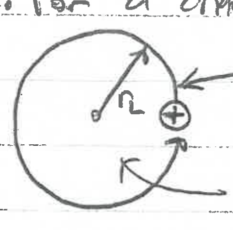
Lecture #6: Magnetic Moment and the Mirror Force Hawes ①

I. Magnetic Moment

A. Magnetic moment due to particle Larmor Motion

1. A current loop has a magnetic moment $\mu = IA$
2. For a charged particle with charge q in Larmor Motion

a. $\underset{\sim}{B} \otimes$



$I = \frac{\text{charge}}{\text{time}} = \frac{q}{(2\pi/\omega_c)} = \frac{\omega_c q}{2\pi}$

$A = \pi r_L^2 = \pi \frac{v_\perp^2}{\omega_c^2}$

b. Thus $\mu = IA = \left(\frac{q\omega_c}{2\pi}\right)\left(\pi \frac{v_\perp^2}{\omega_c^2}\right) = \frac{q v_\perp^2}{2\left(\frac{qB}{m}\right)} = \boxed{\frac{m v_\perp^2}{2B} = \mu}$ Magnetic Moment

II. The Mirror Force

A. What happens when $\nabla B \parallel \underline{B}$?

1. Because Maxwell's Equations demand $\nabla \cdot \underline{B} = 0$, for magnetic field to increase along field line, another component must change.

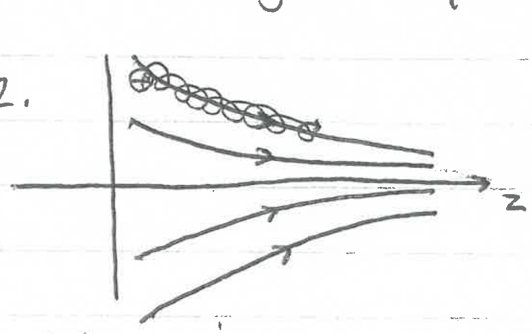
a. Cylindrical Coordinates (NRL p. 6) $\nabla \cdot \underset{\sim}{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0$

b. Take axisymmetric field $\left(\frac{\partial}{\partial \phi} = 0\right)$ with no azimuthal component $(B_\phi = 0)$.

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) = -\frac{\partial B_z}{\partial z}$$

c. Assuming $\frac{\partial B_z}{\partial z}$ is independent of r (valid for small r), we can integrate to yield. $B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$ (Assume constant of integration is zero)

2.



Increasing field along z direction requires a B_r component.

3. What is the particle motion in such a field?

II. (Continued)

B. Force on Particle

1. We want to find $\underset{\sim}{F} = q(\underset{\sim}{v} \times \underset{\sim}{B})$ for this case.

a. $\underset{\sim}{B} = \epsilon B_r \hat{\underset{\sim}{r}} + B_z \hat{\underset{\sim}{z}} = -\epsilon \frac{r}{2}\frac{\partial B_z}{\partial z}\hat{\underset{\sim}{r}} + B_z \hat{\underset{\sim}{z}}$

b. $\underset{\sim}{v} = v_r \hat{\underset{\sim}{r}} + v_\phi \hat{\underset{\sim}{\phi}} + v_z \hat{\underset{\sim}{z}}$

2. In cylindrical coordinates:

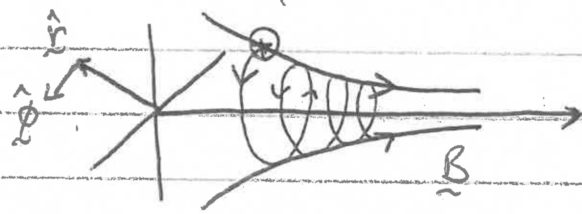
$$\underset{\sim}{F} = q(v_\phi B_z - v_z \cancel{B_\phi})\hat{\underset{\sim}{r}} + q(\epsilon v_z B_r - v_r B_z)\hat{\underset{\sim}{\phi}} + q(v_r \cancel{B_\phi}^{\,0} - \epsilon v_\phi B_r)\hat{\underset{\sim}{z}}$$

large terms small terms

3. $O(1)$: $\underset{\sim}{F} = q v_\phi B_z \hat{\underset{\sim}{r}} - q v_r B_z \hat{\underset{\sim}{\phi}} = q(v_\phi \hat{\underset{\sim}{\phi}} + v_r \hat{\underset{\sim}{r}}) \times (B_z \hat{\underset{\sim}{z}})$

a. These terms are just the usual terms dictating Larmor motion.

b. This is easy to see for a particle with a guiding center at $r=0$.



