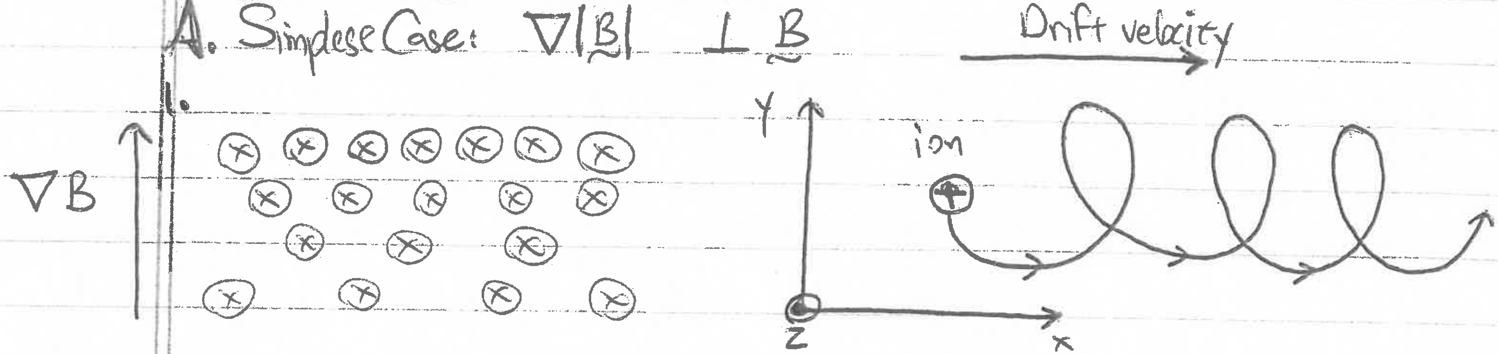
Lecture #5: Particle Motion in Constant, Non-uniform $\underset{\sim}{B}$ Fields Howes

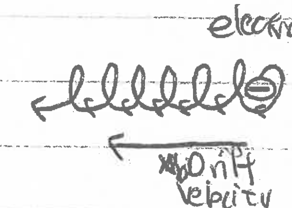
I. The "Grad B" Drift:

A. Simplest Case: $\nabla|\underset{\sim}{B}| \perp \underset{\sim}{B}$

1.



2. Ions and electrons drift in opposite directions!



3. Two Spatial Scales characteristic of this problem:
   a. Larmor radius $r_L = \frac{v_\perp}{\omega_c}$
   b. Magnetic Field Scale Length $L = \left(\frac{\nabla B}{B}\right)^{-1}$

4. This problem can be done using a multiple-scale analysis using an expansion parameter

$$\epsilon = \frac{r_L}{L} \ll 1$$

   a. In this case, $\underset{\sim}{B}$ changes little over the Larmor radius.
   b. In general, for the case of $\underset{\sim}{E} = 0$,

$$m\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}) \leftarrow$$ We can expand in ϵ, solving order by order.

   c. But, the solution in general requires a lot of algebra and averaging, so rather than that we'll take a more "intuitive" approach.

Lecture #5 (Continued) Howes ③

I. (Continued)

B. Intuitive approach based on average force.

1. Analogous to $\underset{\sim}{E}\times\underset{\sim}{B}$ or Gravitational drift, the drift due to a general force $\underset{\sim}{F}$ (for $m\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v}\times\underset{\sim}{B} + \underset{\sim}{F}$) is $\boxed{\underset{\sim}{V}_D = \frac{1}{q}\frac{\underset{\sim}{F}\times\underset{\sim}{B}}{B_0^2}}$

2. We'll use a perturbative approach to find the effective averaged force due to a gradient in B, then use the formula above to find the drift.

