I. Application of Multiple Timescale Methods

A. i. We'll see how to apply this powerful analytical method with a simple example first.

B. Example - Duffing's Equation

1. A simple nonlinear oscillator problem is given by Duffing's Equation $\frac{d^2x}{dt^2} = -x + x^3$

(For more info, see http://mathworld.wolfram.com/DuffingDifferentialEquation.html)

2a. To solve this problem, we will assume the system evolves on two, separable timescales: Short t

Long $\tau = \epsilon^2 t$

b. We will treat these as separate variables.

c. Here $\epsilon \ll 1$ is a small dimensionless number, used for bookkeeping.

d. NOTE: $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon^2 \frac{\partial^2}{\partial t \partial \tau} + \epsilon^4 \frac{\partial^2}{\partial \tau^2}$$

3. As usual with the simple harmonic oscillator, we'll make the assumption of small amplitude oscillations.

Expand Solution $\longrightarrow$ $x(t,\tau) = \epsilon x_1(t,\tau) + \epsilon^2 x_2(t,\tau) + \epsilon^3 x_3(t,\tau) + \ldots$

4. Plug expansion for x and $\frac{d}{dt}$ into original equation:

$$\frac{\partial^2}{\partial t^2}(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + 2\epsilon^2 \frac{\partial^2}{\partial t \partial \tau}(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + \epsilon^4 \frac{\partial^2}{\partial \tau^2}(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3)$$

$$= -(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + (\epsilon^3 x_1^3 + 3\epsilon^4 x_1^2 x_2 + 3\epsilon^5 x_1 x_2^2 + 3\epsilon^5 x_1^2 x_3 + \ldots)$$

Lect #4 (Continued)

I. B. (Continued)

5. Find equations at each power of ϵ:

a. $\mathcal{O}(\epsilon)$: $\frac{\partial^2 x_1}{\partial t^2} = -x_1$

b. $\mathcal{O}(\epsilon^2)$: $\frac{\partial^2 x_2}{\partial t^2} = -x_2$

c. $\mathcal{O}(\epsilon^3)$: $\frac{\partial^2 x_3}{\partial t^2} + 2\frac{\partial^2 x_1}{\partial t \partial \tau} = -x_3 + x_1^3$

⋮

6. Solve $\mathcal{O}(\epsilon)$ equation:

a. General solution for $x_1(t,\tau)$: $\boxed{x_1(t,\tau) = A(\tau)\cos t + B(\tau)\sin t}$

b. On the short timescale t, $A(\tau)$ and $B(\tau)$ are treated as constants. The higher order equations will allow us to solve for $A(\tau)$, $B(\tau)$

7. $\mathcal{O}(\epsilon^2)$ equation does not tell us anything new.

8. Solve $\mathcal{O}(\epsilon^3)$ equation:

a. We have solved for x_1, so we can substitute in:

NOTE: $\frac{\partial^2 x_1}{\partial t \partial \tau} = -\frac{\partial A}{\partial \tau}\sin t + \frac{\partial B}{\partial \tau}\cos t$

b. Thus: $\frac{\partial^2 x_3}{\partial t^2} + x_3 = 2\frac{\partial A}{\partial \tau}\sin t - 2\frac{\partial B}{\partial \tau}\cos t + A^3\cos^3 t + 3A^2B\cos^2 t\sin t + 3AB^2\cos t\sin^2 t + B^3\sin^3 t$

c. 1. We assume the x_3 is periodic over one oscillation $[0, 2\pi]$

2. Therefore, we can annihilate x_3 by averaging over an oscillation.

d. TRICK: Multiply the equation by $\sin t$ and integrate $\int_0^{2\pi} dt$

1. LHS: $\int_0^{2\pi} \sin t\left(\frac{\partial^2 x_3}{\partial t^2} + x_3\right) dt$

i. Integrate by parts twice on first term:

$$\int_0^{2\pi} \sin t \frac{\partial^2 x_3}{\partial t^2} dt = \cancel{\sin t \frac{\partial x_3}{\partial t}\Big|_0^{2\pi}}^{\,0} - \cancel{\cos t\, x_3\Big|_0^{2\pi}}^{\,0} - \int \sin t\, x_3\, dt$$

by periodicity of x_3

ii. Thus $\int_0^{2\pi} \sin t(-x_3 + x_3)dt = 0$

Lect #4 (Continued)