For this case, $v_\phi = -v_\perp$ $v_r = 0$ (for $q>0$)

c. Thus $\underset{\sim}{F} = -q v_\perp B_z \hat{\underset{\sim}{r}}$ provides centripetal acceleration for Larmor motion.

d. $\frac{m v_\perp^2}{r} = q v_\perp B_z$ Solving for $r = \frac{m v_\perp}{qB} = \frac{v_\perp}{\omega_c} = r_L$!

4. $O(\epsilon)$: Axial Component: $F_z = -q v_\phi B_r$

a. Again, we take the case for a particle guiding center at $r=0$. Thus $v_\phi = -v_\perp$ and $r = r_L$.

b. $$F_z = -q(-v_\perp)\left(-\frac{r_L}{2}\frac{\partial B_z}{\partial z}\right) \underset{\uparrow\; r_L = \frac{v_\perp}{\omega_c}}{=} -q\frac{v_\perp^2}{2\omega_c}\frac{\partial B_z}{\partial z} = -q\left(\frac{m}{qB}\right)\frac{v_\perp^2}{2}\frac{\partial B_z}{\partial z} = -\left(\frac{m v_\perp^2}{2B}\right)\frac{\partial B_z}{\partial z}$$

This can be written $\boxed{F_z = -\mu \frac{\partial B_z}{\partial z}}$ Magnetic Mirror Force

II. B. (Continued)

5. The Mirror Force accelerates the particle along the field line in the direction of decreasing magnetic field magnitude.

6. This can be written in general as

$$\boxed{\underset{\sim}{F} = -\mu(\hat{\underset{\sim}{b}}\cdot\nabla)\underset{\sim}{B}}$$

where $\hat{b}\cdot\nabla$ is the gradient along the field $\underset{\sim}{B}$.

a. Compare to the electrostatic force on a charge.
For $\underset{\sim}{E} = -\nabla\phi$ and $\underset{\sim}{F} = q\underset{\sim}{E}$, $\underset{\sim}{F} = -q\nabla\phi$

b. The Mirror Force acts on the particle magnetic moment $\mu = \frac{mv_\perp^2}{2B}$, where the field magnitude B appears like a potential.
$\Rightarrow$ Repels particles from strong field region!

7. $\mathcal{O}(\epsilon)$: Azimuthal Component $F_\phi = q v_z B_r$

a. The presence of an azimuthal component of force means particles can gain energy in the perpendicular component at rate $v_\phi F_\phi$.

b. For perpendicular energy $W_\perp = \frac{1}{2}mv_\perp^2$, we have

$$\frac{dW_\perp}{dt} = v_\phi F_\phi = q(v_\phi) v_z B_r$$

c. For ions, $v_\phi = -v_\perp$ and $B_r = -\frac{r_L}{2}\frac{\partial B_z}{\partial z}$ for particle guiding center at $z=0$,

$$\frac{dW_\perp}{dt} = q(-v_\perp)\left(-\frac{r_L}{2}\frac{\partial B}{\partial z}\right)v_z = \frac{q\, v_\perp^2}{2\omega_c} v_z\frac{\partial B_z}{\partial z} = \frac{mv_\perp^2}{2B}v_z\frac{\partial B_z}{\partial z} = \mu\, v_z\frac{\partial B_z}{\partial z}$$

d. NOTE: $\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{dx}{dt}\frac{\partial B}{\partial x} + \frac{dy}{dt}\frac{\partial B}{\partial y} + \frac{dz}{dt}\frac{\partial B}{\partial z} = \frac{\partial B}{\partial t} + v_x\frac{\partial B}{\partial x} + v_y\frac{\partial B}{\partial y} + v_z\frac{\partial B}{\partial z} = \frac{\partial B}{\partial t} + \underset{\sim}{v}\cdot\nabla B$