3. $\boxed{\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt} = q\left[\underset{\sim}{v}\times\underset{\sim}{B}(\underset{\sim}{r})\right]}$

4. Taylor Expansion: $\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}(\underset{\sim}{r}_0) + (\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\nabla\underset{\sim}{B} + \frac{[(\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\nabla]^2}{2!}\underset{\sim}{B} + \ldots$

a. Note, if we normalize this formula according to $B_0 = |\underset{\sim}{B}(\underset{\sim}{r}_0)|$, we find

$$\frac{\underset{\sim}{B}(\underset{\sim}{r})}{B_0} = \underset{\sim}{\hat{b}} + (\underset{\sim}{r}-\underset{\sim}{r}_0)\cdot\frac{\nabla\underset{\sim}{B}}{B_0}$$

$\mathcal{O}(1)$ $\qquad \mathcal{O}\left(r_L\cdot\frac{(B_0/L)}{B_0}\right) = \mathcal{O}\left(\frac{r_L}{L}\right)$

b. Thus, as usual, our Taylor Expansion is a good approximation if

$$\epsilon = \frac{r_L}{L} \ll 1.$$

5. a. To simplify the algebra, consider $\underset{\sim}{B}(y) = B_0\underset{\sim}{\hat{z}} + \epsilon\, y\frac{\partial B}{\partial y}\hat{z}\ldots$

b. Expand $\underset{\sim}{V} = \underset{\sim}{V}_0 + \epsilon\underset{\sim}{V}_1 + \ldots$

unperturbed orbit for $\nabla B = 0$ $\qquad$ correction to orbit due to ∇B

6. From Lecture #3 for $\underset{\sim}{E}=0$, we know the orbit in the absence of the gradient gives

(take $\underset{\sim}{\hat{e}}_1 = \hat{x}$ & $\underset{\sim}{\hat{e}}_2 = \hat{y}$)

$$\left.\begin{aligned} V_x &= V_\perp \cos\omega_c t \\ V_y &= -V_\perp \sin\omega_c t \\ V_z &= V_{\parallel 0}\end{aligned}\right\}\ \underset{\sim}{V} = V_\perp(\cos\omega_c t\,\hat{x} - \sin\omega_c t\,\hat{y}) + V_{\parallel 0}\underset{\sim}{\hat{z}}$$

I. B. (Continued)

7. $\underset{\sim}{F} = q(\underset{\sim}{v}_0 + \epsilon \underset{\sim}{v}_1) \times \left(B_0 \hat{\underset{\sim}{z}} + \epsilon\, y \frac{\partial B}{\partial y} \hat{\underset{\sim}{z}}\right)$

a. $\mathcal{O}(1)$: $\underset{\sim}{F}_0 = q\left(\underset{\sim}{v}_0 \times B_0 \hat{\underset{\sim}{z}}\right) = q B_0 v_\perp \left(-\cos\omega_c t\, \hat{\underset{\sim}{y}} - \sin\omega_c t\, \hat{\underset{\sim}{x}}\right)$

To find average force, take $\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \underset{\sim}{F}_0\, dt \equiv \langle \underset{\sim}{F}_0 \rangle$

$$\langle \underset{\sim}{F}_0 \rangle = \frac{q B_0 v_\perp \omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \left(\cancelto{0}{-\cos\omega_c t\, \hat{\underset{\sim}{y}}} - \cancelto{0}{\sin\omega_c t\, \hat{\underset{\sim}{x}}}\right) dt = 0$$

b. $\mathcal{O}(\epsilon)$: $\underset{\sim}{F}_1 = \underbrace{q\, \underset{\sim}{v}_1 \times (B_0 \hat{\underset{\sim}{z}})}_{①} + \underbrace{q\, \underset{\sim}{v}_0 \times \left(y \frac{\partial B_y}{\partial y} \hat{\underset{\sim}{z}}\right)}_{②}$

① We annihilate term ① by assuming $\underset{\sim}{v}_1$ is periodic in $T = \frac{2\pi}{\omega_c}$.
To verify this assumption requires substantial algebra, skipped here.

$$\frac{q B_0 \omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \underset{\sim}{v}_1 \times \hat{\underset{\sim}{z}} = 0$$

② From our solution to $\underset{\sim}{v}_0$, we get $y - y_0 = \int \frac{dv_y}{dt} dt = \frac{v_\perp}{\omega_c}\cos\omega_c t$
we'll take $y_0 = 0$ for simplicity.

$$\left\langle q\, \underset{\sim}{v}_0 \times \left(y \frac{\partial B_y}{\partial y} \hat{\underset{\sim}{z}}\right)\right\rangle = \frac{q\omega_c}{2\pi}\frac{\partial B_y}{\partial y}\frac{v_\perp^2}{\omega_c}\int_0^{\frac{2\pi}{\omega_c}} \cos\omega_c t\left(\cos\omega_c t\, \hat{\underset{\sim}{y}} - \sin\omega_c t\, \hat{\underset{\sim}{x}}\right) dt$$