I.B.8d. (Continued)

2. RHS: $2\frac{\partial A}{\partial \tau}\int_0^{2\pi}\sin^2 t\,dt = 2\pi\frac{\partial A}{\partial \tau}$

$$-2\frac{\partial B}{\partial \tau}\int_0^{2\pi}\sin t\cos t\,dt = -2\frac{\partial B}{\partial \tau}\left[\frac{\sin^2 t}{2}\right]_0^{2\pi} = 0$$

$$A^3\int_0^{2\pi}\sin t\cos^3 t\,dt = A^3\left[\frac{-\cos^4 t}{4}\right]_0^{2\pi} = 0$$

$$3A^2B\int_0^{2\pi}\cos^2 t\sin^2 t\,dt = 3A^2B\int_0^{2\pi}(\sin^2 t-\sin^4 t)\,dt = 3A^2B\left(\pi-\frac{3\pi}{4}\right) = \frac{3\pi}{4}A^2B$$

$$3AB^2\int_0^{2\pi}\sin^3 t\cos t\,dt = 3AB^2\left[\frac{\sin^4 t}{4}\right]_0^{2\pi} = 0$$

$$B^3\int_0^{2\pi}\sin^4 t\,dt = \frac{3\pi}{4}B^3$$

NOTE: 1. $\int_0^{2\pi}\sin^2 t\,dt = \pi$

2. $\int_0^{2\pi}\sin^4 t\,dt = \frac{3\pi}{4}$

3. Thus RHS $= 2\pi\frac{\partial A}{\partial \tau} + \frac{3\pi}{4}A^2B + \frac{3\pi}{4}B^3$

4. Thus

$$\frac{\partial A}{\partial \tau} = -\frac{3}{8}(A^2B+B^3) = -\frac{3}{8}(A^2+B^2)B$$

e. We can perform the same trick this time multiplying by $\cos t$ and $\int_0^{2\pi}dt$.

1. Again LHS $=0$

2. RHS $= -2\pi\frac{\partial B}{\partial \tau} + \frac{3\pi}{4}A^3 + \frac{3\pi}{4}AB^2$

3. Thus

$$\frac{\partial B}{\partial \tau} = \frac{3}{8}(A^3+AB^2) = \frac{3}{8}(A^2+B^2)A$$

f. Thus, we have

$$\boxed{\begin{aligned}\frac{\partial A}{\partial \tau} &= -\frac{3}{8}(A^2+B^2)B\\ \frac{\partial B}{\partial \tau} &= \frac{3}{8}(A^2+B^2)A\end{aligned}}$$

C. B. 8. (Continued)

g. These equations for $A(\tau)$ & $B(\tau)$ are also nonlinear, but they may be solved by another trick.

1. TRICK: Multiply $\frac{\partial A}{\partial \tau}$ by A, $\frac{\partial B}{\partial \tau}$ by B and add equations

2. $A\frac{\partial A}{\partial \tau} + B\frac{\partial B}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial \tau}(A^2+B^2) = 0$

3. Therefore $A^2+B^2 = \text{constant}$. Let $C^2 \equiv A^2+B^2$.

h. Thus $\frac{\partial A}{\partial \tau} = -\frac{3C^2}{8}B$ $\qquad \frac{\partial B}{\partial \tau} = \frac{3C^2}{8}A$

i. Solving $\frac{\partial^2 A}{\partial \tau^2} = \frac{-3C^2}{8}\frac{\partial B}{\partial \tau} = -\left(\frac{3C^2}{8}\right)^2 A$