Since $\frac{dB_z}{dt} = \cancel{\frac{\partial B_z}{\partial t}}^{0} + \underset{\sim}{v}\cdot\nabla B_z = v_z\frac{\partial B}{\partial z}$, we get

$$\frac{dW_\perp}{dt} = \mu\frac{dB}{dt}$$

e. But $\mu = \frac{W_\perp}{B}$, so $\frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{dB}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0 \Rightarrow \boxed{\frac{d\mu}{dt} = 0}$

III. Adiabatic Invariance

A. Interpretation:

1. $\frac{d\mu}{dt} = 0$ implies that, as a charged particle moves through a changing field $\mu = \frac{mv_\perp^2}{2B}$ remains constant.

B. Alternative Derivation:

1. First, note $m\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\underset{\sim}{v} \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) = 0$

   a. Therefore, total energy is constant $\mathcal{E} = \frac{1}{2}mv^2 = \frac{1}{2}m(v_\parallel^2 + v_\perp^2)$

   b. Thus $\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) + \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{mv_\parallel^2}{2}\right) = -\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right)$

2. Mirror Force equation: $F_\parallel = -\mu(\hat{\underset{\sim}{b}} \cdot \nabla)B = m\frac{dv_\parallel}{dt}$

   a. Multiply by $v_\parallel$:

   $$m v_\parallel \frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu(v_\parallel \hat{\underset{\sim}{b}}) \cdot \nabla B = -\mu \underset{\sim}{v}_\parallel \cdot \nabla B$$

   b. Again $\frac{dB}{dt} = \cancel{\frac{\partial B}{\partial t}}^{0} + \underset{\sim}{v} \cdot \nabla B = \underset{\sim}{v}_\parallel \cdot \nabla B$, so

   $$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{dB}{dt} = \underbrace{-\frac{mv_\perp^2}{2}\frac{1}{B}\frac{dB}{dt} = -\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right)}$$

   c. ~~Note:~~ Multiply by $\frac{1}{B}$

   $$\frac{1}{B}\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) - \frac{mv_\perp^2}{2}\frac{1}{B^2}\frac{dB}{dt} = 0$$

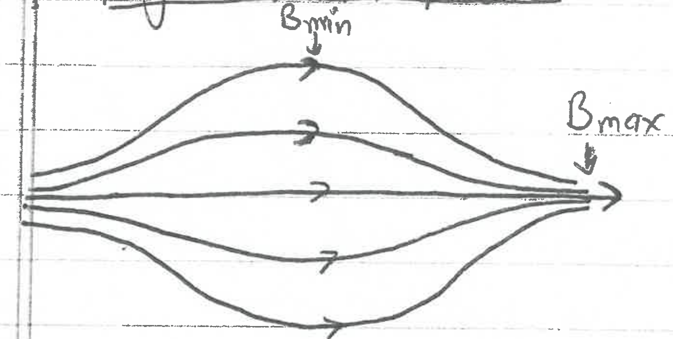
   d. NOTE: $\frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{\frac{mv_\perp^2}{2}}{B}\right) = \frac{1}{B}\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) - \frac{mv_\perp^2}{2}\frac{1}{B^2}\frac{dB}{dt}$

   e. Thus $\boxed{\frac{d\mu}{dt} = 0.}$

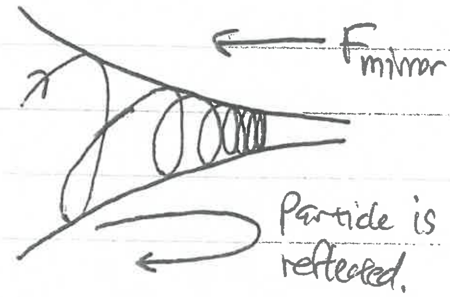
Lecture #6 (Continued) Howes ⑤

IV. Confinement by Magnetic Mirror

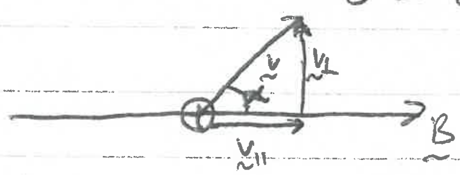
A. Magnetic Mirror Machine:



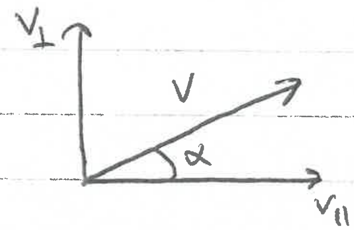
1. Particles are confined by magnetic mirror force at either end of the machine.



