

# PHYS:7729 Homework #6

Due at the beginning of class, Thursday, March 30, 2023.

1. In his 1959 paper, Buneman analyzed the maximum growth rate of a cold electron beam streaming at velocity  $V$  through an equal number density of ions initially at rest. The dispersion relation for this system is given by

$$D(\mathbf{k}, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV - \omega)^2} = 0.$$

- (a) If we assume that  $\omega_{pi}^2 \ll \omega^2$ , show that the dispersion relation can be written

$$kV = \omega + \omega_{pe} + \frac{1}{2} \frac{\omega_{pe} \omega_{pi}^2}{\omega^2}.$$

- (b) Show that the above dispersion relation has a solution for real  $k$  of the form

$$\omega = (\omega_{pe} \omega_{pi}^2 \cos \theta)^{1/3} e^{i\theta}$$

where  $\theta$  is any arbitrary angle.

- (c) Show that the imaginary part of  $\omega$  reaches a maximum when  $\theta = \pi/3$ .  
(d) Show that, when  $\theta = \pi/3$ ,

$$\gamma = \omega_{pe} \left( \frac{m_e}{2m_i} \right)^{1/3} \frac{\sqrt{3}}{2}$$

and

$$\omega_r = \omega_{pe} \left( \frac{m_e}{2m_i} \right)^{1/3} \frac{1}{2}.$$

- (e) Is assumption (a) satisfied?

2. Consider the problem of the counter-streaming Cauchy distribution,

$$F_0(v_z) = \frac{C}{2\pi} \left[ \frac{1}{C^2 + (v_z - V)^2} + \frac{1}{C^2 + (v_z + V)^2} \right].$$

- (a) Sketch the distribution for the choice  $C = V$ .  
(b) Apply Gardner's Theorem to show that stability is guaranteed for  $C > \sqrt{3}V$ .  
(c) Apply the Penrose Criterion to show that this equilibrium is stable for  $C > V$  and unstable for  $C < V$ .