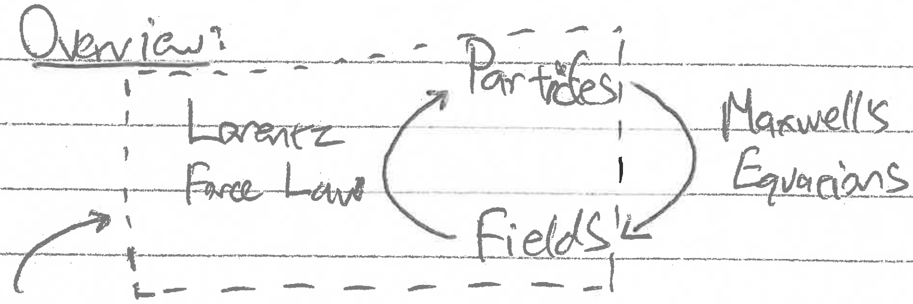
Lecture #3: Single Particle Motion in Constant Uniform Fields — Howes ①

I. Overview:



Today, and for next few weeks, we will focus on "half" of the plasma physics problem, the inconsistent picture of the effect of fields on the motion of particles, neglected the particles effect on fields.

II. Constant, Uniform $\mathbf{B}$ with $\mathbf{E}=0$.

A. 1. Nonrelativistic Limit

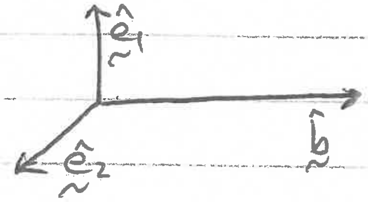
2. Lorentz Force Law $m_s \frac{d\mathbf{v}_s}{dt} = q_s(\mathbf{E} + \mathbf{v}_s \times \mathbf{B})$

3. Drop species subscript "s" because single particle motion does not depend on other species (or any other particles).

4. Thus, for $\mathbf{E}=0$, $\quad m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B}$

B. Solve:

1. Define $\hat{\mathbf{b}} \equiv \frac{\mathbf{B}}{|\mathbf{B}|}$ and unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ to define a right-handed, orthonormal basis



$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$

2a. We can write $\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_\parallel \hat{\mathbf{b}}$

and $\mathbf{B} = B_0 \hat{\mathbf{b}}$

b. Thus $\quad \frac{d\mathbf{v}}{dt} = \frac{qB_0}{m}\,\mathbf{v}\times\hat{\mathbf{b}}$

c. NOTE: $\mathbf{v}\times\hat{\mathbf{b}} = -v_1\hat{\mathbf{e}}_2 + v_2\hat{\mathbf{e}}_1$

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II.B. (Continued)

3. Take dot product with each unit vector to get component equations

$\hat{\underset{\sim}{e}}_1 \cdot \quad \frac{d\underset{\sim}{v}}{dt} = \frac{qB_0}{m} \underset{\sim}{v} \times \hat{\underset{\sim}{b}} \qquad \Rightarrow \quad \frac{dv_1}{dt} = \frac{qB_0}{m} v_2$

$\hat{e}_2 \cdot \qquad \text{"} \qquad \Rightarrow \quad \frac{dv_2}{dt} = -\frac{qB_0}{m} v_1$

$\hat{\underset{\sim}{b}} \cdot \qquad \text{"} \qquad \Rightarrow \quad \frac{dv_\parallel}{dt} = 0$

4. Take $\frac{d}{dt}$ of $\hat{\underset{\sim}{e}}_1$ eq.

$$\frac{d^2 v_1}{dt^2} = \frac{qB_0}{m}\left(\frac{dv_2}{dt}\right) \overset{\text{substitute } \hat{\underset{\sim}{e}}_2 \text{ eq.}}{=} -\left(\frac{qB_0}{m}\right)^2 v_1$$

This is just the equation of a simple harmonic oscillator.

