PHYS:7729 Homework #4

Due at the beginning of class, Thursday, March 9, 2023.

1. (10 pts) A plasma has a "spherical shell" distribution function given by

$$f_0(\mathbf{v}) = \frac{n_0}{4\pi C^2}\delta(|\mathbf{v}| - C)$$

where C is a constant.

   (a) Using the Fourier analysis approach, show that the dispersion relation for electrostatic waves in this plasma is $\omega^2 = \omega_p^2 + k^2C^2$.

   (b) What is the region of validity of this dispersion relation?

2. (6 pts) Show that the Laplace transform of $f(t) = \cosh(at)$ is given by

$$\tilde{f}(p) = \frac{p}{p^2 - a^2}.$$

3. (6 pts) Use the Residue Theorem to evaluate the inverse Laplace transform of

$$\tilde{f}(p) = \frac{1}{p^2 - a^2}.$$

4. (18 pts) Solution of Navier-Stokes Equations:
The Navier-Stokes Equations for the viscous evolution of a hydrodynamic fluid are given by:

$$\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{U}$$

$$\rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla p + \nu \rho \nabla^2 \mathbf{U}$$

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{U}$$

where ν is the coefficient of kinematic viscosity. Assume a wave vector of the form $\mathbf{k} = k\hat{\mathbf{z}}$. The initial conditions for a sound wave in this system at $t = 0$ are $\mathbf{U}(\mathbf{x}, 0) = \overline{U} \cos(kz)\hat{\mathbf{z}}$ and $\mathbf{U}'(\mathbf{x}, 0) = -\overline{U}\omega_0 \sin(kz)\hat{\mathbf{z}}$. Use the Laplace-Fourier transform method (Fourier transform in space, Laplace transform in time) to solve for the velocity $\mathbf{U}(\mathbf{x}, t)$. Note that the z-component of the velocity U_z is the only non-trivial part of the solution.
HINT: This is similar to a linear dispersion relation problem, so your first step is to linearize the Navier-Stokes equations.

(a) Fourier transform the linearized equations and find the differential equation for $U_z(\mathbf{k}, t)$ in terms of time derivatives. Use the definition of the sound speed $c_s^2 = \gamma p_0/\rho_0$ to simplify the equation.

(b) Solve for the Laplace transform $\tilde{U}_z(\mathbf{k}, p)$.

(c) Perform the inverse Laplace transform to find a solution for $U_z(\mathbf{k}, t)$. You may wish to define $\omega^2 = k^2c_s^2 - \nu^2k^4/4$ to simplify notation.

(d) Fourier transform the initial conditions and apply them to the answer above so that you may obtain the final solution $U_z(\mathbf{x}, t)$.

(e) Determine the evolution of the magnitude of the velocity $|U_z(\mathbf{x}, t)|$ for $\nu^2k^2 < 4c_s^2$.

(f) In the weak damping limit $\nu^2k^2 \ll 4c_s^2$, what are the effective real frequency of oscillation (include the small, first order correction) and damping rate?

(g) Qualitatively sketch the solution $U_z(z = 0, t)$ in the case that $\nu^2k^2 < 4c_s^2$.

(h) Qualitatively sketch $U_z(z = 0, t)$ for the cases $\nu^2k^2 = 4c_s^2$ and $\nu^2k^2 > 4c_s^2$ on the same plot (but a different plot from part d).