omega^2 - \omega_p^2 - c^2 k^2 & 0 \\ 0 & 0 & \omega^2 - \omega_p^2 - k^2 c_e^2 (1 - \frac{\omega_{pi}^2}{\omega^2}) \end{pmatrix} \begin{pmatrix} E_{T1} \\ E_{T2} \\ E_L \end{pmatrix} = 0$$

a. NOTE! The only change from the Cold Plasma Equations is in the Longitudinal term.

⇒ The inclusion of electron pressure modifies the plasma oscillation.

b. The transverse Modified Light Waves are unaffected by the pressure gradient term. This is because $\mathbf{k} \cdot \mathbf{E}_T = 0$ for these modes, so there are no density fluctuations and thus no pressure fluctuations for the transverse mode.

6. Longitudinal Term: $(E_L \neq 0, E_T = 0)$ $\boxed{\omega^2 - \omega_p^2 - k^2 c_e^2 (1 - \frac{\omega_{pi}^2}{\omega^2}) = 0}$

a. General Solution: $\boxed{\omega^2 = \frac{(\omega_p^2 + k^2 c_e^2)}{2} \left(1 \pm \sqrt{1 - \frac{4k^2 c_e^2 \omega_{pi}^2}{(\omega_p^2 + k^2 c_e^2)^2}} \right)}$

b. There are two separate roots, corresponding to two wave modes:

1. Langmuir Waves

2. Ion Acoustic Waves

II. D (Continued)

G. (Continued)

c. In general, the solution is difficult to interpret.

⇒ Plasma physicists often take limits of the solution.

Finding Simplified Solutions in limiting cases is an important skill for interpreting physical behavior.

d. For Example: $\omega^2 - \omega_{pe}^2 - k^2 c_e^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0$

$\omega_p^2 \approx \omega_{pe}^2$

1. When $\omega > \omega_{pe}$ (and $\omega_{pe} \gg \omega_{pi}$), $\frac{\omega_{pi}^2}{\omega^2} \ll 1$, so

$$\boxed{\omega^2 - \omega_{pe}^2 - k^2 c_e^2 = 0} \quad \text{High Frequency Limit}$$

2. When $\omega < \omega_{pi}$ (and $\omega_{pe} \gg \omega_{pi}$), $\omega^2 \ll \omega_{pe}^2$, so

$$\boxed{-\omega_{pe}^2 - k^2 c_e^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0} \quad \text{Low Frequency Limit}$$

E. Langmuir Waves High Frequency, $\omega > \omega_{pe}$

1. $\boxed{\omega^2 = \omega_{pe}^2 + k^2 c_e^2}$

2. NOTE: $\omega_{pe}^2 \lambda_{De}^2 = \left(\frac{n_0 e^2}{\epsilon_0 m_e}\right) \left(\frac{\epsilon_0 k T_e}{n_0 e^2}\right) = \frac{k T_e}{m_e} = c_e^2$

Thus

$$\omega^2 = \omega_{pe}^2 (1 + k^2 \lambda_{De}^2)$$

3. Wavelength limits:

a. Long wavelength: $k^2 \lambda_{De}^2 \ll 1$

$$\Rightarrow \frac{2\pi \lambda_{De}^2}{\lambda^2} \ll 1 \quad \text{or } \lambda \gg \lambda_{De}$$

(Non-propagating) $\boxed{\omega^2 = \omega_{pe}^2}$ Usual electron plasma oscillations as in cold-plasma theory

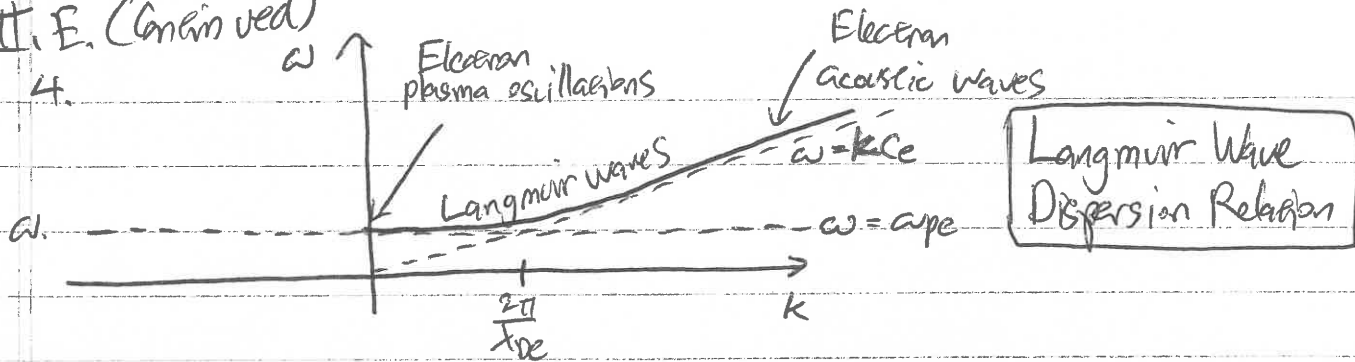
b. Short wavelength: $k^2 \lambda_{De}^2 \gg 1$ (or $\lambda \ll \lambda_{De}$)

$$\boxed{\omega^2 = k^2 c_e^2} \quad \text{Electron acoustic wave (Non-dispersive)}$$