We define the $\boxed{\text{Cyclotron Frequency} \quad \omega_c \equiv \frac{qB_0}{m}}$

Thus, $\boxed{\frac{d^2 v_1}{dt^2} = -\omega_c^2 v_1}$

5. Other properties: a. Take $\underset{\sim}{v} \cdot$ with equation:

$$\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = \frac{1}{2}\frac{dv^2}{dt} = 0$$

b. So $v^2 = \sqrt{v_1^2 + v_2^2 + v_\parallel^2}$ is constant

c. But, we also know from $\hat{\underset{\sim}{b}}$ equation that $\frac{dv_\parallel}{dt} = 0$, so

$\boxed{v_\perp \equiv \sqrt{v_1^2 + v_2^2}}$ is constant.

d. Acceleration is perpendicular to $\underset{\sim}{v}$, so Lorentz Force cannot change the energy of the particles.

6. Complete Solution: a. General Solution $v_1 = Ae^{-i\omega_c t} + Be^{i\omega_c t}$

b. $\hat{\underset{\sim}{e}}_1$ component gives $v_2 = +i(-Ae^{-i\omega t} + Be^{i\omega t})$

c. Take initial conditions $\begin{array}{l} v_{10} = v_\perp \\ v_{20} = 0 \end{array}$ at time $t=0$.

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II.B.6. (Continued)

d. This gives $A = B = \frac{v_\perp}{2}$

e. Solving for velocity: $v_1 = \frac{v_\perp}{2}\left(e^{i\omega_c t} + e^{-i\omega_c t}\right) = v_\perp \cos\omega_c t$

$$v_2 = \frac{i v_\perp}{2}\left(e^{i\omega_c t} - e^{-i\omega_c t}\right) = v_\perp \sin\omega_c t$$

$$v_\parallel = v_{\parallel 0}$$

7. Solve for Position $\underline{x}$.

a. $v_1 = \frac{dx_1}{dt}$ so $x_1 = \frac{v_\perp}{\omega_c}\sin\omega_c t + x_{10}$

$v_2 = \frac{dx_2}{dt}$ $\quad x_2 = \frac{v_\perp}{\omega_c}\cos\omega_c t + x_{20}$

$v_\parallel = \frac{dx_\parallel}{dt}$ $\quad x_\parallel = v_{\parallel 0} t + x_{\parallel 0}$

b. We can define the Larmor radius (note this is not the thermal Larmor radius defined using v_t)

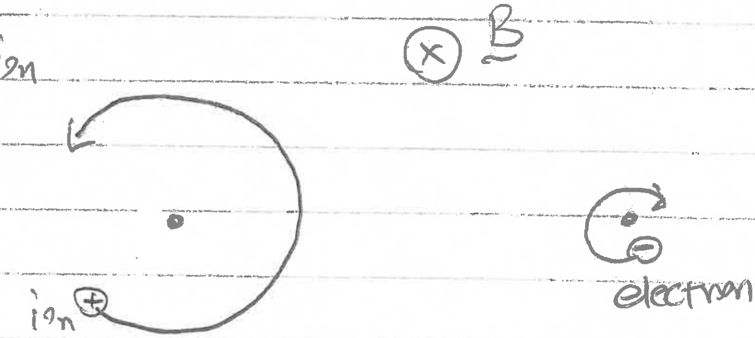
$$\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{q B_0}}$$

c. The perpendicular components x_1 & x_2, can then be written

$$x_1 = r_L \sin\omega_c t + x_{10}$$

$$x_2 = r_L \cos\omega_c t + x_{20}$$

8. Larmor Motion



Note the the Magnetic field created by Larmor Motion opposes the mean field

$\Rightarrow$ DIAMAGNETIC behavior

III. Constant, Uniform $\underset{\sim}{E}$ and $\underset{\sim}{B}$

A. 1. In this case $m \frac{d\underset{\sim}{v}}{dt} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

2. First, what velocity $\underset{\sim}{v}$ leads to a RHS $= 0$. This component will have no acceleration, corresponding to a drift.