NOTE: $\int_0^{\frac{2\pi}{a}} \cos^2 ax\, dx = \frac{\pi}{a}$, $\int_0^{2\pi} \cos x \sin x\, dx = 0$

$$\boxed{\langle \underset{\sim}{F}_1 \rangle = \frac{-q v_\perp^2}{2\omega_c}\frac{\partial B_y}{\partial y}\hat{\underset{\sim}{y}}}$$

Thus, there is an average force in the direction of the gradient.

8.

For a general gradient $\nabla B \perp \underset{\sim}{B}$, $\langle \underset{\sim}{F}_1 \rangle = \frac{-q v_\perp^2}{2\omega_c}\nabla B$

I. B. (Continued)

9. Therefore, the ∇B drift is $\underset{\sim}{V}_{\nabla B} = \frac{1}{q}\frac{-q v_\perp^2}{2\omega_c}\frac{\nabla B \times B}{B^2}$

$$\boxed{\underset{\sim}{V}_{\nabla B} = \frac{-v_\perp^2}{2\omega_c}\frac{\nabla B \times \underset{\sim}{B}}{B_0^2}}$$

C. Properties of ∇B drift

1. Since $r_L = \frac{v_\perp}{\omega_c}$, $\underset{\sim}{V}_{\nabla B} = \frac{-1}{2} v_\perp r_L \frac{\nabla B \times \underset{\sim}{B}}{B_0^2}$

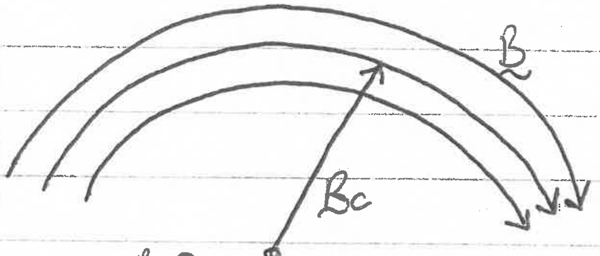
   a. Magnitude of drifts depends on
      1. $v_\perp$
      2. r_L or Perpendicular Energy $\frac{1}{2}m v_\perp^2$
      3. ∇B

2. Ions and electrons drift in opposite directions.

3. Can also be written $\boxed{\underset{\sim}{V}_{\nabla B} = \frac{v_\perp^2}{2\omega_c}\underset{\sim}{b} \times \frac{\nabla B}{B}}$

II. Curvature Drift:

A. 1. Another drift occurs when the field lines are curved.



2. When $\underset{\sim}{B}$ field lines are curved, there is also typically a gradient in $|\underset{\sim}{B}|$, so both ∇B and curvature drifts will be important.

3. To focus on curvature effects, we consider perfectly circular field lines with a radius of curvature $\underset{\sim}{R}_c$, and no gradient in field strength.

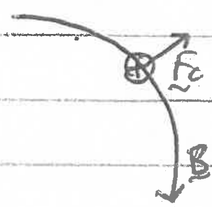
<u>NOTE</u>: Such a field violates $\nabla \cdot \underset{\sim}{B} = 0$, but we can most easily isolate the curvature drift this way.

II. A. (Continued)

4. Centrifugal Force:

a. For a particle to move along a circular path, a centrifugal force will be felt by the particle

$$\underset{\sim}{F}_c = \frac{m v_\parallel^2}{R_c}\hat{\underset{\sim}{r}} = m v_\parallel^2 \frac{\underset{\sim}{R}_c}{R_c^2}$$



b. We may now see what drift will be caused by $\underset{\sim}{F_c}$

$$\underset{\sim}{V}_c = \frac{1}{q}\frac{\underset{\sim}{F}_c \times \underset{\sim}{B}}{B^2}$$

5. a. $$\boxed{\underset{\sim}{V}_c = \frac{m v_\parallel^2}{q B^2}\frac{\underset{\sim}{R}_c \times \underset{\sim}{B}}{R_c^2}}$$ Curvature Drift

b. Noting $\omega_c = \frac{qB}{m}$, $\underset{\sim}{V}_c = \frac{v_\parallel^2}{\omega_c B}\frac{\underset{\sim}{R}_c \times \underset{\sim}{B}}{R_c^2}$

6. Properties

a. Depends on parallel energy $\frac{1}{2} m v_\parallel^2$.

b. Drift is in opposite direction for ions and electrons.