2. Pitch Angle:



$\alpha \equiv$ angle between velocity vector and magnetic field.



$v_\parallel = v\cos\alpha$

$v_\perp = v\sin\alpha$

$\underset{\sim}{v} \cdot \underset{\sim}{B} = vB\cos\alpha$

3. Parallel Equation of motion $F_\parallel = -\mu\, \underset{\sim}{\hat{b}} \cdot \nabla B$ can be written

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s}$$ where "s" is distance along field line.

a. $\dfrac{dv_\parallel}{dt} = \cancelto{0}{\dfrac{\partial v_\parallel}{\partial t}} + \underset{\sim}{v}\cdot\nabla v_\parallel = v_\parallel \dfrac{\partial v_\parallel}{\partial s}$ along field line.

b. $m v_\parallel \dfrac{\partial v_\parallel}{\partial s} = \dfrac{\partial}{\partial s}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\dfrac{\partial B}{\partial s} = -\dfrac{\partial \mu B}{\partial s}$ since $\mu =$ constant.

c. Thus $\dfrac{d}{ds}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right) = 0$ along field line

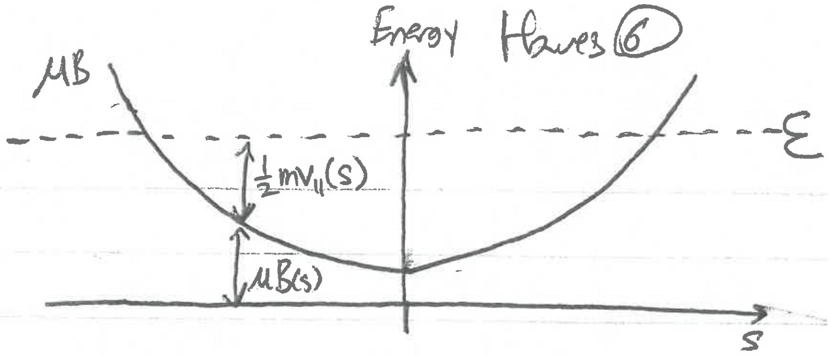
Therefore $\boxed{\mathcal{E} = \frac{1}{2}m v_\parallel^2(s) + \mu B(s)}$

$\mathcal{E} =$ const Conservation of Energy

$\mu =$ const Adiabatic Invariant

Lecture #6 (Continued)

IV. A. (Continued)



4. Potential Interpretation:

a. A charged particle in an electrostatic field $\underline{E} = -\nabla\phi$ has conserved energy $\mathcal{E} = \frac{1}{2}mv^2 + q\phi$

b. Here conservation involves parallel velocity and magnetic field magnitude $\mathcal{E} = \frac{1}{2}mv_{\parallel}^2 + \mu B$

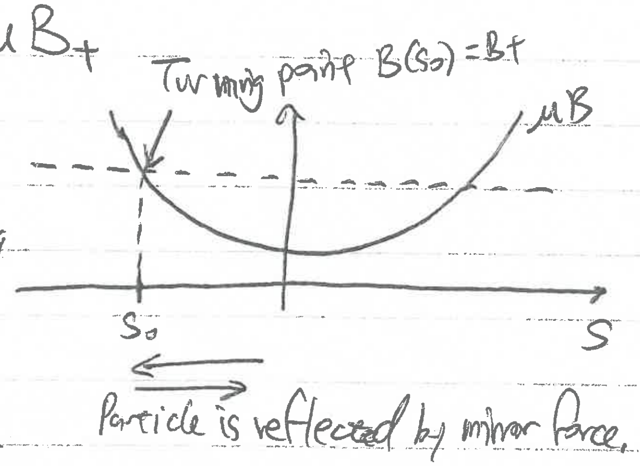
5. We can solve for $v_{\parallel}(s)$

$$v_{\parallel}(s) = \pm\sqrt{\frac{2}{m}\left[\mathcal{E} - \mu B(s)\right]}$$ where $\mathcal{E}$ and μ are constants

a. When $v_{\parallel}(s_0) = 0$, the particle reaches a turning point.