a. $q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) = 0 \Rightarrow \underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B}$

b. Cross with $\underset{\sim}{B}$ to solve for $\underset{\sim}{v}$: $\underset{\sim}{E} \times \underset{\sim}{B} = -(\underset{\sim}{v} \times \underset{\sim}{B}) \times \underset{\sim}{B} = B_0^2 [\underset{\sim}{v} - v_\parallel \hat{\underset{\sim}{b}}]$

c. Thus $\underset{\sim}{v} - v_\parallel \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B_0^2}$

d. Define the "E cross B" velocity $\boxed{\underset{\sim}{v}_E \equiv \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B_0^2}}$

B. Solve for motion in $\underset{\sim}{E} \times \underset{\sim}{B}$ frame:

1. The velocity $\underset{\sim}{u}$ in the $\underset{\sim}{E} \times \underset{\sim}{B}$ frame is $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{v}_E$

2. Plugging in $\underset{\sim}{v}$:

a. $m \frac{d\underset{\sim}{u}}{dt} + \cancel{m \frac{d \underset{\sim}{v}_E}{dt}}^{\,0} = q(\underset{\sim}{E} + \underset{\sim}{v}_E \times \underset{\sim}{B} + \underset{\sim}{u} \times \underset{\sim}{B})$

b. NOTE: $\underset{\sim}{v}_E \times \underset{\sim}{B} = \frac{(\underset{\sim}{E} \times \hat{\underset{\sim}{b}}) \times \hat{\underset{\sim}{b}} \cancel{B_0^2}}{\cancel{B_0^2}} = -\underset{\sim}{E} + E_\parallel \hat{\underset{\sim}{b}}$

c. Thus $m \frac{d\underset{\sim}{u}}{dt} = q(E_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{u} \times \underset{\sim}{B})$

3. Take $\hat{\underset{\sim}{b}} \cdot$ to solve for parallel motion

$$m \frac{du_\parallel}{dt} = qE_\parallel \Rightarrow \boxed{u_\parallel = \frac{qE_\parallel}{m} t + u_{\parallel 0}}$$

II. B. (Continued)

4. Perpendicular Motion: $\mathbf{u}_\perp = \mathbf{u} - u_\parallel \hat{\mathbf{b}}$

a. $$m\frac{d\mathbf{u}_\perp}{dt} = q(\mathbf{u}_\perp \times \mathbf{B})$$

b. This is identical to the motion with $\mathbf{E}=0$ in II.

c. Thus $\mathbf{u}_\perp = v_\perp\left(\cos(\omega_c t)\,\hat{\mathbf{e}}_1 - \sin(\omega_c t)\,\hat{\mathbf{e}}_2\right)$

d. In the $\mathbf{E}\times\mathbf{B}$ drift frame, you have standard Larmor motion.

5. Total velocity $\mathbf{v} = \mathbf{u} + \mathbf{v}_E$

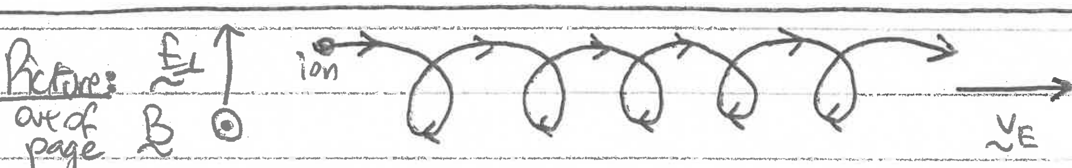
$$\boxed{\mathbf{v} = \left(\frac{qE_\parallel}{m}t + u_{\parallel 0}\right)\hat{\mathbf{b}} + v_\perp\left(\cos\omega_c t\,\hat{\mathbf{e}}_1 - \sin\omega_c t\,\hat{\mathbf{e}}_2\right) + \mathbf{v}_E}$$