Lecture #24 (Continued)

Homework 7

II. E. (Continued)



5. IMPORTANT NOTE: a. At wavelengths such that $k\lambda_{De} \gg 1$, the phase velocity of the waves $v_p = \frac{\omega}{k} \approx c_e$ is very near the electron thermal velocity $v_{Te} = \sqrt{\frac{2kT_e}{m_e}}$.

- b. The waves are resonant with the electrons. This leads to strong collisionless damping of the electron acoustic wave called Landau damping.
- c. Thus, electron acoustic waves do not really occur.
- d. This is a failure of the two-fluid theory. \Rightarrow Kinetic theory is required to describe collisionless damping.

F. Ion Acoustic Waves: Low Frequency, $\omega < \omega_{pi}$

$$1. -\omega_{pe}^2 - k^2 c_e^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0 \Rightarrow \omega^2 = \frac{\omega_{pi}^2 k^2 c_e^2}{\omega_{pe}^2 + k^2 c_e^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2} \frac{k^2 c_e^2}{(1 + k^2 \lambda_{De}^2)}$$

a. NOTE: $\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{\left(\frac{n_0 q_i^2}{\epsilon_0 m_i}\right)}{\left(\frac{n_0 e^2}{\epsilon_0 m_e}\right)} = \frac{m_e}{m_i} \Rightarrow \omega^2 = \frac{k^2 \left(\frac{kT_e}{m_e}\right) \frac{m_e}{m_i}}{1 + k^2 \lambda_{De}^2}$

b. Ion Acoustic Dispersion Relation $\omega^2 = \frac{k^2 c_i^2}{1 + k^2 \lambda_{De}^2}$

c. DEF: Ion Acoustic Speed $c_i^2 = \frac{kT_e}{m_i}$ \leftarrow Electrons provide pressure \leftarrow Ions provide inertia.

II. F. (Continued)

2. Wavelength Limits:

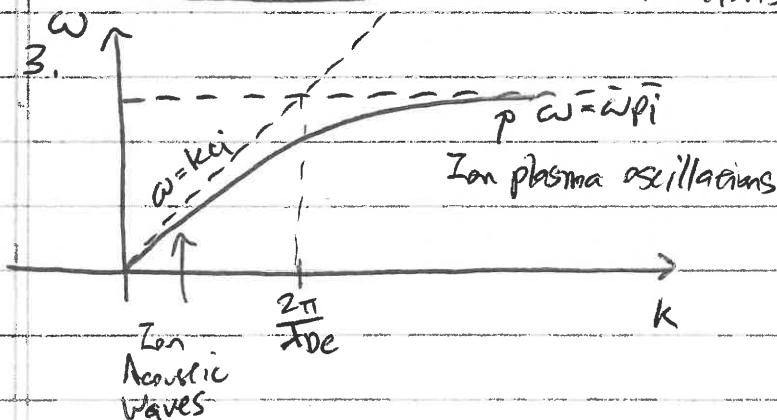
a. Long wavelength: $k^2 \lambda_{De}^2 \ll 1$

$$\boxed{\omega^2 = k^2 c_i^2} \quad \text{Ion Acoustic Wave} \quad (\text{Non-dispersive})$$

$$\frac{\omega}{k} = \pm c_i$$

b. Short wavelength: $k^2 \lambda_{De}^2 \gg 1$

$$1. \omega^2 = \frac{k^2 c_i^2}{k^2 \lambda_{De}^2} = \frac{\left(\frac{k T_e}{m_i}\right)}{\left(\frac{\epsilon_0 k T_e}{n_0 e^2}\right)} = \frac{n_0 e^2}{\epsilon_0 m_i} = \frac{n_0 q_i^2}{\epsilon_0 m_i} = \omega_{pi}^2$$

2. $\boxed{\omega^2 = \omega_{pi}^2}$ Plasma oscillations at ion plasma frequency.

3a. If $T_i = T_e$ (instead of $T_i \ll T_e$ as we assumed), then the ion acoustic phase speed $\frac{\omega}{k} = c_i^* = \sqrt{\frac{k T_e}{m_i}} \approx \sqrt{\frac{2 k T_i}{m_i}} = v_{Ti}$.

b. Thus, when ion temperature is equal to electron temperature, ions' thermal velocity is resonant with the wave phase speed.

c. Ion Acoustic waves experience strong collisionless Landau damping when $T_i = T_e$. (Again, kinetic description is necessary)

d. Only when $T_i \ll T_e$ do ion acoustic waves propagate with little damping.