PHYS:7729 Homework #5

Due at the beginning of class, Thursday, March 23, 2023.

1. We have shown in class that the Cauchy distribution

$$F_{0C}(v_z) = \frac{C}{\pi}\frac{1}{C^2 + v_z^2}$$

yields a dispersion relation

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C)^2}$$

which has a solution $\omega = \pm\omega_p$ and $\gamma = -|k|C$. Show that in the high phase velocity limit ($|k| \to 0$), the weak growth rate approximation gives the same result.

Note: For this problem, you can start with the dispersion relation expression above (you do not need to rederive the dispersion relation from the Cauchy distribution).

2. Large Argument Expansion of the Plasma Dispersion Function

   (a) If the pole at $\xi = x + iy$ is very close to the z axis ($|y| \ll |x|$), show that the plasma dispersion function is given by

$$Z(\xi) = i\frac{k}{|k|}\sqrt{\pi}e^{-\xi^2} + \frac{1}{\sqrt{\pi}}P\int_{-\infty}^{\infty}\frac{e^{-z^2}}{z-\xi}dz.$$

   (b) By writing

$$\frac{1}{z-\xi} = \frac{-1}{\xi(1-z/\xi)} = -\frac{1}{\xi}\left[1 + \left(\frac{z}{\xi}\right) + \left(\frac{z}{\xi}\right)^2 + \cdots\right]$$

   and integrating term by term, show that in the limit of large ξ the plasma dispersion function is given by the following power series

$$Z(\xi) = i\frac{k}{|k|}\sqrt{\pi}e^{-\xi^2} - \left[\frac{1}{\xi} + \frac{1}{2\xi^3} + \frac{3}{4\xi^5} + \cdots\right].$$

3. Using the Error Function representation of the plasma dispersion function

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2}[1 + \mathrm{erf}(i\xi)]$$

where

$$\mathrm{erf}(i\xi) = \frac{2}{\sqrt{\pi}}\int_0^{i\xi} e^{-z^2}dz,$$

show that for small ξ,

$$Z(\xi) = i\sqrt{\pi}e^{-\xi^2} - 2\xi + \frac{4}{3}\xi^3 - \frac{8}{15}\xi^5 + \cdots$$

Hint: Use a Taylor Series expansion for $\exp(-\xi^2)$ and $\exp(-z^2)$, integrate term by term, and then collect like powers of ξ.

4. For a plasma consisting of protons and electrons, both with Maxwellian velocity distributions, the dispersion relation can be written

$$D(\mathbf{k}, p) = 1 - \frac{1}{k^2 \lambda_{De}^2} \frac{1}{2} \left[Z'(\xi_e) + \frac{T_e}{T_i} Z'(\xi_i) \right] = 0,$$

where T_e is the electron temperature, T_i is the ion temperature, and the derivative of the Plasma Dispersion Function is denoted by $Z'(\xi) = \partial Z(\xi)/\partial \xi$.

HINT: Use the weak growth rate approximation to solve this problem.

(a) Use the large-argument expansion of the plasma dispersion function for the ions and the small argument expansion for the electrons to simplify the dispersion relation and obtain the analytical solutions

$$\frac{\omega}{k} = \pm \sqrt{\frac{T_e}{m_i}} \frac{1}{(1 + k^2 \lambda_{De}^2)^{1/2}},$$

and

$$\gamma/\omega = -\sqrt{\frac{\pi}{8}} \left[\sqrt{\frac{m_e}{m_i}} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp\left(-\frac{T_e}{2T_i} \frac{1}{(1 + k^2 \lambda_{De}^2)} \right) \right] \frac{1}{(1 + k^2 \lambda_{De}^2)^{3/2}}.$$

(b) In what limit of the real frequency ω is this solution valid?

5. (a) To model a hot beam, one can use a shifted Cauchy distribution of the form

$$F_0(v_z) = \frac{C}{\pi} \frac{1}{C^2 + (v_z - U)^2}$$

where U is the beam velocity. Show that the dispersion relation for this plasma is

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C + ikU)^2} = 0$$

(b) Solve for the real frequency and damping rate of such a plasma.