Here $B(s_0) = \dfrac{\mathcal{E}}{\mu} \equiv B_+$

Thus $\mathcal{E} = \frac{1}{2}mv_{\parallel}^2 + \mu B = \mu B_+$



6. Physical Interpretation:

a. The particle experiences a changing $|\underline{B}|$ as it moves along field.

b. Induced azimuthal force F_ϕ does work on particle, increasing $v_\perp$

c. Total energy $\frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2 = \mathcal{E}$ is conserved, so $v_{\parallel}$ must decrease

d. Eventually, $v_{\parallel} = 0$, so the particle turns around, having been "mirrored"

7. How does pitch angle $\alpha(s)$ change? $\mathcal{E} = \frac{1}{2}mv_{\parallel}^2 + \mu B$ and $\mathcal{E} = \frac{1}{2}mv^2$

a. $v_{\parallel} = v\cos\alpha$, so $\mathcal{E} = \frac{1}{2}mv^2\cos^2\alpha + \mu B = \mathcal{E}\cos^2\alpha + \mu B$.

b. Thus $1 - \cos^2\alpha = \dfrac{\mu B}{\mathcal{E}}$, or $\sin^2\alpha = \dfrac{\mu B}{\mathcal{E}} = \dfrac{\mu B}{\mu B_+} = \dfrac{B}{B_+}$

Lecture #6 (Continued) Hawes ⑦

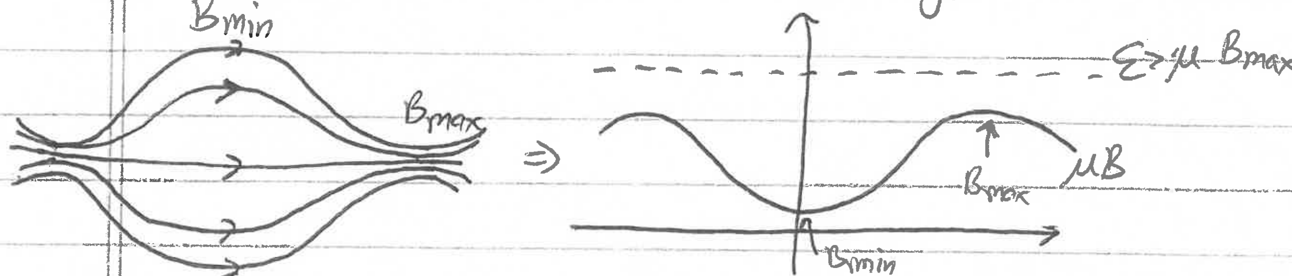
VI A. 7. (Continued)

c. Thus $\sin^2 \alpha(s) = \frac{B(s)}{B_t}$

~~As $B(s)$ increases to B_t, $\sin^2\alpha(s) \to 1$, or $\alpha(s) \to \frac{\pi}{2}$.~~

8. Practical Considerations:

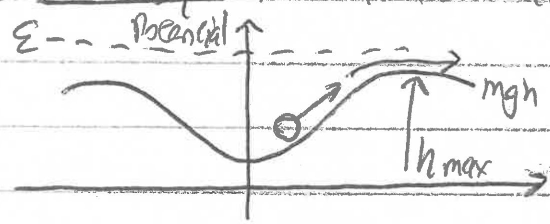
- a. There is a limit to the maximum field strength:



b. For Particles with $\varepsilon > \mu B_{max}$ $\quad V_{\parallel} = \sqrt{\frac{2}{m}(\varepsilon - \mu B_{max})} > 0$

Thus, $V_{\parallel}$ never reaches zero ⇒ particles are not reflected.

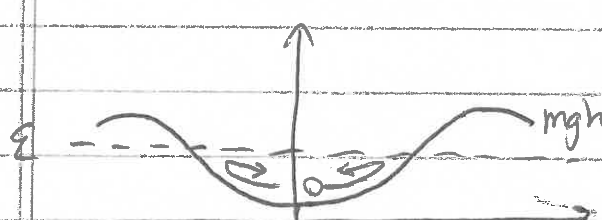
c. Analogy: Frictionless ball on a hill/valley.