2. General solution can be written

$$A(\tau) = A_0 \cos\left(\frac{3C^2}{8}\tau + \phi_0\right)$$

where A_0 is amplitude, ϕ_0 is phase

3. Thus $B(\tau) = A_0 \sin\left(\frac{3C^2}{8}\tau + \phi_0\right)$

j. Plugging in $A(\tau)$ and $B(\tau)$ to get full x_1 solution:

$$x_1(t,\tau) = A_0\cos\left(\frac{3C^2}{8}\tau + \phi_0\right)\cos t + A_0 \sin\left(\frac{3C^2}{8}\tau + \phi_0\right)\sin t$$

$$\boxed{x_1(t,\tau) = A_0 \cos\left(t - \frac{3C^2}{8}\tau - \phi_0\right)}$$

Using $\cos(F-G) = \cos F \cos G + \sin F \sin G$

9. Let's check our assumption that x_3 is periodic over oscillation in t.

a. $\frac{\partial^2 x_3}{\partial t^2} + x_3 = -2\frac{\partial^2 x_1}{\partial t \partial \tau} + x_1^3$ This is a driven harmonic oscillator.

b. $\frac{\partial^2 x_1}{\partial t \partial \tau} = \frac{\partial}{\partial \tau}\left[-A_0 \sin\left(t - \frac{3C^2}{8}\tau - \phi_0\right)\right] = \frac{3C^2 A_0}{8}\cos\left(t - \frac{3C^2}{8}\tau - \phi_0\right)$

c. Thus, $\frac{\partial^2 x_3}{\partial t^2} + x_3 = \frac{-6C^2A_0}{8}\cos\left(t - \frac{3C^2}{8}\tau - \phi_0\right) + A_0^3\cos^3\left(t - \frac{3C^2}{8}\tau - \phi_0\right)$

d. Since Forcing term (RHS) is periodic in t, so must be solution $x_3(t,\tau)$.

Lect #4 (Continued)

I. B. (Continued)

10. Final Solution.

a. NOTE: $C^2 = A^2 + B^2 = A_0^2$

b. Set our bookkeeping term $\epsilon = 1$.

$$\boxed{x_1(t) = A_0 \cos\left(t - \frac{3A_0^2}{8}t - \phi_0\right)}$$

c. Small amplitude assumption means $t \gg \frac{3A_0^2}{8}t$

11. Interpretation: a. System is, to lowest order, a harmonic oscillator

b. Over long times, $x_1(t)$ builds up a large phase shift $-\frac{3A_0^2 t}{8}$ due to the nonlinear term.

II. Dimensionless Equations

A. Conversion of equations to a dimensionless form often leads to less cumbersome notation when working with the equations.

B. Example: Linearized Hydrodynamic Momentum Equation

1. $$\rho_0 \frac{\partial \underset{\sim}{u}}{\partial t} = -\nabla p$$

2. Choose characteristic values by which to normalize:

a. $\underset{\sim}{u}' = \frac{\underset{\sim}{u}}{c_s}$ $\qquad p' = \frac{p}{p_0}$

b. $\nabla = \frac{\partial}{\partial \underset{\sim}{x}}$, so $\underset{\sim}{x}' = \frac{\underset{\sim}{x}}{L_0}$ $\qquad t' = \frac{t}{\left(\frac{L_0}{c_s}\right)} = \frac{c_s t}{L_0}$

3. Substitute normalized values into equation

a. $$\rho_0 \frac{\partial(\underset{\sim}{u}' c_s)}{\partial\left(t' \frac{L_0}{c_s}\right)} = -\frac{\partial(p' p_0)}{\partial(\underset{\sim}{x}' L_0)} \Rightarrow \frac{c_s^2}{L_0}\frac{\partial \underset{\sim}{u}'}{\partial t'} = -\left(\frac{p_0}{\rho_0}\right)\frac{1}{L_0} - \nabla' p'$$

b. NOTE: Speed of sound $c_s^2 \equiv \left(\frac{p_0}{\rho_0}\right)$, so this leaves $\boxed{\frac{\partial \underset{\sim}{u}'}{\partial t'} = -\nabla' p'}$