7. One may also show that $\frac{\hat{\underset{\sim}{R}}_c}{R_c} = -\hat{\underset{\sim}{b}}\cdot\nabla\hat{\underset{\sim}{b}}$, (HW)

So $$\boxed{\underset{\sim}{V}_c = \frac{v_\parallel^2}{\omega_c}\hat{\underset{\sim}{b}}\times(\hat{\underset{\sim}{b}}\cdot\nabla)\hat{\underset{\sim}{b}}}$$

III. Examples of ∇B and Curvature Drift

A. Current Carrying Wire

1. Consider the field due to a wire carrying a current $\underset{\sim}{I} = I_0 \hat{\underset{\sim}{z}}$

a. In cylindrical (r, ϕ, z) coordinates,

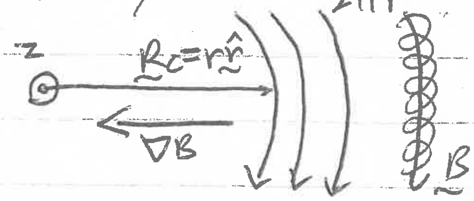
$$\underset{\sim}{B} = \frac{\mu_0 I_0}{2\pi r}\hat{\underset{\sim}{\phi}}$$

Lecture #5 (Continued)

III. A. Continued

2. Using NRL p.6, $\nabla B = \frac{\partial B}{\partial r}\underset{\sim}{\hat{r}} + \frac{1}{r}\cancel{\frac{\partial B}{\partial \phi}}\underset{\sim}{\hat{\phi}} + \cancel{\frac{\partial B}{\partial z}}\underset{\sim}{\hat{z}} = -\frac{\mu_0 I_0}{2\pi r^2}\underset{\sim}{\hat{r}}$

a. Gradient points inward as field decreases outward



3. ∇B Drift:

$$\underset{\sim}{v}_{\nabla B} = -\frac{v_\perp^2}{2\omega_c}\frac{\left(-\frac{\mu_0 I_0}{2\pi r^2}\hat{r}\right)\times\frac{\mu_0 I_0}{2\pi r}\hat{\phi}}{\left(\frac{\mu_0 I_0}{2\pi r}\right)^2} = +\frac{v_\perp^2}{2\omega_c r}\underset{\sim}{\hat{z}}$$

4. Curvature Drift:

$$\underset{\sim}{v}_c = \frac{v_\parallel^2}{\omega_c \left(\frac{\mu_0 I_0}{2\pi r}\right)}\frac{\cancel{r}\hat{r}\times\frac{\mu_0 I_0}{2\pi r}\hat{\phi}}{r^2} = \frac{v_\parallel^2}{\omega_c r}\underset{\sim}{\hat{z}}$$

5. Thus, the sum of the drifts is

a. $\underset{\sim}{v} = \underset{\sim}{v}_B + \underset{\sim}{v}_c = \frac{1}{\omega_c r}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\underset{\sim}{\hat{z}}$

b. NOTE: $\frac{1}{\omega_c r} = \frac{m}{qBr} = \frac{m 2\pi \cancel{r}}{q\mu_0 I_0 \cancel{r}}$, so $\boxed{\underset{\sim}{v} = \frac{2\pi}{q\mu_0 I_0}\left(\frac{m v_\perp^2}{2} + m v_\parallel^2\right)\underset{\sim}{\hat{z}}}$

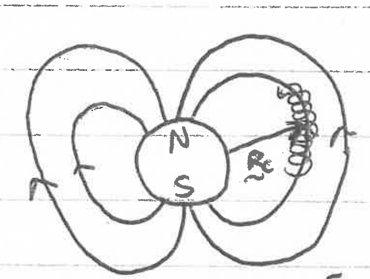
c. The drift velocity does not depend on r! A steady current is caused by the drift of ions and electrons (opposite directions) in $\underset{\sim}{\hat{z}}$