6. Integrate to get total position $\mathbf{x}$ given initial condition $\mathbf{x}_0$.

a. $$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

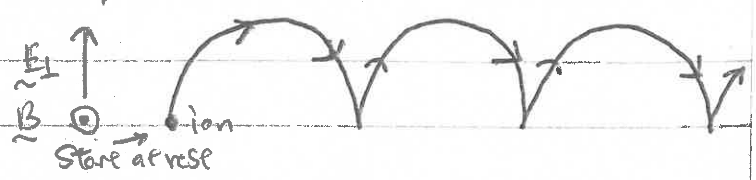
b. $$\boxed{\mathbf{x} = \mathbf{x}_0 + \left(\mathbf{v}_E + u_{\parallel 0}\hat{\mathbf{b}}\right)t + \frac{qE_\parallel}{m}\frac{t^2}{2}\hat{\mathbf{b}} + r_L\left(\sin\omega_c t\,\hat{\mathbf{e}}_1 + \cos\omega_c t\,\hat{\mathbf{e}}_2\right)}$$

7. Physical Picture:

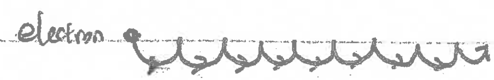


a. As the particle is accelerated by electric field r_L increases
" " " or decelerated " " " r_L decreases.
This asymmetry leads to a drift.

b. This can be more clearly understood starting with $v_0=0$.



c. For electrons, $\mathbf{E}$ acceleration is in opposite direction, but so is $\mathbf{v}\times\mathbf{B}$ Lorentz force, leading to a drift in the same direction!



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d. The $\underset{\sim}{E} \times \underset{\sim}{B}$ drift is independent of charge!

C. Other Forces:

1. Other forces acting on particle also lead to drifts, for example g.

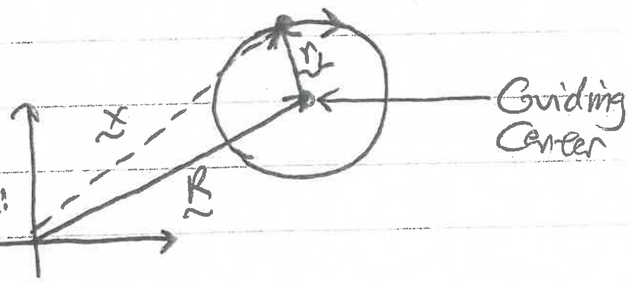
a. $m \frac{d\underset{\sim}{v}}{dt} = q(\cancel{\underset{\sim}{E}}^{0} + \underset{\sim}{v} \times \underset{\sim}{B}) + mg \Rightarrow \boxed{\underset{\sim}{V_g} = \frac{m}{q} \frac{g \times \underset{\sim}{B}}{B_0^2} \text{ Gravitational Drift}}$

b. But, for gravity, electrons & ions drift in opposite directions.

c. This leads to a net current driven by the drift.

D. Guiding Center:

1. Position can be split into Guiding Center $\underset{\sim}{R}$ + Larmor motion $\underset{\sim}{r_L}$:



$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{r_L}$$

a. $\underset{\sim}{R} = \underset{\sim}{x_0} + \underset{\sim}{V_E} t + \left(v_{\parallel 0} t + \frac{q E_\parallel}{m} \frac{t^2}{2}\right) \hat{b}$

b. $\underset{\sim}{r_L} = r_L (\sin \omega_c t \, \underset{\sim}{\hat{e}_1} + \cos \omega_c t \, \underset{\sim}{\hat{e}_2})$

2. Note: $\underset{\sim}{r_L} = \frac{\hat{b} \times \underset{\sim}{v}}{\omega_c}$

3. IMPORTANT CONCEPT:

a. Decomposition of motion:
1) Rapid Larmor motion about field line, $\underset{\sim}{r}$
2) Slow drift across field line, $\underset{\sim}{R}$

b. NOTE: If one averages the motion over $T = \frac{2\pi}{\omega_c}$,

$\int_0^{\frac{2\pi}{\omega}} dt \, \underset{\sim}{r_L} = 0$, leaving only guiding center drift $\underset{\sim}{R}(t)$.

c. This is the fundamental concept underlying Multiple Time Scale Methods $\Rightarrow$ next lecture.