$\varepsilon = \frac{1}{2}mv^2 + mgh$

i) If $\varepsilon > mgh_{max}$, ball passes over hill

ii) If $\varepsilon < mgh_{max}$, ball is trapped in valley, oscillating back & forth.



d. 1. We know pitch angle α increases as B increases. $\sin^2\alpha(s) = \frac{B(s)}{B_t}$

2. Thus, at $B = B_{min}$, pitch angle is at a minimum.

3. For a particle which reaches $\alpha = \frac{\pi}{2}$ ($V_{\parallel} = 0$) at $B = B_{max}$, what is its pitch angle at B_{min}? ⇒ $\sin^2\alpha(s) = \frac{B(s)}{B_{max}}$

For $B(s) = B_{min}$,

$$\boxed{\sin^2 \alpha_{min} = \frac{B_{min}}{B_{max}}}$$

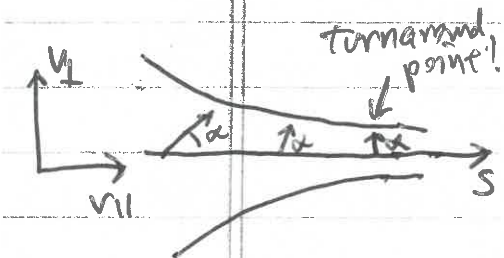
Lecture #6 (Continued) Hawes ⑧

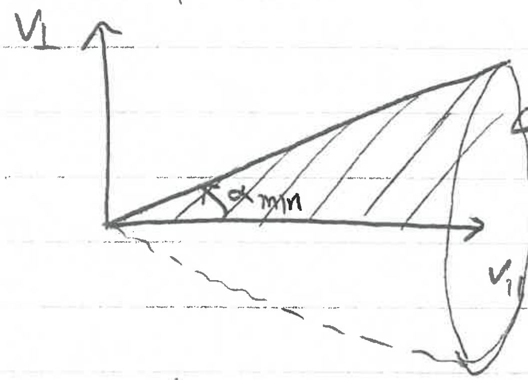
IV. A. 8. (Continued)

e. $\boxed{\text{Thus, for particles with } \alpha < \alpha_{min}\text{, particles will escape from magnetic mirror.}}$

f. The <u>Mirror Ratio</u> $\boxed{R_m \equiv \frac{B_{max}}{B_{min}}}$. Thus $\boxed{\sin^2 \alpha_{min} = \frac{1}{R_m}}$

g. Looking in velocity space





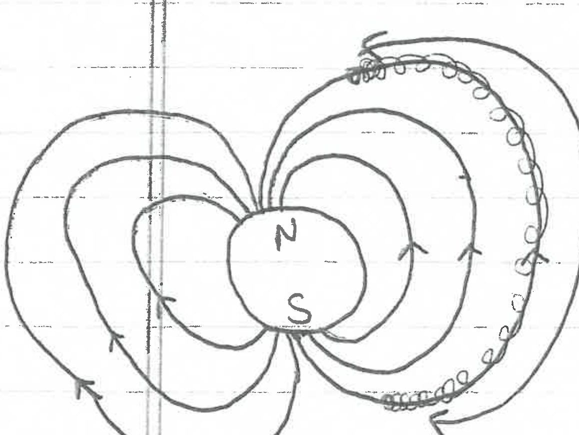
Particles with $\alpha < \alpha_{min}$ will be lost.

$\Rightarrow$ $\boxed{\text{Loss Cone}}$

h. 1. In a collisionless plasma, all particles with $\alpha < \alpha_{min}$ will be lost from mirror.

2. In a collisional plasma, particle collisions will scatter particles into the loss cone, and eventually much of the plasma will be lost.

B. Earth's Magnetosphere:



Bounce Motion

1. Dipole field of earth behaves as a magnetic mirror
- Weak field at equator
- Strong field at poles

2. Particles trapped on field lines will bounce from pole to pole.

(Don't forget ∇B & curvature drifts also lead to motion westward around the earth)