B. Earth's Magnetosphere

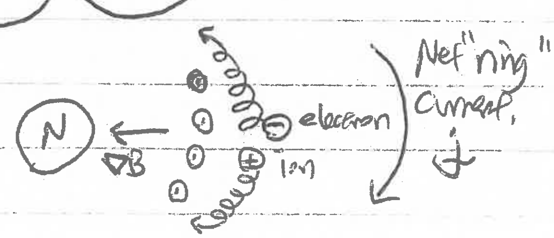
1. Particles trapped in Earth's dipole field experience ∇B and curvature drifts.
2. Produces the "ring current" in the westward direction
3. Strength of ring current is proportional to energy of particles.

⇒ Magnetic Storms!

Side view:



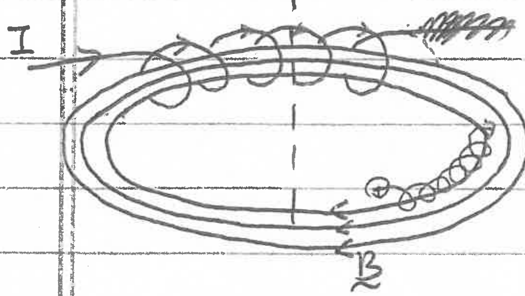
Polar view



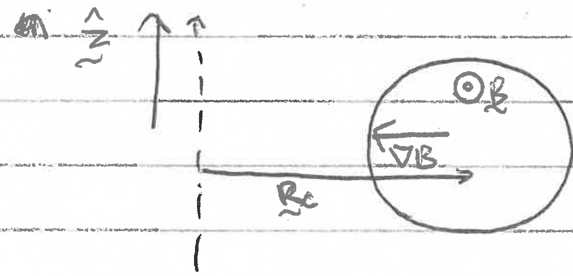
Lecture #5 (Continued) Hawes (7)

II (Continued)

C. Particle Confinement in a Toroidal Magnetic Field



1. Toroidal Magnetic Field can be produced by winding a current-carrying wire around a torus.
2. Lowest order motion is helical Larmor motion around a field line which closes on itself. ⇒ Good Confinement?
3. Consider particle drifts due to ∇B and curvature:



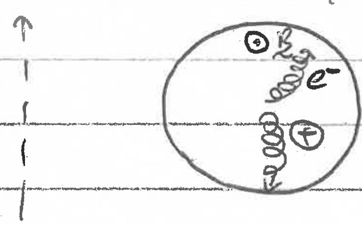
a. ∇B drift

$$\underset{\sim}{v}_{\nabla B} \propto -\nabla B \times \underset{\sim}{B} \propto -\hat{\underset{\sim}{z}}$$

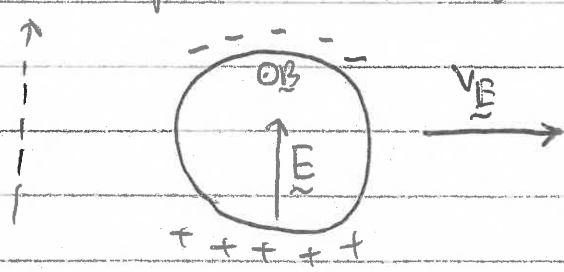
b. Curvature

$$\underset{\sim}{v}_c \propto \underset{\sim}{R}_c \times \underset{\sim}{B} \propto -\hat{\underset{\sim}{z}}$$

c. For ions, drifts add in $-\hat{z}$ direction, electrons in $+\hat{z}$ direction.

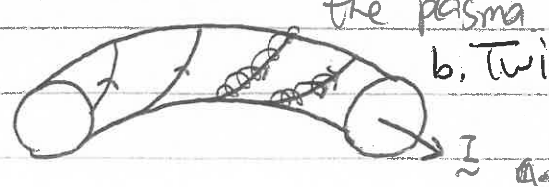


d. Drifts leads to a polarization charge



e. $\underset{\sim}{E} \times \underset{\sim}{B}$ drift will cause plasma to move outward and be lost.

4. a. This loss can be stopped by sending a toroidal current through the plasma



b. Twisted magnetic field prevents buildup of polarization charge.